APPENDIX ONE

further reading

This appendix presents several suggestions for further reading, including a few detailed references. Only a few works, all of relatively general interest, are listed. Unless stated otherwise, the books below are at a level which should be accessible to the reader of this text. No attempt has been made to indicate the extensive literature pertaining to particular fields of application.

References are listed by author and date, followed by a brief description. Complete titles are tabulated at the end of this appendix. For brevity, the present volume is referred to as FAPT in the annotations.

1 *Some Relevant Philosophy and the History of Probability Theory*

DAVID (1962) Engaging history of some of the earliest developments in probability theory. Attention is also given to the colorful personalities involved.

KYBURG AND SMOKLER (1963) Essays by Borel, de Finetti, Koopman, Ramsey, Savage, and Venn on a matter of significance in applied probability theory, the topic of *subjective* probability.

LAPLACE (1825) Interesting discussions of philosophical issues related to probability theory and its applications to real world issues.

TODHUNTER (1865) The classic reference for the early history of probability theory.

2 *Introductory Probability Theory and Its Applications*

FELLER (1957) Thorough development of the discrete case with a vast supply of interesting topics and applications. Contains a large body of fundamental material on combinatorial analysis and the use of transforms in the study of discrete renewal processes which is not included in FAPT.

FISZ (1963) Large, scholarly, and relatively complete text treating probability theory and classical mathematical statistics. Tightly written. A very desirable reference work.

GNEDENKO (1962) Respected text with much coverage common to FAPT, but at a more advanced mathematical level. Includes a brief introduction to mathematical statistics.

KOLMOGOROV (1933) A short, original, and definitive work which established the axiomatic foundation of modern mathematical probability theory. Every student of applied probability theory will profit from spending at least several hours with this exceptional document. Although many sections are presented at an advanced level, the reader will rapidly achieve some understanding of the nature of those topics which are required for a rigorous theoretical foundation but neglected in a volume such as FAPT. As one significant example, he will learn that our third axiom of probability theory (known formally as the axiom of *finite additivity*) must be replaced by another axiom (specifying *countable additivity*) in order to deal properly with probability in continuous sample spaces.

LOÈVE (1955) Significantly more advanced than Gnedenko, this is a mathematical exposition of probability theory. Limited concern with applications and physical interpretation.

PAPOULIS (1965) An effective, compact presentation of applied probability theory, followed by a detailed investigation of random processes with emphasis on communication theory. Especially recommended for electrical engineers.

PARZEN (1960) More formal, appreciably more detailed presentation at a mathematical level slightly above FAPT. More concern with mathematical rather than physical interpretation. A lucid, valuable reference work.

PFEIFFER (1965) A more formal development at about the same level as FAPT. With care, patience, and illustration the author introduces matters of integration, measure, etc., not mentioned in FAPT. A recommended complement to FAPT for readers without training in theoretical mathematics who desire a somewhat more rigorous foundation. Contains an annotated bibliography at the end of each chapter.

PITT (1963) A concise statement of introductory mathematical probability theory, for readers who are up to it. Essentially self-contained, but the information density is very great.

3 *Random Processes*

COX (1962) Compact, readable monograph on the theory of renewal processes with applications.

COX and MILLER (1965) General text on the theory of random processes with applications.

COX and SMITH (1961) Compact, readable monograph which introduces some aspects of elementary queuing problems.

DAVENPORT and ROOT (1958) Modern classic on the application of random process theory to communication problems.

DOOB (1953) Very advanced text on the theory of random processes for readers with adequate mathematical prerequisites. (Such people are unlikely to encounter FAPT.)

FISZ (1963) Cited above. Contains a proof of the ergodic theorem for discrete-state discrete-transition Markov processes stated in Chap. 5 of FAPT.

HOWARD (1960) An entirely clear, brief introduction to the use of Markov models for decision making in practical situations with economic consequences.

HOWARD (in preparation) Detailed investigation of Markov models and their applications in systems theory.

LEE (1960) Lucid introductory text on communication applications of random process theory.

MORSE (1958) Clear exposition of Markov model applications in queuing theory aspects of a variety of practical operational situations.

PAPOULIS (1965) Cited above.

