BEH. 410 - February 2003

Problem set # 1 - SOLUTION

Problem 1 - Boltzmann's relation and cell micromechanics

a) Expression for the mean squared displacement relative to the average position

Credit was given for these two approaches:

- Prove that $\langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$

$$\langle (x-\langle x\rangle)^2 \rangle = \langle x^2 - 2x\langle x\rangle + \langle x\rangle^2 \rangle = \langle x^2\rangle - 2\langle x\rangle\langle x\rangle + \langle x\rangle^2 = \langle x^2\rangle - \langle x\rangle^2$$

the "average" function is linear
$\langle\langle x \rangle\rangle = \langle x \rangle$

□ Analytical:

Boltzmann's distribution law with energy $\mathcal{E}_x$ associated to the x-coordinate (work by spring)

$$p(x) = \frac{1}{Q}\exp\left(\frac{-\mathcal{E}_x}{k_B T}\right) \quad \text{where} \quad \begin{cases} \mathcal{E}_x = \frac{1}{2}k_s(x-x_0)^2, & x_0 = \text{equilibrium length on } x\text{-axis} \\ Q = \int_{-\infty}^{+\infty} \exp\left(\frac{-\mathcal{E}_x}{k_B T}\right) dx \end{cases}$$

Introducing $\sigma^2 = \frac{k_B T}{k_s}$, $Q = \int_{-\infty}^{+\infty} \exp\left(\frac{-(x-x_0)^2}{2\sigma^2}\right) dx = \sigma\sqrt{2\pi}$

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{-(x-x_0)^2}{2\sigma^2}\right)$$

$$\langle x\rangle = \int_{-\infty}^{+\infty}\frac{x}{\sigma\sqrt{2\pi}}\exp\left(\frac{-(x-x_0)^2}{2\sigma^2}\right)dx = \frac{-\sigma}{\sqrt{2\pi}}\left[\exp\left(\frac{-(x-x_0)^2}{2\sigma^2}\right)\right]_{-\infty}^{+\infty} = 0$$

$$\langle x^2\rangle = \int_{-\infty}^{+\infty}\frac{x^2}{\sigma\sqrt{2\pi}}\exp\left(\frac{-(x-x_0)^2}{2\sigma^2}\right)dx \underset{\text{integration by parts}}{=} \left[\frac{-\sigma}{\sqrt{2\pi}}\,x\,\exp\left(\frac{-(x-x_0)^2}{2\sigma^2}\right)\right]_{-\infty}^{+\infty} + \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left(\frac{-(x-x_0)^2}{2\sigma^2}\right)dx$$

$$\langle x^2\rangle = 0 + \frac{\sigma}{\sqrt{2\pi}}\sigma\sqrt{2\pi} = \sigma^2 = \frac{k_B T}{k_s}$$

$$\boxed{\langle (x-\langle x\rangle)^2\rangle = \frac{k_B T}{k_s}}$$

□ Dimensional

Thermal energy $\sim k_B T$

Elastic energy $\sim F_{spring}\cdot d$ where
$$\begin{cases} d = \text{bead diameter} = 40\text{ nm} = \text{characteristic length} \\ F_{spring} = k_s r \quad \text{with} \quad r = \sqrt{(x-\langle x\rangle)^2 + (y-\langle y\rangle)^2} \\ \text{because } y \text{ equivalent to } x \quad r \sim \sqrt{2(x-\langle x\rangle)^2} \end{cases}$$

From $k_B T \sim k_s r d$, we get

$$\boxed{\langle (x-\langle x\rangle)^2\rangle \sim \frac{1}{2}\left(\frac{k_B T}{k_s d}\right)^2}$$

HOWEVER, SUCH CONSIDERATIONS OF DIMENSIONAL ANALASIS ARE ONLY EXPECTED FROM YOU LATER IN THE SEMESTER. SO DON'T GET BOTHERED AND CONFUSED IF THEY DON'T MAKE IMMEDIATE SENSE TO YOU TODAY.

