22.101

Solutions PS2

(10pts)

1) Consider the Reaction:

$$n^1 + H^1 = H^2 + \gamma(2.225\ \mathrm{MeV})$$

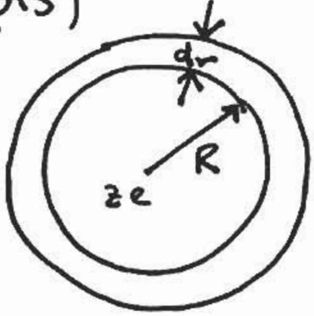
$$n^1 = MH^2 - MH^1 + \frac{\gamma(2.225\,\mathrm{MeV})}{931.5\ \frac{\mathrm{MeV}}{\mathrm{amu}}}$$

Given: $MH^2 = 2.0141022\,\mathrm{amu}$

$MH^1 = 1.0078252\ \mathrm{amu}$

$$\Rightarrow \boxed{n^1 = 1.0086587\ \mathrm{amu}}$$

2) (20pts)



Sphere has some charge density (ρ) uniformly distributed throughout.

$$E = \int_0^R \frac{Q_1 Q_2}{r}, \quad Q_1 = \tfrac{4}{3}\pi r^3 \rho, \quad Q_2 = 4\pi r^2 \rho\, dr$$

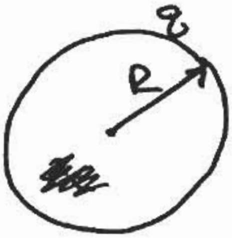
(From Note

need to determine: $\rho = \dfrac{ze}{\frac{4}{3}\pi R^3}$

$$\Rightarrow E = \int_0^R \frac{(\frac{4}{3}\pi r^3 \rho)(4\pi r^2 \rho)}{r}\, dr = \frac{16}{3}\pi^2 \rho^2 \int_0^R r^4\, dr$$

$$= \frac{16}{3}\pi^2 \rho^2 \frac{R^5}{5} \text{ (plug in } \rho\text{) get: } \boxed{E = \frac{3}{5}\frac{(ze)^2}{R}}$$

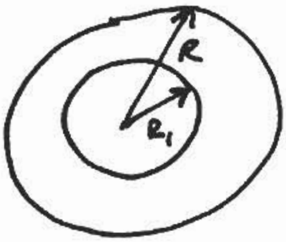
For surface charge case:



$$E = \int_0^{Ze} \frac{q}{R}\,dq = \frac{1}{2}\frac{q^2}{R}\Big|_0^{Ze} = \boxed{\frac{1}{2}\frac{(Ze)^2}{R}}$$

This energy is equal to the energy needed to move charges (dq) from infinity to the surface of the sphere.

For a shell:



$$\rho = \frac{Ze}{4/3\,\pi(R^3 - R_1^3)}$$

$$\Rightarrow E = \int_{R_1}^{R} \frac{\left(4/3\,\pi\rho\,(r^3 - R^3)\right)\left(4\pi\rho r^2\right)}{r}\,dr$$

$$= \frac{16}{3}\pi^2\rho^2 \int_{R_1}^{R} r(r^3 - R^3)\,dr = \frac{16}{3}\pi^2\rho^2 \int_{R_1}^{R} r^4\,dr - R_1^3 \int_{R_1}^{R} r\,dr$$

$$= \frac{16}{3}\pi^2\rho^2\left(\frac{R^5}{5} - \frac{R_1^5}{5} - \frac{R_1^3 R^2}{2} + \frac{R_1^5}{2}\right)$$

$$\Rightarrow \boxed{E = 3\left(\frac{Ze}{R^3 - R_1^3}\right)^2\left(\tfrac{1}{5}R^5 - \tfrac{1}{2}R_1^3 R^2 + \tfrac{3}{10}R_1^5\right)}$$

Anything like this is fine...

3) (20pts) If the mass discrepancy can be assumed to be due exclusively to the Coulomb term, we can look at the mass error:

$$\frac{\left(\frac{3}{5}-\frac{1}{2}\right)\frac{(Ze)^2}{R}}{M} = \frac{\left(\frac{3}{5}-\frac{1}{2}\right)(Ze)^2}{M r_0 A^{1/3}} \approx 0.074\%$$

Since the mass error caused by the semi-empirical formula itself is 0.01%, this completely rules out the surface charge model.

