Introduction to Simulation - Lecture 24

Model-Order Reduction

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Thanks to Luca Daniel, Guillaum Lassauxx, Jing Li, Mark Reichelt, Deepak Ramaswamy, Michal Rewienski, Mary Tolikas, Karen Veroy and Karen Willcox

MOR Outline

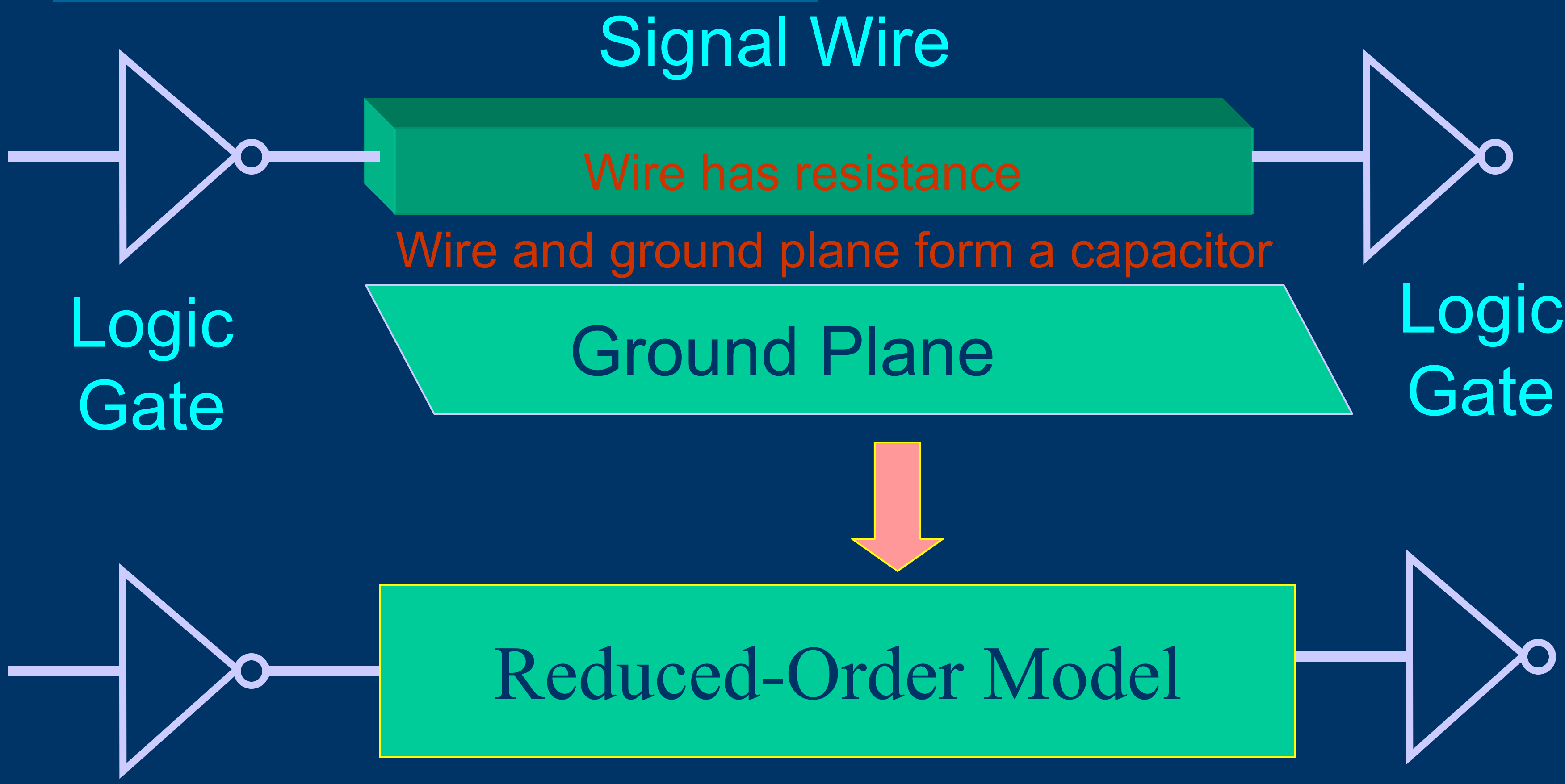
- Need For Model Reduction
  - Circuits, MEMS, Jet Engines
- Steady-State Case (linear and nonlinear)
- Dynamic Linear Case
  - Eigenmodes and Rational Functions
  - Projection Framework, Krylov, TBR
- Nonlinear Case
  - Projection Framework

