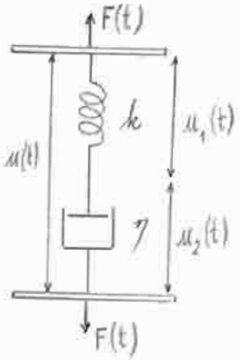
BEH. 410. April 2003

Problem set # 5. SOLUTIONS

Problem 1



a) Maxwell model.

$$\begin{cases} F = F_1 = F_2 \\ F_1 = k\, u_1 \\ F_2 = \eta \dfrac{du_2}{dt} \end{cases} \qquad u = u_1 + u_2 \text{ yields } \boxed{\frac{du}{dt} = \frac{1}{k}\cdot\frac{dF}{dt} + \frac{F}{\eta}}$$

b) Fitting experimental data:

We know $F = 250 \times 10^{-12}\, t$ and $\dfrac{dF}{dt} = 250 \times 10^{-12}$ N.s^{-1}

We need to evaluate $\dfrac{du}{dt}$, the slope of the tangent to the plot, for two distinct ts to solve the two-unknown system of equations

$$t_1 = 0.75\, s \qquad \left.\frac{du}{dt}\right|_{t_1} = \frac{(0.13 - 0.05)\times 10^{-6}}{1 - 0.5} = 1.6 \times 10^{-7}\ m.s^{-1} = \frac{1}{k}\frac{dF}{dt} + \frac{250\times 10^{-12}}{\eta}\times 0.75 \qquad (1)$$

$$t_2 = 1.75\, s \qquad \left.\frac{du}{dt}\right|_{t_2} = \frac{(0.44 - 0.28)\times 10^{-6}}{2 - 1.5} = 3.2 \times 10^{-7}\ m.s^{-1} = \frac{1}{k}\frac{dF}{dt} + \frac{250\times 10^{-12}}{\eta}\times 1.75 \qquad (2)$$

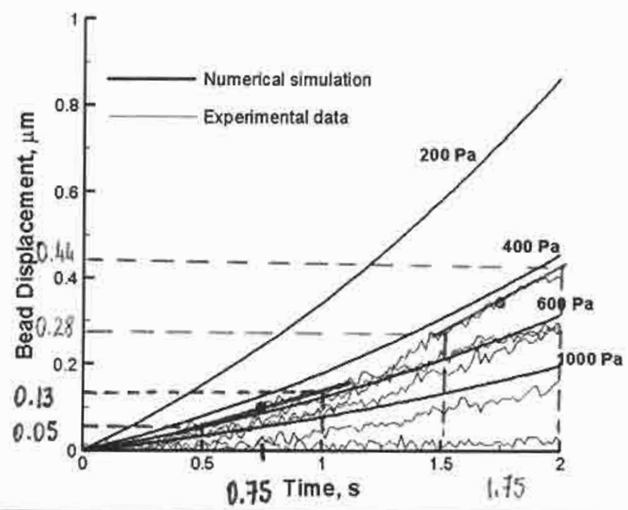
(1) − (2) yields

$$\boxed{\eta = \frac{250\times 10^{-12}\,(t_2 - t_1)}{\left.\frac{du}{dt}\right|_{t_2} - \left.\frac{du}{dt}\right|_{t_1}}} = \underline{0.16\ Pa.s.m}$$

(1) $\times\, t_2$ − (2) $\times\, t_1$ yields

$$\boxed{k = \frac{dF}{dt}\cdot\frac{t_2 - t_1}{t_2 \left.\frac{du}{dt}\right|_{t_1} - t_1 \left.\frac{du}{dt}\right|_{t_2}}}$$

$\underline{k = 6.25 \times 10^{-3}\ Pa.m}$



c) The experimental graph is fairly well fitted by the Maxwell model: it captures the first slow, then steeper increase in displacement as time goes by. However, this model seems to fail at describing the plateau displacement expected at longer times (which makes biological sense). Nonetheless, nor the Voigt model, nor the standard linear solid model are capable of fitting this asymptotical displacement. Moreover the two latter would predict a faster increase in bead displacement, which does not seem to correspond to the tissue response.