Using the expression for energy from problem 2 and making $x = \frac{R_1}{R}$

$$E = \frac{3}{5}\frac{(Ze)^2}{R}\frac{3x^5-5x^3+2}{2(x^3-1)^2}$$

$$= \frac{3}{5}\frac{(Ze)^2}{R}\frac{3x^3+6x^2+4x+2}{2(x^2+x+1)^2}$$

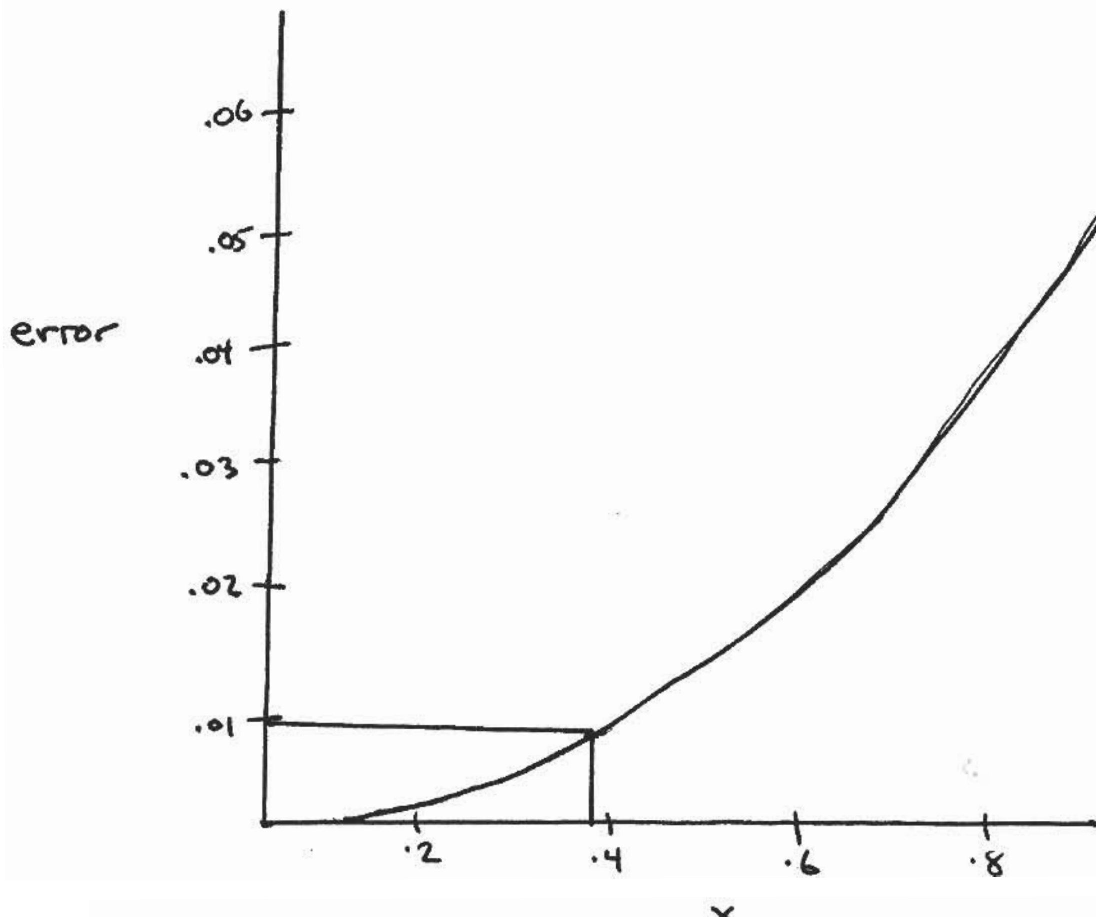
The mass error, even if caused exclusively by the Coulomb term, can be calculated as above:

$$\text{error} = \frac{\left(\frac{3}{5}\frac{(Ze)^2}{R} - E\right)}{M} \times 100\%$$

$$= \frac{3}{5} \frac{(Ze)^2}{MR} \left(1 - \frac{3x^3+6x^2+4x+2}{2(x^2+x+1)^2}\right) \times 100\%$$

$$\text{error} = 0.44\% \left(1 - \frac{3x^3+6x^2+4x+2}{2(x^2+x+1)^2}\right)$$

Plot error vs. x (I used Maple)