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Problem set # 1. SOLUTION

Problem 1 (cont'd)

b) Estimation of the spring constant k_s

Both approaches give the same order of magnitude for k_s:

- Find an estimate for $\langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 = \langle x^2 \rangle$

Time (ms)	0	1000	2000	3000	4000
x (nm)	-50	-25	25	15	0
x^2 (nm^2)	2500	625	625	225	0

$\langle x^2 \rangle \simeq 8 \cdot 10^{-16}\ m^2$ (same order of magnitude if y used instead of x)

- $k_B = 1.38 \cdot 10^{-23}\ J.K^{-1}$
  $T = 310\ °K$ (body temperature)

□ Analytical $\quad k_s = \dfrac{k_B T}{\langle (x - \langle x \rangle)^2 \rangle} \simeq 5 \cdot 10^{-6}\ J.m^{-2}$

□ Dimensional $\quad k_s \simeq \dfrac{k_B T}{d\sqrt{2\langle (x - \langle x \rangle)^2 \rangle}} \simeq 3 \cdot 10^{-6}\ J.m^{-2}$

c) Principle of equipartition of energy

Suppose $\mathcal{E}_x = a x^2$, a constant. Reproducing the reasoning of part a):

$$p(x) = \frac{\exp\left(\frac{-ax^2}{k_B T}\right)}{\int_{-\infty}^{+\infty} \exp\left(\frac{-ax^2}{k_B T}\right) dx} = \sqrt{\frac{a}{\pi k_B T}} \exp\left(\frac{-ax^2}{k_B T}\right)$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 p(x)\, dx = \frac{k_B T}{2a}$$ (same derivations as in a)) or $\begin{cases} a\langle x^2 \rangle = \frac{1}{2} k_B T \\ \text{equipartition of energy} \end{cases}$

Each degree of freedom of the system contributes to its energy by $\frac{1}{2} k_B T$.

Note: this could be extended to the case $\mathcal{E}_i = a x_i^n$, $n > 0$, and x_i = coordinate

With $\beta = \dfrac{1}{k_B T}$,

$$\langle \mathcal{E}_i \rangle = \frac{\int_{-\infty}^{+\infty} e^{-\beta \mathcal{E}_i} \mathcal{E}_i\, dx_i}{\int_{-\infty}^{+\infty} e^{-\beta \mathcal{E}_i}\, dx_i} = -\frac{\partial}{\partial \beta} \ln \int_{-\infty}^{+\infty} e^{-\beta \mathcal{E}_i} dx_i = -\frac{\partial}{\partial \beta} \ln \int_{-\infty}^{+\infty} e^{-\beta a x_i^n} dx_i$$

$$= -\frac{\partial}{\partial \beta} \ln \left(\beta^{-1/n} a^{-1/n} \int_{-\infty}^{+\infty} e^{-u^n} du \right.$$ with $u = (\beta a)^{1/n} x_i$

$$= \frac{1}{n\beta} = \frac{kT}{n}$$

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Problem 2 - Microrheology & the Langevin equation

a) Expression for the autocorrelation function $\langle x(t+\tau)\,x(t)\rangle$

□ Here's the most direct way to get to the result :

$$\langle x(t+\tau)\,x(t)\rangle = \left\langle \int_{-\infty}^{t+\tau} \exp\left(\frac{-\kappa(t+\tau-t'_1)}{\zeta}\right)\frac{f(t'_1)}{\zeta}\,dt'_1 \cdot \int_{-\infty}^{t} \exp\left(\frac{-\kappa(t-t'_2)}{\zeta}\right)\frac{f(t'_2)}{\zeta}\,dt'_2 \right\rangle$$

$$= \left\langle \int_{-\infty}^{t} \exp\left(\frac{-\kappa(t-t'_2)}{\zeta}\right)\left\{\int_{-\infty}^{t+\tau} \exp\left(\frac{-\kappa(t+\tau-t'_1)}{\zeta}\right)\cdot\frac{f(t'_1)}{\zeta}\cdot\frac{f(t'_2)}{\zeta}\,dt'_1\right\} dt'_2 \right\rangle$$

$$= \int_{-\infty}^{t} \exp\left(\frac{-\kappa(t-t'_2)}{\zeta}\right)\left\{\int_{-\infty}^{t+\tau} \exp\left(\frac{-\kappa(t+\tau-t'_1)}{\zeta}\right)\langle f(t'_1)f(t'_2)\rangle\cdot\frac{1}{\zeta^2}\,dt'_1\right\} dt'_2$$

$$= \int_{-\infty}^{t} \exp\left(\frac{-\kappa(t-t'_2)}{\zeta}\right)\int_{-\infty}^{t+\tau} \exp\left(\frac{-\kappa(t+\tau-t'_1)}{\zeta}\right)\frac{2\zeta kT}{\zeta^2}\,\delta(t'_1-t'_2)\,dt'_1\,dt'_2$$