Today's Outline

- Need For Model Reduction
  - Circuits, MEMS, Optics, Jet Engines
- Simple Example Problem
  - Heat Conducting bar example
- Steady-State Case (linear and nonlinear)
- Dynamic Linear Case
  - Truncating Eigenmodes
  - Rational Function Fitting

Application Example

Signal Wire in an Integrated Circuit

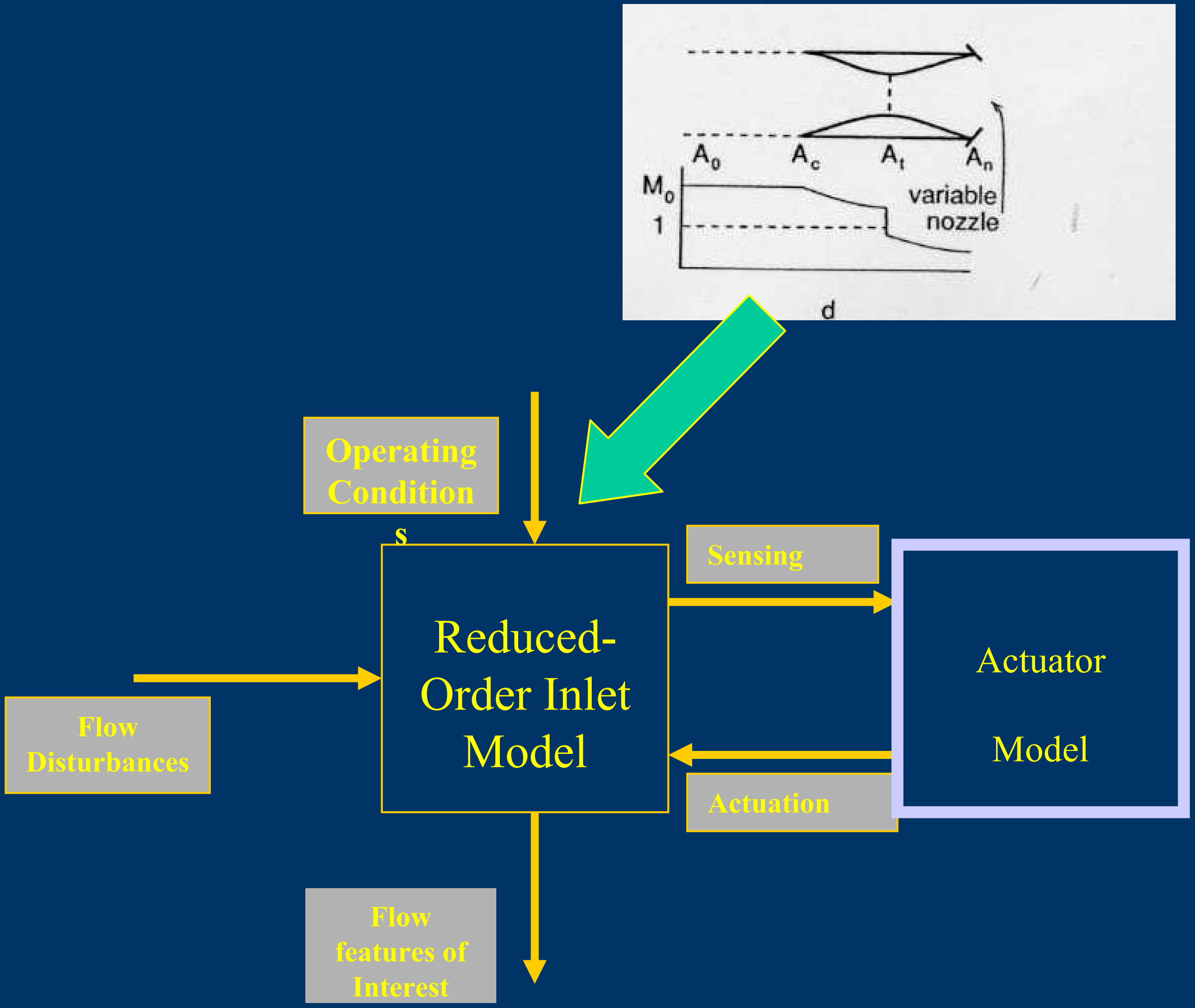


- Assess wiring impact on IC performance
- Wire Model must preserve terminal behavior

