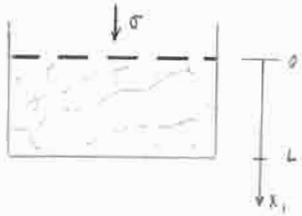
BEH. 410 . April 2003

Homework # 6 . SOLUTIONS

Problem 1



a) <u>Differential equation for the displacement $u_1(x_1, t)$</u>

- stress-strain relation in 1D

$$\left.\begin{aligned} \varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} \\ \sigma_{11} &= (2G + \lambda)\,\varepsilon_{11} - p = H\,\varepsilon_{11} - p \end{aligned}\right\} \quad (1)$$

total measured stress

- Darcy's law $\quad U_1 = -k\,\dfrac{\partial p}{\partial x_1}$ $\quad (2)$

where U_1 relative fluid velocity with respect to the solid network

k Darcy's permeability

Moreover $U_1 = \phi\,(v_f - v_s)$ $\quad (3)$

where ϕ porosity of the medium and $\phi = \dfrac{A_f}{A_s + A_f}$

v_f local fluid velocity

v_s " solid "

- Conservation of mass: let's write that the fluid accumulation during Δt is equal to the net increase in volume of the elemental slice of tissue depicted below.

$$\left\{ A_f\,(v_f - v_s)\big|_{x_1} - A_f\,(v_f - v_s)\big|_{x_1+\Delta x_1} \right\}\Delta t = V(t+\Delta t) - V(t)$$

$$= A_{TOT}\left\{ \Delta x_1\,(1+\varepsilon_{11})\big|_{t+\Delta t} - \Delta x_1\,(1+\varepsilon_{11})\big|_{t} \right\}$$

Dividing both sides of this last equation by $A_{TOT} * \Delta t * \Delta x_1$ and using (3), we get:

$$\frac{U_1|_{x_1} - U_1|_{x_1+\Delta x_1}}{\Delta x_1} = \frac{\varepsilon_{11}(t+\Delta t) - \varepsilon_{11}(t)}{\Delta t} \quad \text{and for } \Delta x_1 \to 0 \text{ and } \Delta t \to 0$$

$$\frac{\partial U_1}{\partial x_1} = -\frac{\partial \varepsilon_{11}}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{\partial u_1}{\partial x_1}\right) = -\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial t}\right)$$

, integrating (assuming the constant fluid flow to be zero)

$$U_1 = -\frac{\partial u_1}{\partial t} \quad (4)$$

- Conservation of momentum $\quad \rho\,\dfrac{\partial^2 u_1}{\partial t^2} = \nabla\cdot\underline{\underline{\sigma}} = 0$ at low frequencies $\quad (5)$

($\rho_{mass}\,\dfrac{\partial^2 u_1}{\partial t^2}$ has the same order of magnitude than $\nabla\,\underline{\underline{\sigma}}$ for $\omega \sim 10^{3.5}\,s^{-1}$ taking $\rho_{mass} \sim \rho_{water} = 10^3\ \text{kg m}^{-3}$, $u_1 = \frac{L}{10} \simeq 10^{-3}$ m, $\sigma_{11} \simeq 2G\,\varepsilon_{11}$, $G \sim 10^6$ Pa, $\varepsilon_{11} \sim 0.1$)

▶ Using (5) and (1), in 1D

$$\frac{\partial \sigma_{11}}{\partial x_1} = 0 = H\,\frac{\partial \varepsilon_{11}}{\partial x_1} - \frac{\partial p}{\partial x_1} \underset{(2)}{=} H\,\frac{\partial^2 u_1}{\partial x_1^2} - \frac{U_1}{-k} \underset{(4)}{=} H\,\frac{\partial^2 u_1}{\partial x_1^2} + \frac{1}{k}\left(-\frac{\partial u_1}{\partial t}\right)$$

which can be written as a diffusion equation with diffusivity Hk.

$$\boxed{(Hk)\,\frac{\partial^2 u_1}{\partial x_1^2} = \frac{\partial u_1}{\partial t}}$$

b) <u>Equation for fluid pressure p</u>

- from (1), one gets $\quad \varepsilon_{11} = \dfrac{1}{H}\,(\sigma_{11} + p)$ $\quad (6)$
- differentiating (4) yields

$$\frac{\partial U_1}{\partial x_1} = -\frac{\partial}{\partial t}\,\frac{\partial u_1}{\partial x_1} \underset{(1)(6)}{=} -\frac{\partial}{\partial t}\left(\frac{\sigma_{11}+p}{H}\right) = -\frac{1}{H}\,\frac{\partial}{\partial t}\,(\sigma_{11}+p) \underset{(2)}{=} -k\,\frac{\partial^2 p}{\partial x_1^2}$$

$$\boxed{\frac{\partial}{\partial t}\left[\sigma_{11}(t) + p(x_1, t)\right] = (Hk)\,\frac{\partial^2 p}{\partial x_1^2}}$$

The equivalent "diffusivity" is Hk here too.