PARZEN (1962) Relatively gentle introduction to random process theory with a wide range of representative examples.

4 *Classical and Modern Statistics*

CHERNOFF and MOSES (1959) An elementary, vivid introduction to decision theory.

CRAMER (1946) A thorough, mathematically advanced text on probability and mathematical statistics.

FISZ (1963) Cited above.

FRASER (1958) Clear presentation of elementary classical statistical theory and its applications.

FREEMAN (1963) The last half of this book is a particularly logical, readable presentation of statistical theory at a level somewhat more advanced than Fraser. Contains many references and an annotated bibliography of texts in related fields.

MOOD and GRAYBILL (1963) One of the most popular and successful basic treatments of the concepts and methods of classical statistics.

PRATT, RAIFFA, and SCHLAIFER (1965) From elementary probability theory through some frontiers of modern statistical decision theory with emphasis on problems with economic consequences.

RAIFFA and SCHLAIFER (1961) An advanced, somewhat terse text on modern Bayesian analysis. Lacks the interpretative material and detailed explanatory examples found in the preceding reference.

SAVAGE (1954) An inquiry into the underlying concepts of statistical theory. Does not require advanced mathematics.

WILLIAMS (1954) A gentle, animated introduction to game theory—the study of decision making in competitive, probabilistic situations.

5 *Complete Titles of Above References*

CHERNOFF, H., and L. MOSES: *"Elementary Decision Theory,"* John Wiley & Sons, Inc., New York, 1959.

COX, D. R.: *"Renewal Theory,"* John Wiley & Sons, Inc., New York, 1962.

———, and H. D. MILLER: *"The Theory of Stochastic Processes,"* John Wiley & Sons, Inc., New York, 1965.

———, and W. L. SMITH: *"Queues,"* John Wiley & Sons, Inc., New York, 1961.

CRAMER, H.: *"Mathematical Methods of Statistics,"* Princeton University Press, Princeton, N.J., 1946.

DAVENPORT, W. B., JR., and W. L. ROOT: *"Random Signals and Noise,"* McGraw-Hill Book Company, New York, 1958.

DAVID, F. N.: *"Games, Gods, and Gambling,"* Hafner Publishing Company, Inc., New York, 1962.

DOOB, J. L.: *"Stochastic Processes,"* John Wiley & Sons, Inc., New York, 1953.

FELLER, W.: *"An Introduction to Probability Theory and Its Applications,"* vol. 1, 2d ed., John Wiley & Sons, Inc., New York, 1957.

FISZ, M.: *"Probability Theory and Mathematical Statistics,"* John Wiley & Sons, Inc., New York, 1963.

FRASER, D. A. S.: *"Statistics: An Introduction,"* John Wiley & Sons, Inc., New York, 1958.

FREEMAN, H.: *"Introduction to Statistical Inference,"* Addison-Wesley Publishing Company, Inc., Reading, Mass., 1963.

GNEDENKO, B. V.: *"Theory of Probability,"* Chelsea, New York, 1962.

HOWARD, R. A.: *"Dynamic Programming and Markov Processes,"* The M.I.T. Press, Cambridge, Mass., 1960.

———: (in preparation), John Wiley & Sons, Inc., New York.

KOLMOGOROV, A. N.: *"Foundations of the Theory of Probability"* (Second English Edition), Chelsea, New York, 1956.

KYBURG, H. E., JR., and H. E. SMOKLER: *"Studies in Subjective Probability,"* John Wiley & Sons, Inc., New York, 1964.

LAPLACE, *A Philosophical Essay on Probabilities*, 1825 (English translation), Dover Publications, Inc., New York, 1951.

LEE, Y. W.: *"Statistical Theory of Communication,"* John Wiley & Sons, Inc., New York, 1960.

LOÈVE, M.: *"Probability Theory,"* D. Van Nostrand Company, Inc., Princeton, N.J., 1963.

MOOD, A. M., and F. A. GRAYBILL: *"Introduction to the Theory of Statistics,"* McGraw-Hill Book Company, New York, 1963.

MORSE, P. M.: "*Queues, Inventories, and Maintenance,*" John Wiley & Sons, Inc., New York, 1958.