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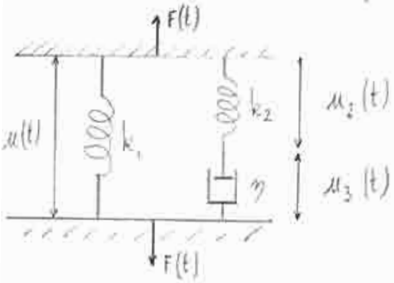
Problem 2

a) Differential equation satisfied by the 3-element model for a standard viscoelastic solid:



$$\begin{cases} e(t) = e_1(t) = e_2(t) + e_3(t) & (1) \\ F(t) = \underbrace{F_{S_1}(t)}_{\text{spring 1}} + \underbrace{F_{S2D}(t)}_{\text{spring 2 + dashpot}} & (2) \end{cases} \quad \text{with} \quad \begin{cases} F_{S1}(t) = k_1 u_1(t) & (3) \\ F_{S2D}(t) \underset{(4)}{=} k_2 u_2(t) \underset{(5)}{=} \eta \dfrac{du_3}{dt} \end{cases}$$

We can write

$$\begin{cases} F \underset{(2,3,4)}{=} k_1 u_1 + \eta \dfrac{du_3}{dt} \overset{(1)}{=} k_1 u + \eta\left(\dfrac{du}{dt} - \dfrac{du_2}{dt}\right) \\ \underset{(4)}{=} k_1 u + \eta\left(\dfrac{du}{dt} - \dfrac{1}{k_2}\dfrac{dF_{S2D}}{dt}\right) \underset{(2)}{=} k_1 u + \eta\left[\dfrac{du}{dt} - \dfrac{1}{k_2}\left(\dfrac{dF}{dt} - \dfrac{dF_{S1}}{dt}\right)\right] \\ F \underset{(3)}{=} k_1 u + \eta\left[\dfrac{du}{dt} - \dfrac{1}{k_2}\left(\dfrac{dF}{dt} - k_1\dfrac{du}{dt}\right)\right] \end{cases}$$

Thus:

$$\boxed{F + \frac{\eta}{k_2}\frac{dF}{dt} = k_1 u + \eta\left(1 + \frac{k_1}{k_2}\right)\frac{du}{dt} \qquad \text{or } \alpha = \eta / k_2 \text{ and } \beta = \eta\left(1 + \frac{k_1}{k_2}\right)}$$

b) $F + \alpha \dot{F} = k_1 u + \beta \dot{u}$ (equ 1)

Assume that $u(t)$ is the forcing term, imposed on the system. The difference of any two solutions to (equ. 1) is solution of $F + \alpha \dot{F} = 0$ (equ 2)

- Consequently, the general solution of (equ. 1) is the sum of any particular solution to (equ 1) and the general solution of the associated homogeneous equation (equ. 2)
- The general solution to (equ 2) will contain decaying exponentials (dissipative system) representing a transient behavior that we'll neglect. We concentrate instead on the <u>particular steady state solution to (equ. 1)</u>
- The response to a steady state sinusoidal strain is a steady state sinusoidal stress at the same frequency, but of possibly different phase.
- Using the complex representation, $\frac{d}{dt} \leftrightarrow * \, i\omega$ and (equ 1) becomes $\hat{F} + i\omega\alpha\hat{F} = k_1\hat{u} + i\omega\beta\hat{u}$

with $\begin{cases} \hat{u} = u_0 \exp(i\omega t) \\ \hat{F} = F_0 \exp(i\omega t + \phi) \end{cases}$

We define a complex shear modulus

$$\hat{G} = \frac{\hat{F}}{\hat{u}} = \frac{k_1 + i\beta\omega}{1 + i\alpha\omega} = \boxed{\underbrace{\frac{k_1 + \alpha\beta\omega^2}{1 + \alpha^2\omega^2}}_{G'} + i\underbrace{\frac{\beta\omega - \alpha k_1\omega}{1 + \alpha^2\omega^2}}_{G''}}$$

multiply numerator & denominator by $(1 - i\alpha\omega)$

G': solid-like behavior, energy storage, elasticity

G'': liquid-like behavior, energy dissipation, viscosity

c) $G' = \dfrac{k_1 + \alpha\beta\omega^2}{1 + \alpha^2\omega^2} \xrightarrow[\omega \to 0]{} k_1$, $\xrightarrow[\omega \to \infty]{} \dfrac{\beta}{\alpha} = k_1 + k_2$

$G'' = \dfrac{\beta\omega - \alpha k_1\omega}{1 + \alpha^2\omega^2} \xrightarrow[\omega \to 0]{} 0$ in $(\beta - \alpha k_1)\omega$, $\xrightarrow[\omega \to \infty]{} 0$ in $\dfrac{\beta - \alpha k_1}{\alpha^2\omega}$

- The denominator of G'' is always positive.