Problem 1 (cont'd)

c) Stress relaxation : Boundary conditions
$$\boxed{\begin{array}{l} u_1\,(x_1=0,\ t) = u_0 \\ u_1\,(x_1=L,\ t) = 0 \\ u_1\,(x_1=0,\ t=0) = u_0 \\ u_1\,(x_1>0,\ t=0) = 0 \end{array}}$$
Initial conditions

d) Creep : Boundary conditions.
here we start by applying a given stress and would like to know how it relates to a displacement.
Using Hooke's law (1) at $x_1 = 0$, we have $\sigma_{11}^{*} = H\,\varepsilon_{11} - p = H\,\varepsilon_{11} = H\,\dfrac{\partial u_1}{\partial x_1}$ or $\boxed{\dfrac{\partial u_1}{\partial x_1} = \dfrac{\sigma_0}{H}}$

and $\boxed{u_1\,(x_1 = L,\ t) = 0}$ movement prevented by container

Initial conditions $\boxed{u_1\,(x_1,\ t=0) = 0}$ tissue takes time to deform

e) Solve the differential equation for $u_1\,(x_1, t)$

Diffusion equation for stress relaxation (see a)) : $\dfrac{\partial u_1}{\partial t} = Hk\,\dfrac{\partial^2 u_1}{\partial x_1^2}$ (7)

Separation of variables

(i) assume / hope that the solution $u_1\,(x_1, t)$ to (7) can be written as the product of a function of x_1 and a function of t : $u_1\,(x_1, t) = X(x_1)\ \Theta(t)$

(ii) substitute in (7) $\dfrac{\partial}{\partial t}\left[X(x_1)\,\Theta(t)\right] = Hk\,\dfrac{\partial^2}{\partial x_1^2}\left[X(x_1)\,\Theta(t)\right]$

$$X(x_1)\,\frac{d\Theta(t)}{dt} = (Hk)\ \Theta(t)\ \frac{d^2X(x_1)}{dx_1^2}$$

(iii) group terms of equation above (assume they're non-null)

$$\frac{1}{Hk}\ \frac{1}{\Theta}\ \frac{d\Theta}{dt} = \frac{1}{X}\ \frac{d^2X}{dx_1^2} = A, \quad A \text{ constant} \qquad (8)$$

(iv) solve separately for $X(x_1)$ and $\Theta(t)$

- If we choose $A = 0$, we actually consider the steady state situation for the problem :

$$\begin{cases} \dfrac{d\Theta}{dt} = 0 \\ \dfrac{d^2X}{dx_1^2} = 0 \end{cases} \Rightarrow X(x_1) = u_0\left(1 - \frac{x_1}{L}\right) \text{ from BC in c)}$$

$$\boxed{\begin{array}{l}\text{at steady state} \\ u_1\,(x_1, t) = u_0\left(1 - \dfrac{x_1}{L}\right)\end{array}}$$

- let's now solve (7) for $u_T = u_1\,(x_1, t) - u_{\text{steady state}}$

(7) becomes $Hk\left[\dfrac{\partial^2 u_1}{\partial x_1^2} - \dfrac{\partial^2 u_{ss}}{\partial x_1^2}\right] = \dfrac{\partial u_1}{\partial t} - \dfrac{\partial u_{ss}}{\partial t}$ because $\dfrac{\partial^2 u_{ss}}{\partial x_1^2} = \dfrac{\partial u_{ss}}{\partial t} = 0$

$$\boxed{\begin{array}{l} (Hk)\ \dfrac{\partial^2 u_T}{\partial x_1^2} = \dfrac{\partial u_T}{\partial t} \\ \text{with } u_T\,(x_1=0,\ t) = u_0 - u_0 = 0 \\ \phantom{\text{with }} u_T\,(x_1=L,\ t) = 0 - 0 = 0 \\ \phantom{\text{with }} u_T\,(x_1,\ t=0) = 0 \end{array}} \qquad (9)$$

Now choose $B = -\lambda^2$ with $\lambda \in \mathbb{R}_+^*$ and $B \in \mathbb{R}_-^*$
and solve (9) for $u_T\,(x_1, t) = X_T\,(x_1)\ \Theta_T\,(t)$

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Problem 1 (cont'd)