For an error of 0.01% $\boxed{\frac{R_c}{R} = 0.395}$

4) (15 pts) We know

$$Z_{stable} = \frac{-k_2}{2k_3}, \quad k_2 = -(4a_a + (M_N - M_H))$$

$$k_3 = \frac{4a_a}{A}\left(1 + \frac{A^{2/3}}{4a_a/a_c}\right)$$

we know,

$$a_a = 19.0 \text{ MeV}$$

$$a_c = 0.0595 \text{ MeV}$$

$$M_N = 939.55 \text{ MeV} \qquad {}_{54}Xe^{142}$$

$$M_H = 938.767 \text{ MeV}$$

$$k_2 = -(4(19) + (939.55 - 938.767))$$

$$= -76.78 \text{ MeV}$$

$$k_3 = \frac{4(19)}{142}\left(1 + \frac{(142)^{2/3}}{4\left(\frac{19}{0.595}\right)}\right) = 0.649$$

$$Z_{stable} = 59.13$$

Since $Z_{stable} > 54$ ${}_{54}Xe^{142}$ will want to shed electrons $\Rightarrow \boxed{\beta^- \text{ unstable}}$

5) (20 pts) we have,

$$M(Z,A) = ZM_H + (A-Z)M_n - B(Z,A)$$

$$M(2,4) = 2M_H + 2M_n - B(2,4)$$

$$M(Z-2,A-4) = (Z-2)M_H + (A-Z+2)M_n - B(Z-2,A-4)$$

$$Q = (M(Z-2,A-4) + M(2,4)) - M(Z,A)$$

$$= B(Z-2,A-4) + B(2,4) - B(A,Z)$$

use semi-empirical mass formula.

$$B(A,Z) = a_v A - a_c \frac{Z^2}{A^{1/3}} - a_s A^{2/3} - a_a \frac{(A-2Z)^2}{A} + a_p A^{-3/4}$$

$$B(A-4,Z-2) = a_v(A-4) - a_c \frac{(Z-2)^2}{(A-4)^{1/3}} - a_s(A-4)^{2/3}$$
$$- a_a \frac{[(A-4) - 2(Z-2)]^2}{A-4} + a_p (A-4)^{-3/4}$$

