Homework # 8 SOLUTIONS

Problem 3 microbead of diameter d, subjected to $F = F_o \sin(\omega t)$ experiencing a displacement of amplitude u_o.

a) Elastic CSK : since the medium is elastic, the force and resulting displacement are in phase

b) Order of magnitude for strain ε

characteristic length : d
distance that varies : u_o $\quad \Big\} \quad \varepsilon \sim \dfrac{u_o}{d}$

c) Scaling relationship between F_o & u_o : $\quad F_o \sim \sigma d^2 \sim E \varepsilon d^2 \sim E \dfrac{u_o}{d} d^2$ or $\boxed{F_o \sim E d u_o}$

d) Viscous CSK : displacement lags applied force by a phase angle of $\pi/2$

scaling relationship : $F_o \sim d^2 \sigma^V \sim d^2 \mu \dfrac{d\varepsilon}{dt} \sim d^2 \mu \omega \dfrac{u_o}{d}$ or $\boxed{F_o \sim \mu d \omega u_o}$

e) Solid-like vs liquid-like behavior

$$\frac{\text{elastic / solid}}{\text{viscous / liquid}} \sim \frac{F_o(c)}{F_o(d)} = \frac{E d u_o}{\mu d \omega u_o} = \frac{E}{\mu \omega} = \frac{100 \text{ Pa}}{100 \text{ kg} \cdot \text{m}^{-1} \cdot \text{s}^{-1} \times 1 \text{ s}^{-1}} = 1$$

comparable importance of 2 effects

Problem 5

a) Contracted force generated by the cell :

$$F \sim D^2 \sigma \sim D^2 E_g \varepsilon \sim D^2 E_g \frac{\delta}{L}$$

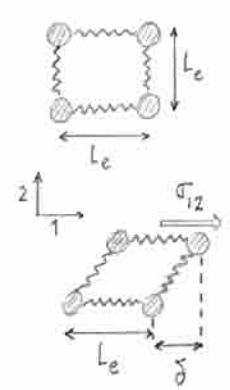
b) Beads perpendicular to direction of cell contraction.

If the cell is modeled as thin disk, we can consider its contraction to be an unconfined compression process for which $\varepsilon_\perp = -\nu \, \varepsilon_{//}$ (ν Poisson's ratio, $//$ index for direction $//$ contraction, $\perp$ " " $\perp$ ")

Hence we expect those beads to move further apart.

Problem 1 Red blood cell membrane mechanics

a) Shear modulus K_S



Since $L_e \ll L$ (contour length) we can treat each spectrin polymer as a Gaussian chain and use the linear region of the freely jointed chain model for the force / extension relationship : $\underline{F}_{2D} = \dfrac{2 k_B T}{2 L_p} \cdot \dfrac{\underline{r}}{L}$ with $\begin{cases} r : \text{extension} \\ L_p : \text{persistence length} \end{cases}$

Let's call $\begin{cases} \text{this extension } \delta = |\underline{r}| \\ \text{the membrane thickness } t. \end{cases}$

The tension across the (1,2) plane is obtained by dividing $|\underline{F}_{2D}|$ by the average spacing of the spectrin chains, L_e : $\quad N_{12} = \dfrac{2 k_B T}{b} \cdot \dfrac{\delta}{L L_e}$ (1)

And from the class notes, $N_{12} = t \, \sigma_{12} = K_S \, \varepsilon_{12}$

$= K_S \cdot \frac{1}{2} \left(\partial u_2 / \partial x_1 + \partial u_1 / \partial x_2 \right) = \frac{1}{2} K_S \dfrac{\delta}{L_e}$ (2)

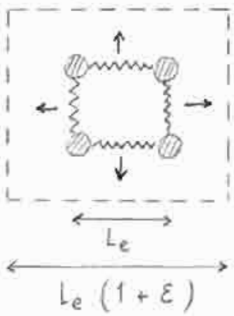
Equating (1) and (2) yields $\boxed{K_S = \dfrac{2 k_B T}{L_p L}}$

Problem 1 (cont'd)

b) for $L = 200 \cdot 10^{-9}$ nm & $L_p = 10 \cdot 10^{-9}$ nm , $K_S \approx 4 \cdot 10^{-6}$ N/m