PAPOULIS, A.: "*Probability, Random Variables, and Stochastic Processes,*" McGraw-Hill Book Company, New York, 1965.

PARZEN, E.: "*Modern Probability Theory and Its Applications,*" John Wiley & Sons, Inc., New York, 1960.

———: "*Stochastic Processes,*" Holden-Day, San Francisco, 1962.

PFEIFFER, P. E.: "*Concepts of Probability Theory,*" McGraw-Hill Book Company, New York, 1965.

PITT, H. R.: "*Integration, Measure, and Probability,*" Hafner Publishing Company, Inc., New York, 1963.

PRATT, J. W., H. RAIFFA, and R. SCHLAIFER: "*Introduction to Statistical Decision Theory*" (preliminary edition), McGraw-Hill Book Company, New York, 1965.

RAIFFA, H., and R. SCHLAIFER: "*Applied Statistical Decision Theory,*" Harvard Business School, Division of Research, Boston, 1961.

SAVAGE, L. J.: "*The Foundations of Statistics,*" John Wiley & Sons, Inc., New York, 1954.

TODHUNTER, I.: "*A History of a Mathematical Theory of Probability from the Time of Pascal to that of Laplace*" 1865, Chelsea, New York, 1949.

WILLIAMS, J. D.: "*The Compleat Strategyst, Being a Primer on the Theory of Games of Strategy,*" McGraw-Hill Book Company, New York, 1954.

APPENDIX TWO

common PDF's, PMF's, and their means, variances, and transforms

The most common elementary PDF's and PMF's are listed below. Only those transforms which may be expressed in relatively simple forms are included.

In each entry involving a single random variable, the random variable is denoted by x. For compound PDF's and PMF's, the random variables are denoted by two or more of the symbols u, v, w, x, and y. Symbols k, m, and n are used exclusively for nonnegative integer parameters. Unless there is a special reason for using other symbols, all parameters are denoted by the symbols a, b, and c.

The nomenclature used for PDF's and PMF's is not universal. Several of the entries below are, of course, special cases or generalizations of other entries.

Bernoulli PMF

$$p_x(x_0) = \begin{cases} 1 - P & x_0 = 0 \\ P & x_0 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$0 < P < 1$

$E(x) = P \qquad \sigma_x^2 = P(1 - P)$

$p_x^T(z) = 1 - P + zP$

Beta PDF

$$f_x(x_0) = \begin{cases} c(a,b)x_0^{a-1}(1 - x_0)^{b-1} & 0 < x_0 < 1 \\ 0 & \text{otherwise} \end{cases}$$

$a > 0 \qquad b > 0$

$$c(a,b) = \frac{(a + b - 1)!}{(a - 1)!(b - 1)!}$$

$$E(x) = \frac{a}{a + b} \qquad \sigma_x^2 = \frac{ab}{(a + b)^2(a + b + 1)}$$

Binomial PMF

$$p_x(x_0) = \begin{cases} \binom{n}{x_0} P^{x_0}(1 - P)^{n-x_0} & x_0 = 0,1,2, \ldots ,n \\ 0 & \text{otherwise} \end{cases}$$

$0 < P < 1 \qquad n = 1, 2, 3, \ldots$

$E(x) = nP \qquad \sigma_x^2 = nP(1 - P)$

$p_x^T(z) = (1 - P + zP)^n$

Bivariate-normal PDF

$$f_{x,y}(x_0,y_0) = \frac{\exp\left(-\frac{1}{2(1-\rho^2)}\left\{\left[\frac{x - E(x)}{\sigma_x}\right]^2 + \left[\frac{y - E(y)}{\sigma_y}\right]^2 - 2\rho\frac{x - E(x)}{\sigma_x}\frac{y - E(y)}{\sigma_y}\right\}\right)}{2\pi\sigma_x\sigma_y\sqrt{1 - \rho^2}}$$

$-\infty < x_0 < \infty \qquad -\infty < y_0 < \infty$

$\sigma_x > 0 \qquad \sigma_y > 0 \qquad -1 < \rho < 1$

$$f_{x,y}^T(s_1,s_2) = E(e^{-s_1x}e^{-s_2y}) = \exp[-s_1E(x) - s_2E(y) + \tfrac{1}{2}(s_1^2\sigma_x^2 + 2\rho s_1s_2\sigma_x\sigma_y + s_2^2\sigma_y^2)]$$