Similarly to (8), we get
$$\boxed{\begin{array}{ll} \dfrac{d^2 X_T}{dx_1^2} + \lambda^2 X_T = 0 & (10) \\ \dfrac{d\theta_T}{dt} + \lambda^2 Hk\, \theta_T = 0 & (11) \\ X_T(x_1 = L) = X_T(x_1 = 0) = 0 & \end{array}}$$

Solving (10) $X_T(x_1) = C \cos(\lambda x_1) + D \sin(\lambda x_1)$

From the boundary conditions $C = 0$

$\lambda_n = \dfrac{n\pi}{L}, \quad n \in \mathbb{N}^*$

Solving (11) $\theta_T(t) = E \exp(-\lambda_n^2 Hk\, t)$

Eventually
$$\boxed{\begin{array}{l} u_T(x_1, t) = \displaystyle\sum_{n=1}^{+\infty} E D_n \exp(-\lambda_n^2 Hk\, t) \sin(\lambda_n x_1) \\ \lambda_n = \dfrac{n\pi}{L} \end{array}}$$

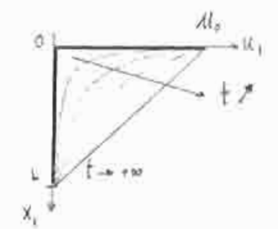
(v) We now need to find the Fourier coefficients ED_n, $n \in \mathbb{N}^*$. Given that

$$u_1(x_1, 0) = 0 = u_0\left(1 - \frac{x_1}{L}\right) + \sum_{n=1}^{+\infty} ED_n \sin(\lambda_n x_1) \quad (12)$$

Multiply by $\sin(\lambda_n x_1)$ and the orthogonality of the sine function

$$\int_0^L ED_n \sin^2(\lambda_n x_1)\, dx_1 = -\int_0^L \sin(\lambda_m x_1)\, u_0\left(1 - \frac{x_1}{L}\right) dx_1 \quad (13)^*$$

$$\frac{1}{2}\lambda_n\, ED_n = -\frac{u_0}{L} \Rightarrow ED_n = -\frac{2u_0}{n\pi},$$



$$\boxed{\begin{array}{l} u_1(x_1, t) = u_0\left(1 - \dfrac{x_1}{L}\right) + \displaystyle\sum_{n=1}^{+\infty} \frac{-2u_0}{n\pi} \exp(-\lambda_n^2 Hk\, t) \sin(\lambda_n x_1) \\ \lambda_n = \dfrac{n\pi}{L} \end{array}}$$

- The relaxation times τ_n are given by $\tau_n = (\lambda_n^2 Hk)^{-1}$ or $\boxed{\tau_n = \left(\dfrac{L}{n\pi}\right)^2 \cdot \dfrac{1}{Hk}, \quad n > 0}$

  The slowest decay time is $\boxed{\tau_1 = \dfrac{L^2}{\pi^2 Hk}}$

* How to get to (13) from (12):

$\displaystyle\sum_{n=1}^{+\infty} ED_n \sin(\lambda_n x_1) = -u_0\left(1 - \frac{x_1}{L}\right)$ which you multiply by $\sin(\lambda_m x_1)$ and integrate:

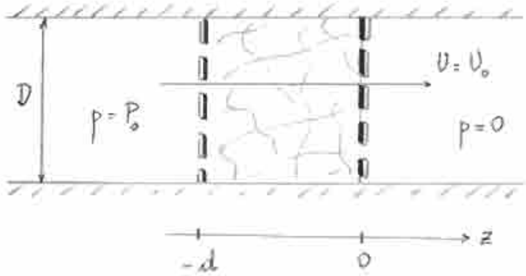
$$\int_0^L \sum_{n=1}^{+\infty} ED_n \sin(\lambda_n x_1) \sin(\lambda_m x_1)\, dx_1 = -\int_0^L u_0\left(1 - \frac{x_1}{L}\right) \sin(\lambda_m x_1)\, dx_1 \quad (14)$$

Because $\displaystyle\int_0^L \sin\left(\frac{n\pi}{L} x_1\right) \sin\left(\frac{m\pi}{L} x_1\right) dx_1 = \frac{1}{2}$ and using