Variations of G''

$$\frac{dG''}{d\omega} = \frac{(\beta - k_1\alpha)(1 + \alpha^2\omega^2) - \omega(\beta - k_1\alpha)(2\alpha^2\omega)}{1 + \alpha^2\omega^2} = \frac{N(\omega)}{D(\omega)}$$

$$\frac{dG''}{d\omega} = 0 \iff N(\omega) = 0 = \beta - k_1\alpha + \alpha^2\omega^2(\beta - k_1\alpha) - 2\alpha^2(\beta - k_1\alpha)\omega^2 = \underbrace{(\beta - k_1\alpha)}_{>0 \text{ physical constraint}}(1 - \omega^2\alpha^2)$$

Maximum of G'' reached for $\frac{dG''}{d\omega} = 0$, ω solution of $1 - \alpha^2\omega^2 = 0$, $\boxed{\omega = 1/\alpha}$

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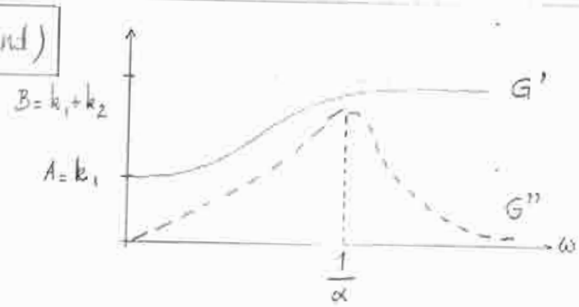
Problem 2 (end)

We find



- When the oscillation frequency is very low, the dashpot can slide almost freely and the right hand side of the 3-element model is not felt. The material behaves as the left hand side spring.
- At very high frequency, the dashpot becomes unable to follow the excitation. The material behaves as two springs in parallel.

Problem 3

a) Differential equation satisfied by the 3-element model for a standard viscoelastic fluid:

$$F + \alpha \dot{F} = \beta \dot{u} + \gamma \ddot{u} \quad (1)$$

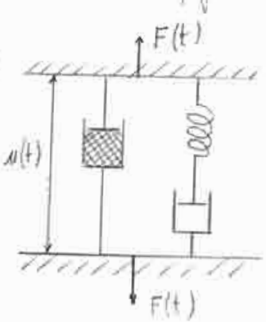
Using the complex representation $\frac{d}{dt} \longleftrightarrow \dot{} \longleftrightarrow i\omega$ and $\frac{d^2}{dt^2} \longleftrightarrow \ddot{} \longleftrightarrow (i\omega^2) = -\omega^2$

(1) becomes $\hat{F} + i\alpha\omega \hat{F} = i\beta\omega \hat{u} - \gamma\omega^2 \hat{u}$

and $$\hat{G} = \frac{\hat{F}}{\hat{u}} = \frac{-\gamma\omega^2 + i\beta\omega}{1 + i\alpha\omega} = \boxed{\underbrace{\frac{(\alpha\beta - \gamma)\,\omega^2}{1 + \alpha^2\omega^2}}_{G'} + i\underbrace{\frac{\omega\beta + \alpha\gamma\omega^3}{1 + \alpha^2\omega^2}}_{G''}}$$

b) To make physical sense, the storage modulus G' has to be positive. $\boxed{\begin{array}{l}\alpha\beta - \gamma > 0 \\ G'' > 0\end{array}}$ (class notes)

c)



$$G' = \frac{(\alpha\beta - \gamma)\,\omega^2}{1 + \alpha^2\omega^2} \xrightarrow{\omega \to 0} 0 \text{ in } (\alpha\beta - \gamma)\,\omega^2$$
$$\xrightarrow{\omega \to +\infty} \frac{\alpha\beta - \gamma}{\alpha^2}$$

$$G'' = \frac{\omega\beta + \alpha\gamma\omega^3}{1 + \alpha^2\omega^2} \xrightarrow{\omega \to 0} 0 \text{ in } \beta\omega$$
$$\xrightarrow{\omega \to +\infty} +\infty \text{ in } \frac{\gamma}{\alpha}\omega$$

note: no other maximum for $G'(\omega)$: $\frac{dG'}{d\omega} = 0 \iff 2\omega(\alpha\beta - \gamma)(1 + \alpha^2\omega^2) - \omega^2(\alpha\beta - \gamma)(2\omega\alpha^2) = 0$

$\iff \underbrace{\omega}_{\geqslant 0}\,\underbrace{(\alpha\beta - \gamma)}_{> 0} = 0$ only "flattens" at $\omega = 0$

- At very low frequency, the two dashpots slide with very little resistance. Strain induces very little stress.
- At very high frequency, the dark dashpot dominates the model, unable to transform the applied strain into stress, making the material essentially rigid (huge loss of energy).