Cauchy PDF

$$f_x(x_0) = \frac{1}{\pi}\frac{a}{a^2 + (x_0 - b)^2} \qquad -\infty < x_0 < \infty$$

$a > 0 \qquad -\infty < b < \infty$

$E(x) = b \qquad \sigma_x^2 = \infty$

[The above value of $E(x)$ is a common definition. Although $E(x) = b$ seems intuitive from the symmetry of $f_x(x_0)$, note that

$$\int_{-\infty}^{\infty} x_0 f_x(x_0)\, dx_0$$

has no unique value for the Cauchy PDF.]

$f_x^T(s) = e^{-bs-a|s|}$

Chi-square PDF

$$f_x(x_0) = \begin{cases} \left[\left(\frac{n}{2} - 1\right)!\right]^{-1} 2^{-n/2}x_0^{(n/2)-1}e^{-x_0/2} & x_0 > 0 \\ 0 & \text{otherwise} \end{cases}$$

$n = 1, 2, 3, \ldots$

$E(x) = n \qquad \sigma_x^2 = 2n$

$f_x^T(s) = (1 + 2s)^{-n/2}$

Erlang PDF

$$f_x(x_0) = \begin{cases} \frac{a^n x_0^{n-1}e^{-ax_0}}{(n - 1)!} & x_0 > 0 \\ 0 & \text{otherwise} \end{cases}$$

$a > 0 \qquad n = 1, 2, 3, \ldots$

$E(x) = na^{-1} \qquad \sigma_x^2 = na^{-2}$

$f_x^T(s) = a^n(s + a)^{-n}$

Exponential PDF

$$f_x(x_0) = \begin{cases} ae^{-ax_0} & x_0 > 0 \\ 0 & \text{otherwise} \end{cases}$$

$a > 0$

$E(x) = a^{-1} \qquad \sigma_x^2 = a^{-2}$

$f_x^T(s) = a(s + a)^{-1}$

Gamma PDF

$$f_x(x_0) = \begin{cases} \dfrac{x_0^a e^{-x_0/b}}{a!\,b^{a+1}} & x_0 > 0 \\ 0 & \text{otherwise} \end{cases}$$

$a > -1 \qquad b > 0$

$E(x) = (a+1)b \qquad \sigma_x^2 = (a+1)b^2$

$f_x^T(s) = (1 + bs)^{-a-1}$

Geometric PMF

$$p_x(x_0) = \begin{cases} P(1-P)^{x_0-1} & x_0 = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$0 < P < 1$

$E(x) = P^{-1} \qquad \sigma_x^2 = (1-P)P^{-2}$

$p_x^T(z) = zP[1 - z(1-P)]^{-1}$

Hypergeometric PMF

$$p_x(x_0) = \begin{cases} \dbinom{m}{x_0}\dbinom{n}{k-x_0}\Big/\dbinom{m+n}{k} & x_0 = 0, 1, 2, \ldots, k \\ 0 & \text{otherwise} \end{cases}$$

$m = 1, 2, 3, \ldots \qquad n = 1, 2, 3, \ldots \qquad k = 1, 2, 3, \ldots, (m+n)$

$$E(x) = \frac{mk}{m+n} \qquad \sigma_x^2 = \frac{mnk(m+n-k)}{(m+n)^2(m+n-1)}$$

Laplace PDF

$$f_x(x_0) = \frac{a}{2} e^{-a|x_0-b|} \qquad -\infty < x_0 < \infty$$

$a > 0 \qquad -\infty < b < \infty$

$E(x) = b \qquad \sigma_x^2 = 2a^{-2}$

$f_x^T(s) = a^2 e^{-bs}(a^2 - s^2)^{-1}$

Log-normal PDF

$$f_x(x_0) = \begin{cases} \dfrac{\exp\{-[\ln(x_0-a) - b]^2/2\sigma^2\}}{\sqrt{2\pi}\,\sigma(x_0-a)} & x_0 > a \\ 0 & \text{otherwise} \end{cases}$$