$$\left\{\begin{array}{l} \displaystyle\int_0^L u_0 \sin(\lambda_m x_1)\, dx_1 = -\frac{u_0}{\lambda_m}\left[\cos\left(\frac{m\pi}{L} x_1\right)\right]_0^L = -\frac{u_0}{\lambda_m}\left[\cos(m\pi) - 1\right] \begin{array}{l} \nearrow = 0 \quad m \text{ even} \\ \searrow = \dfrac{+2Lu_0}{m\pi} \quad m \text{ odd} \end{array} \\ \displaystyle\int_0^L \frac{u_0}{L} x_1 \sin(\lambda_m x_1)\, dx_1 = \frac{u_0}{L}\left\{\left[\frac{-x_1}{\lambda_m}\cos(\lambda_m x_1)\right]_0^L + \frac{1}{\lambda_m}\int_0^L \cos(\lambda_m x_1)\, dx_1\right\} \\ \qquad = \dfrac{u_0}{L}\left\{\dfrac{-L^2}{m\pi}\cos(m\pi) - \dfrac{L^2}{(m\pi)^2}\left[\sin\left(\dfrac{m\pi x_1}{L}\right)\right]_0^L\right\} \begin{array}{l} \longrightarrow -\dfrac{u_0 L}{m\pi} \text{ even} \\ \searrow \dfrac{u_0 L}{m\pi} \text{ odd} \end{array} \end{array}\right.$$

for both m odd & m even, the right hand side of (14) is equal to $-\dfrac{2u_0}{m\pi}$, as in (13).

Problem 2



a) To get the following equations, we'll use the same reasoning as in problem 1.
$A_{TOT} = \pi \frac{D^2}{4}$ here

conservation of momentum in 1D $\frac{\partial \sigma_{zz}}{\partial z} = 0$ (1)

Darcy's law in 1D $U = -k \frac{\partial p}{\partial z}$ (2)

constitutive equation in 1D $\sigma_{zz} = H\, \varepsilon_{zz} - p = H \frac{\partial u_z}{\partial z} - p$ (3)

mass conservation in 1D $U = -\frac{\partial u_z}{\partial t} + U_0$ (4)

b) differential equation $\frac{\partial \sigma_{zz}}{\partial z} \underset{(1)}{=} 0 \underset{(3)}{=} H \frac{\partial^2 u_z}{\partial z^2} - \frac{\partial p}{\partial z} \underset{(2)}{=} H \frac{\partial^2 u_z}{\partial z^2} - \frac{U}{-k} \underset{(4)}{=} H \frac{\partial^2 u_z}{\partial z^2} + \frac{1}{k}\left(U_0 - \frac{\partial u_z}{\partial t}\right)$

$$\boxed{(Hk) \frac{\partial^2 u_z}{\partial z^2} + U_0 = \frac{\partial u_z}{\partial t}} \quad (5)$$

c) Steady flow : if $\frac{\partial}{\partial t} = 0$, (5) becomes

$$\boxed{\begin{array}{l} \frac{\partial^2 u_z}{\partial z^2} = -\frac{U_0}{Hk} \\ (i)\ u_z(z=0) = 0 \\ (ii)\ \frac{\partial u_z}{\partial z}(z=-d) = 0 \end{array}} \quad (6)$$

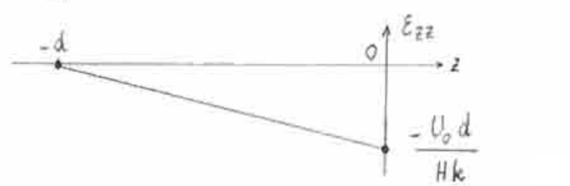
boundary conditions

From (6) $u_z = -\frac{U_0}{2Hk} z^2 + Az + B$

From (i) $B = 0$

From (ii) $A = -\frac{U_0\, d}{Hk}$

d) $$\boxed{\begin{array}{l} u_z(z) = -\frac{U_0}{Hk}\left(\frac{z^2}{2} + dz\right) \\ \varepsilon_{zz}(z) = \frac{d u_z}{dz} = \frac{-U_0}{Hk}(z + d) \end{array}}$$ at steady flow



Note : • the BC $\frac{\partial u_z}{\partial z}(z = -d)$ is NOT OBVIOUS

$\left.\begin{array}{l} p(z=-d) = P_0 \\ \sigma_{zz}(z=-d) = -P_0 \quad \text{no mechanical stress other than pressure} \end{array}\right\}$ plug into (3) $\Rightarrow \left.\frac{\partial u_z}{\partial z}\right|_{z=-d} = 0$

• σ_{zz} independent of z : $\frac{\partial \sigma_{zz}}{\partial z} = 0$ (equation (1))

hence $\sigma_{zz}(-d) = \sigma_{zz}(z=0)$

$$\cancel{H \left.\frac{\partial u_z}{\partial z}\right|_{-d}} - P_0 = H \left.\frac{\partial u_z}{\partial z}\right|_0 - \cancel{p(z=0)}$$

therefore $\left.\frac{\partial u_z}{\partial z}\right|_{z=0} = \frac{-P_0}{H}$.