Application Example

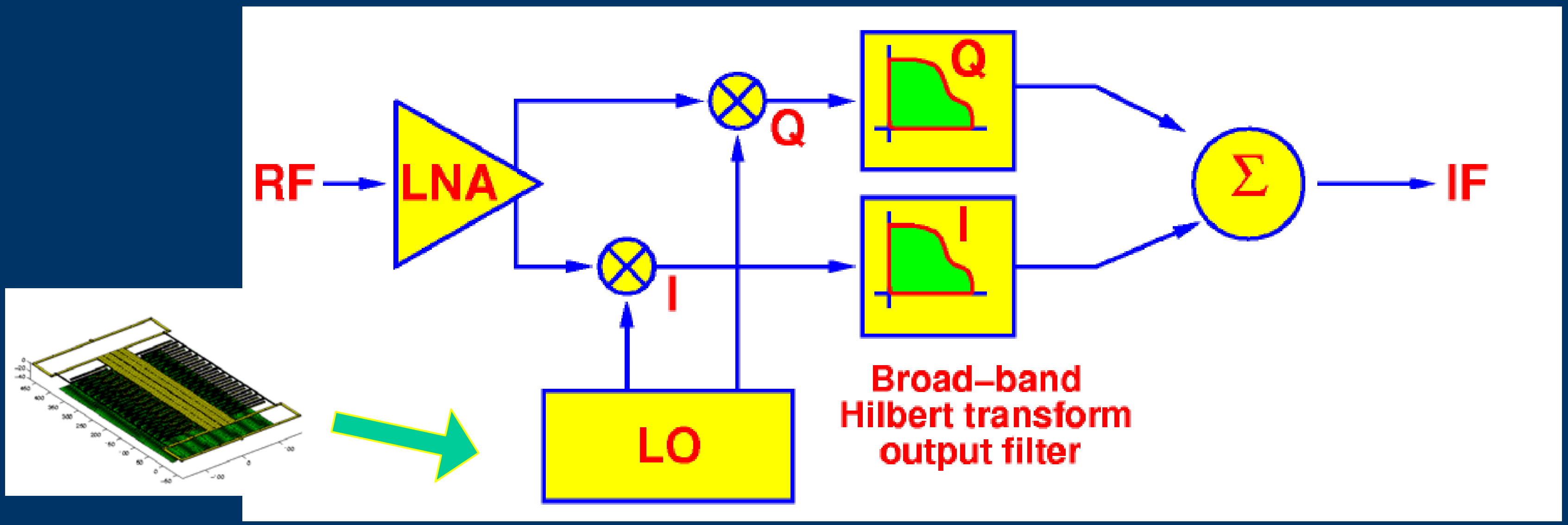
Jet Engine Design

- Generate Low-order models directly from Navier-Stokes Equation based physical simulators.
- Reduced model must preserve instabilities.