$\sigma > 0 \qquad -\infty < a < \infty \qquad -\infty < b < \infty$

$E(x) = a + e^{b+0.5\sigma^2} \qquad \sigma_x^2 = e^{2b+\sigma^2}(e^{\sigma^2} - 1)$

Maxwell PDF

$$f_x(x_0) = \begin{cases} \sqrt{2/\pi}\, a^3 x_0^2 e^{-a^2x_0^2/2} & x_0 > 0 \\ 0 & \text{otherwise} \end{cases}$$

$a > 0$

$E(x) = \sqrt{8/\pi}\, a^{-1} \qquad \sigma_x^2 = (3 - 8/\pi)a^{-2}$

Multinomial PMF

$$p_{u,v,\ldots,y}(u_0, v_0, \ldots, y_0) = \frac{n!\, p_u^{u_0} p_v^{v_0} \cdots p_y^{y_0}}{u_0!\, v_0! \cdots y_0!}$$

$u_0 = 0, 1, \ldots, n \qquad v_0 = 0, 1, \ldots, n \quad \cdots \quad y_0 = 0, 1, \ldots, n$

$u_0 + v_0 + \cdots + y_0 = n$

$p_u + p_v + \cdots + p_y = 1 \qquad 0 < p_u, p_v, \ldots, p_y < 1$

$E(u) = np_u \qquad E(v) = np_v \quad \cdots \quad E(y) = np_y$

$\sigma_u^2 = np_u(1-p_u) \qquad \sigma_v^2 = np_v(1-p_v) \quad \cdots \quad \sigma_y^2 = np_y(1-p_y)$

Normal PDF

$$f_x(x_0) = \frac{e^{-[x_0 - E(x)]^2/2\sigma_x^2}}{\sqrt{2\pi}\,\sigma_x} \qquad -\infty < x_0 < \infty$$

$\sigma_x > 0 \qquad -\infty < E(x) < \infty$

$f_x^T(s) = e^{-sE(x) + (s^2\sigma_x^2/2)}$

Pascal PMF

$$p_x(x_0) = \begin{cases} \dbinom{x_0-1}{n-1} P^n(1-P)^{x_0-n} & x_0 = n, n+1, n+2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$0 < P < 1 \qquad n = 1, 2, 3, \ldots$

$E(x) = nP^{-1} \qquad \sigma_x^2 = n(1-P)P^{-2}$

$p_x^T(z) = (zP)^n[1 - z(1-P)]^{-n}$

Poisson PMF

$$p_x(x_0) = \frac{a^{x_0}e^{-a}}{x_0!} \qquad x_0 = 0, 1, 2, \ldots$$

$a > 0$

$E(x) = a \qquad \sigma_x^2 = a$

$p_x^T(z) = e^{a(z-1)}$

Rayleigh PDF

$$f_x(x_0) = \begin{cases} a^2 x_0 e^{-a^2 x_0^2/2} & x_0 > 0 \\ 0 & \text{otherwise} \end{cases}$$

$a > 0$

$E(x) = \sqrt{\pi/2}\, a^{-1} \qquad \sigma_x^2 = (2 - \pi/2)a^{-2}$

Uniform PDF

$$f_x(x_0) = \begin{cases} \dfrac{1}{b-a} & a < x_0 < b \\ 0 & \text{otherwise} \end{cases}$$

$-\infty < a < b < \infty$

$E(x) = (a + b)/2 \qquad \sigma_x^2 = (b - a)^2/12$

$f_x^T(s) = (e^{-as} - e^{-bs})[s(b - a)]^{-1}$

Weibull PDF

$$f_x(x_0) = \begin{cases} abx_0^{b-1} e^{-ax_0^b} & x_0 > 0 \\ 0 & \text{otherwise} \end{cases}$$

$a > 0 \qquad b > 0$

$$E(x) = \left(\frac{1}{a}\right)^{1/b} \Gamma(1 + b^{-1})$$

$$\sigma_x^2 = \left(\frac{1}{a}\right)^{2/b} \{\Gamma(1 + 2b^{-1}) - [\Gamma(1 + b^{-1})]^2\}$$

$$\Gamma(c) \equiv \int_0^\infty x^{c-1} e^{-x}\, dx$$

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