In class, we mentioned $2.5\ \mu N/m \approx K_s$ for RBC. Our model gives us a close result, which suggest that the spectrin cortex is mainly responsible for the RBC shear modulus (not the lipid bilayer , which is easily sheared !)

c) Expansion modulus K_e



From the FJC model $F \simeq \frac{k_B T}{L_p} \cdot \frac{\varepsilon L_e}{L}$

Further $N = \frac{F}{L_e} = K_e \frac{\Delta A}{A_0} \simeq K_e\ 2\varepsilon$

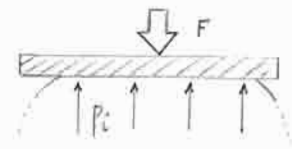
($\frac{F}{L_e}$: same argument as in a) ; $\simeq$: small strain approximation)

$$\boxed{K_e = \frac{k_B T}{2 L_p L}}$$

d) From the data mentioned in b) , $K_e \simeq 10^{-6}$ N/m

The expansion modulus for a RBC is about 0.45 N/m . There exists a large discrepancy between our model and the actual RBC behavior. This is because we did not take into account the role of the lipid bilayer , which cannot be easily expanded.

Problem # 2

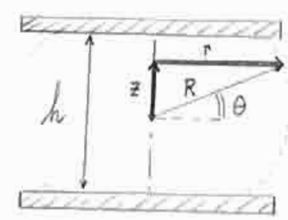


- From a force balance of the upper plate, $p_i A - F = 0$ $\boxed{p_i = \frac{F}{A}}$
- Assume that
  (i) the squished cell remains spherical in shape where not constrained by the plates.
  (ii) the membrane can be treated as a surface tension , using the formula derived in class

$$N = \frac{F}{R} = \frac{E_m t}{2(1-\nu)} \cdot \frac{\Delta A}{A_0}$$

- Geometry ...

$$A_{final} = 2A + A_{free} = 2A + \int_{-\theta}^{\theta} 2\pi r \cdot R d\theta = 2A + 4\pi R^2 \sin\theta$$
$$= 2A + 2\pi R h$$



with $R = \left[\frac{A}{\pi} + \left(\frac{h}{2}\right)^2 \right]^{1/2}$

$A_{initial} = 4\pi R_o^2$ with $R_o = \left(\frac{3}{4\pi} V_o \right)^{1/3}$

h is given by solving $V_o = V_{initial} = V_{final}$:

$$V_{final} = \int_{-h/2}^{h/2} \pi (R^2 - z^2)\, dz = \pi \left(R^2 h - \frac{h^3}{12} \right) = V_o$$

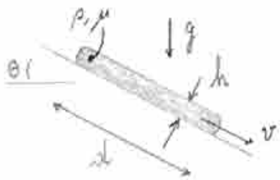
further we know $\frac{A}{\pi} = R^2 - \left(\frac{h}{2}\right)^2$

$$\Rightarrow\ Ah + \pi \frac{h^3}{6} = V_o$$

- Finally , using R, R_o, h defined above

$$\boxed{F = \frac{R E_m t}{(1-\nu)} \cdot \frac{\pi R h + A - 2\pi R_o^2}{2\pi R_o^2}}$$

Problem 4



The rate at which work is done by gravity, Γ_g, must be balanced by the rate at which energy is dissipated inside the viscous cell, Γ_v:

$$\Gamma_g \sim \underbrace{F_g}_{\text{force exerted on cell by gravity}} \cdot v \sim g \sin\theta \cdot \rho \underbrace{V_{cell}}_{\text{cell volume}} \cdot v \sim g \sin\theta \, \rho \, d^2 h \, v$$

$$\Gamma_v \sim \mu \, \dot{\varepsilon}^2 \, V_{cell} \sim \mu \left(\frac{v}{h}\right)^2 d^2 h \sim \mu v^2 \frac{d^2}{h}$$

Equating Γ_g and Γ_v: $\boxed{v \sim \frac{\rho g h^2 \sin\theta}{\mu}}$

note: you could get the result from a force balance, at the surface, between shear & gravitational force:

$$F_g \sim g \sin\theta \, \rho \, d^2 h$$

$$F_s \sim \mu \frac{v}{h} d^2$$