Because for $t>0$ and g existing on $]-\infty, t]$, we have $\int_{-\infty}^{t} g(u)\,\delta(u)\,du = g(0)$:

$$\langle x(t+\tau)\,x(t)\rangle = \int_{-\infty}^{t} \exp\left(\frac{-\kappa(t-t'_2)}{\zeta}\right)\exp\left(\frac{-\kappa(t+\tau-t'_2)}{\zeta}\right)\frac{2kT}{\zeta}\,dt'_2$$

$$= \frac{2kT}{\zeta}\exp\left(\frac{-\kappa\tau}{\zeta}\right)\int_{-\infty}^{t}\exp\left(-\frac{2\kappa(t-t'_2)}{\zeta}\right)dt'_2$$

$$= \frac{2kT}{\zeta}\exp\left(\frac{-\kappa\tau}{\zeta}\right)\exp\left(\frac{-2\kappa t}{\zeta}\right)\left[\frac{\zeta}{2\kappa}\exp\left(\frac{2\kappa t'_2}{\zeta}\right)\right]_{t'_2\to-\infty}^{t'_2=t}$$

$$= \frac{kT}{\kappa}\exp\left(\frac{-\kappa\tau}{\zeta}\right)\exp\left(\frac{-2\kappa t}{\zeta}\right)\left\{\exp\left(\frac{2\kappa t}{\zeta}\right) - 0\right\}$$

$$= \frac{kT}{\kappa}\exp\left(\frac{-\kappa\tau}{\zeta}\right)$$

One can see this result as $x(t+\tau)$ being correlated to $x(t)$ in an "exponential decay in τ" manner

b) Expression for $\Delta(\tau)$

$$\Delta(\tau) = \langle (x(t+\tau)-x(t))^2\rangle = \langle (x(t+\tau))^2 - 2\,x(t+\tau)\,x(t) + (x(t))^2\rangle$$

$$= \langle (x(t+\tau))^2\rangle - 2\langle x(t+\tau)\,x(t)\rangle + \langle (x(t))^2\rangle$$

from part a) with $\tau=0$ $\quad \langle (x(t))^2\rangle = \langle (x(t+\tau))^2\rangle = \frac{kT}{\kappa}$

$$\boxed{\Delta(\tau) = \frac{2kT}{\kappa}\left(1-\exp\left(-\frac{\kappa\tau}{\zeta}\right)\right)}$$

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Problem 2 (cont'd)

c) <u>Limits for $\Delta(\tau)$</u>

short τ : $\tau \to 0$ or $\tau \ll \frac{\zeta}{\kappa}$

$$\exp\left(-\frac{\tau\kappa}{\zeta}\right) \sim 1 + \left(-\frac{\tau\kappa}{\zeta}\right) \quad \text{and} \quad \Delta(\tau) \to \frac{2kT}{\kappa} \cdot \frac{\tau\kappa}{\zeta}$$

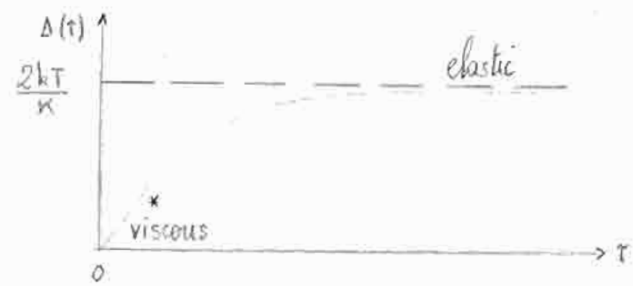
$$\Delta(\tau) \longrightarrow \frac{2kT}{\zeta}\,\tau \quad \underline{\text{linear in } \tau}$$

long τ $\tau \to +\infty$ or $\tau \gg \frac{\zeta}{\kappa}$

$$\exp\left(-\frac{\tau\kappa}{\zeta}\right) \longrightarrow 0$$

$$\Delta(\tau) \longrightarrow \frac{2kT}{\kappa} \quad \underline{\text{constant}}$$

d)



* slope of $\frac{2kT}{\zeta}$

- Viscosity from the linear region of the curve (small τ)
The slope of this region is equal to the diffusion coefficient D of the bead in the complex medium (D indeed relates the squared displacement to time).