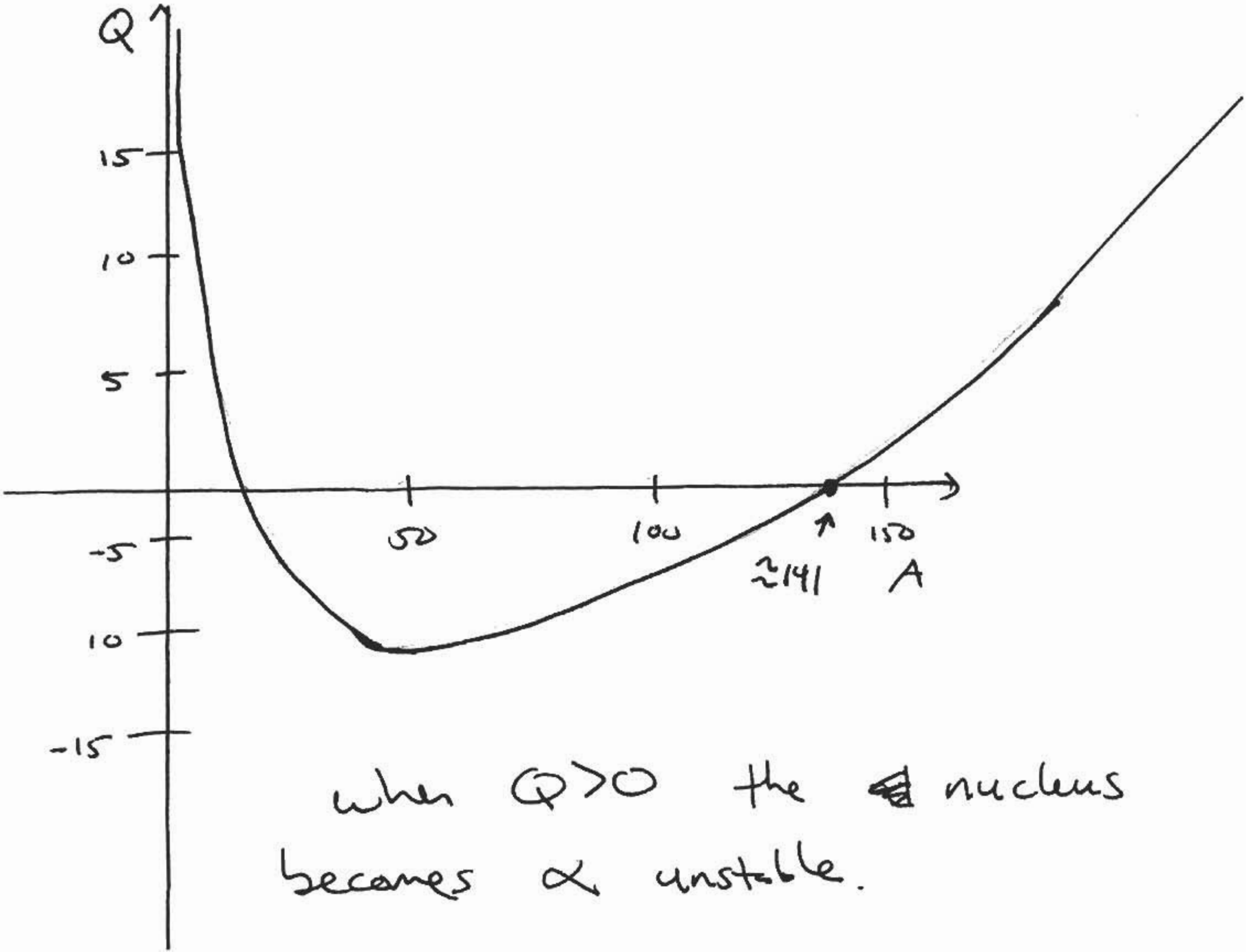
$$B(2,4) = 28.3 \text{ MeV}$$

Putting it all together:

$$Q = 28.3 - 4(14.1) + 0.595\left[\frac{-(Z-2)^2}{(A-4)^{1/3}} + \frac{Z^2}{A^{1/3}}\right] + 13\left[-(A-4)^{2/3} + A^{2/3}\right.$$
$$\left. + 19.0\left[\frac{-(A-2Z)^2}{A-4} + \frac{(A-2Z)^2}{A}\right] + 33.5\left[(A-4)^{-3/4} - A^{-3/4}\right]\right.$$

Plot Q vs. A (Arbitrary Z) (I used Maple)

$Z \simeq .4A$ or so



when $Q > 0$ the nucleus becomes α unstable.

Consider the following examples:

$$ {}_{92}U^{235} \longrightarrow {}_{90}Th^{231} + {}_{2}\alpha^{4} $$

Define a Q value such that: $B = -Q$

$$ Q = \left[M(He^{4}) + M(Th^{231}) - M(U^{235}) \right] \frac{931.5\ \text{MeV}}{\text{amu}} $$

$$ = \boxed{-4.67\ \text{MeV}} $$

$$ {}_{94}Pu^{239} \longrightarrow {}_{92}U^{235} + {}_{2}\alpha^{4} $$

$$ Q = \left[M({}_{2}He^{4}) + M({}_{92}U^{235}) - M({}_{94}Pu^{239}) \right] \frac{931.5\ \text{MeV}}{\text{amu}} $$

$$ = \boxed{-5.24\ \text{MeV}} $$

6) (15 pts) ${}_{Z}X^{A} + n^{1} \longrightarrow {}_{Z}X^{A+1} + Q_n$

$$M(Z,A) = Z M_H + (A-Z) M_n - B(A,Z)$$

$$M(Z,A+1) = Z M_H + (A+1-Z) M_n - B(A+1,Z)$$

$$Q_n = M(Z,A) + M(n) - M(Z,A+1)$$

$$\boxed{= B(A+1,Z) - B(A,Z)}$$

b) ${}_{92}U^{235}$, $\delta = 0$

use semi-empirical Mass formula, Plug & chug..

$$B(235, 92) = 1792.06 \text{ MeV}$$

${}_{92}U^{236}$, $\delta = a_p (A)^{-3/4}$

$$B(236, 92) = 1799.076 \text{ MeV}$$

$$Q_n = 1799.076 \text{ MeV} - 1792.06 \text{ MeV} = 7.016 \text{ MeV}$$

c) ${}_{92}U^{238}$ $\delta = a_p (A)^{-3/4}$

$B(238, 92) = 1811.753$ MeV

${}_{92}U^{239}$ $\delta = 0$

$B(239, 92) = 1817.25$ MeV

$Q_n = \boxed{5.497 \text{ MeV}}$ ~ 1.4 MeV less