$$\text{slope} = \frac{2kT}{\zeta} = D = \frac{kT}{6\pi\mu R} \qquad \text{with } \begin{cases} \mu & \text{viscosity} \\ R & \text{bead radius} \end{cases}$$

$$\mu = \frac{kT}{6\pi R \ \text{slope}}$$

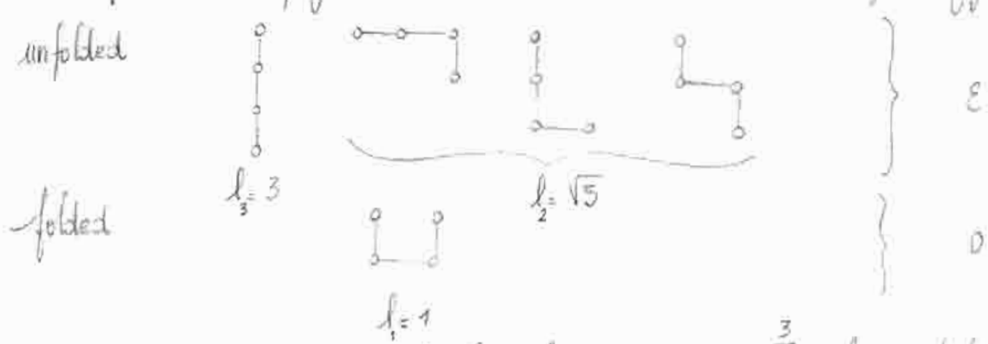
- Spring constant from the limit of $\Delta(\tau)$ as $\tau \longrightarrow +\infty$, which we'll call Δ_∞
Dimensional analysis in 1-D (similar to problem 1 a) !)

$$kT \sim \frac{1}{2} k_s \Delta_\infty$$

$$k_s \simeq \frac{2kT}{\Delta_\infty}$$

Problem 3 - Collapse of a macromolecule

- Consider a four-bead polymer, which has two macrostates of energy ε and 0:

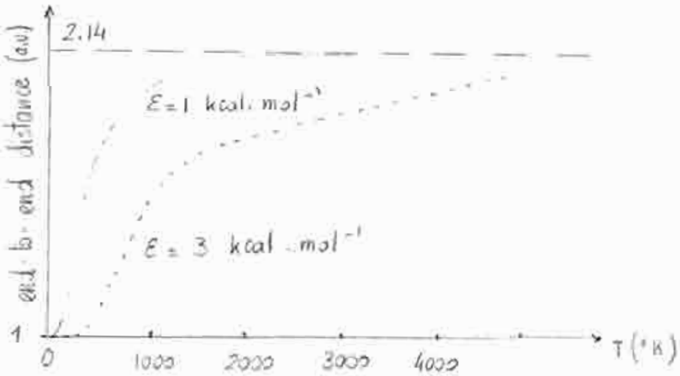


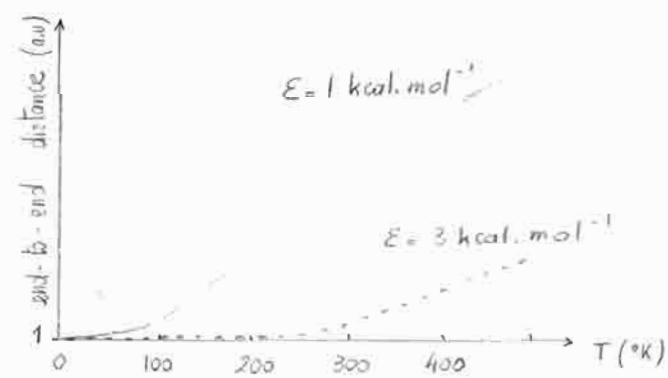
- Average end-to-end distance of the chain $\langle L \rangle = \sum_{i=1}^{3} \ell_i \, p(\ell_i)$

$$\begin{cases} p(\ell_1) = p_{folded} = \dfrac{1}{Q} & \text{from the Boltzmann distribution law, with} \\ Q = 1 + 4 \exp\left(\dfrac{-\varepsilon}{kT}\right) & \\ p(\ell_2) = \dfrac{3}{Q} \exp\left(-\dfrac{\varepsilon}{kT}\right) & \\ p(\ell_3) = \dfrac{1}{Q} \exp\left(\dfrac{-\varepsilon}{kT}\right) & \end{cases}$$

Hence $\langle L \rangle = \dfrac{1}{1 + 4\exp\left(\dfrac{-\varepsilon}{kT}\right)} \left(1 + 3(1+\sqrt{5}) \exp\left(\dfrac{-\varepsilon}{kT}\right)\right)$

- Limits of $\langle L \rangle$
  - $T \to 0^+$ $\quad \langle L \rangle \to 1$ all molecules folded
  - $T \to +\infty$ $\quad \langle L \rangle \to \dfrac{4+3\sqrt{5}}{5} = 2.14$ unfolded macrostates more populated





Although the two energies lead to similar shapes for the plots, they make a substantial difference at a typical temperature of 300 °K, where
- if $\varepsilon = 1$ kcal. mol, the unfolded macrostate is largely represented
- if $\varepsilon = 3$ " " , almost all the molecules are folded.