Application Examples

Micromechanical Resonators in a Wireless transceiver

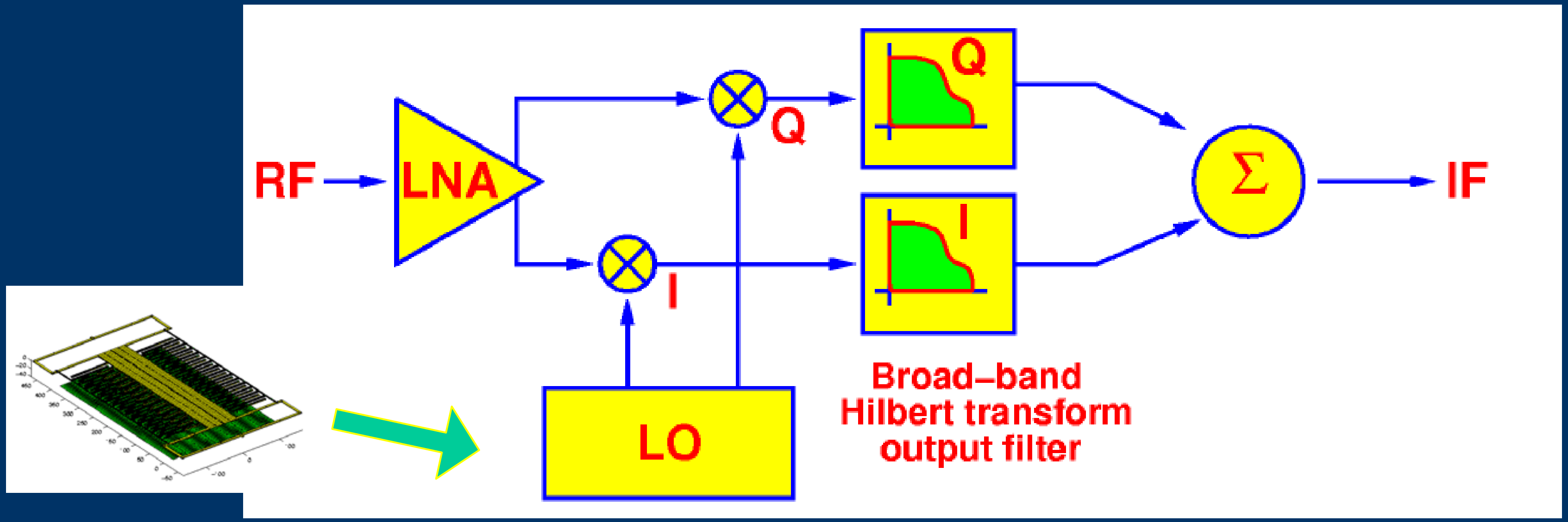


RF Front end with micromachined resonators for the oscillator

- What is system performance (noise, distortion, etc).
- Will poly-substrate separation (changes Q) matter?
- How tight must manufacturing tolerances be?

Application Examples

Micromechanical Resonators in a Wireless transceiver (cont)



Need to simulate ENTIRE system with dynamically accurate macromodels for all the components

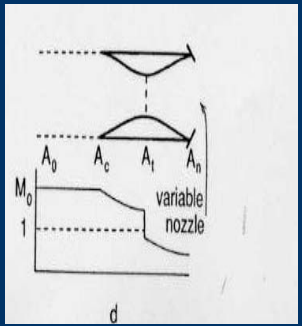
Application Examples

Common features

- Devices have Well-defined Inputs and Outputs
  - Signal Transmission on a Wire
    - Left and right end Voltages and Currents
  - Jet Engine Nozzle Design
    - Nozzle in and out flow, Nozzle size
  - Microresonator
    - Comb finger voltages and currents
- Dynamics is important
  - Internal state must be somehow represented

Application Examples

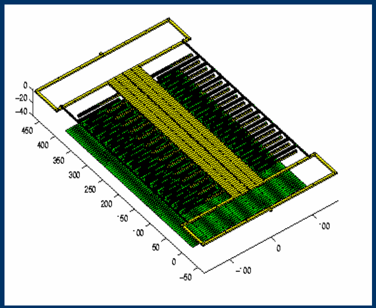
Traditional Approach to Generating Models



6 months…

$$\frac{dx_r(t)}{dt} = F(x_r(t)) + b_r u(t)$$

$$y(t) = c_r^T x_r(t)$$

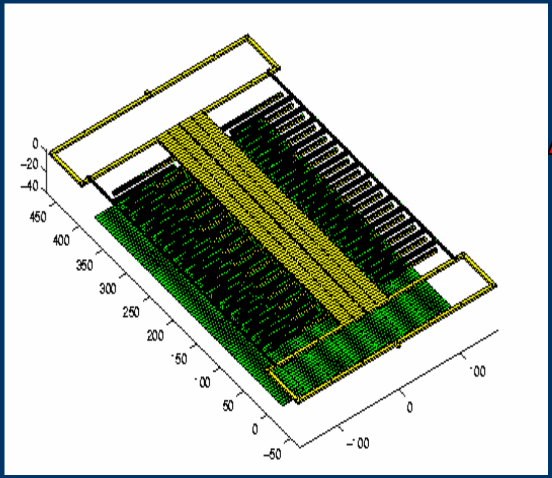


6 months…

Model for the System Simulator

The Numerical Macromodeling or Model Reduction Paradigm

Generate a Reduced-Order Model Directly from 3-D Geometry and Physics



Complicated Geometry, Coupled Electrostatics, Fluids, Elastics

Automatic →

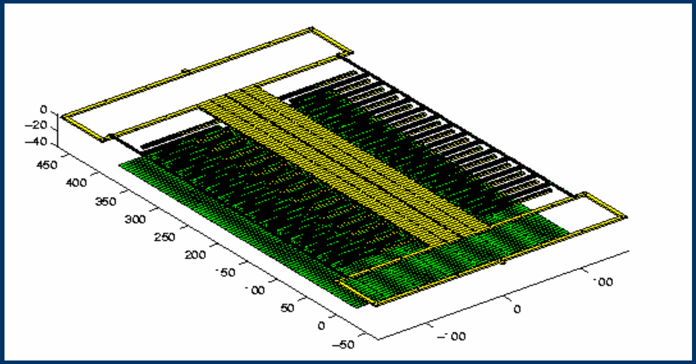
$$\frac{dx_r(t)}{dt} = F(x_r(t)) + b_r u(t)$$

$$y(t) = c_r^T x_r(t)$$

Cheap to evaluate model which captures input (u)/output(y) behavior

The Numerical Macromodeling or Model Reduction Paradigm

What's Needed



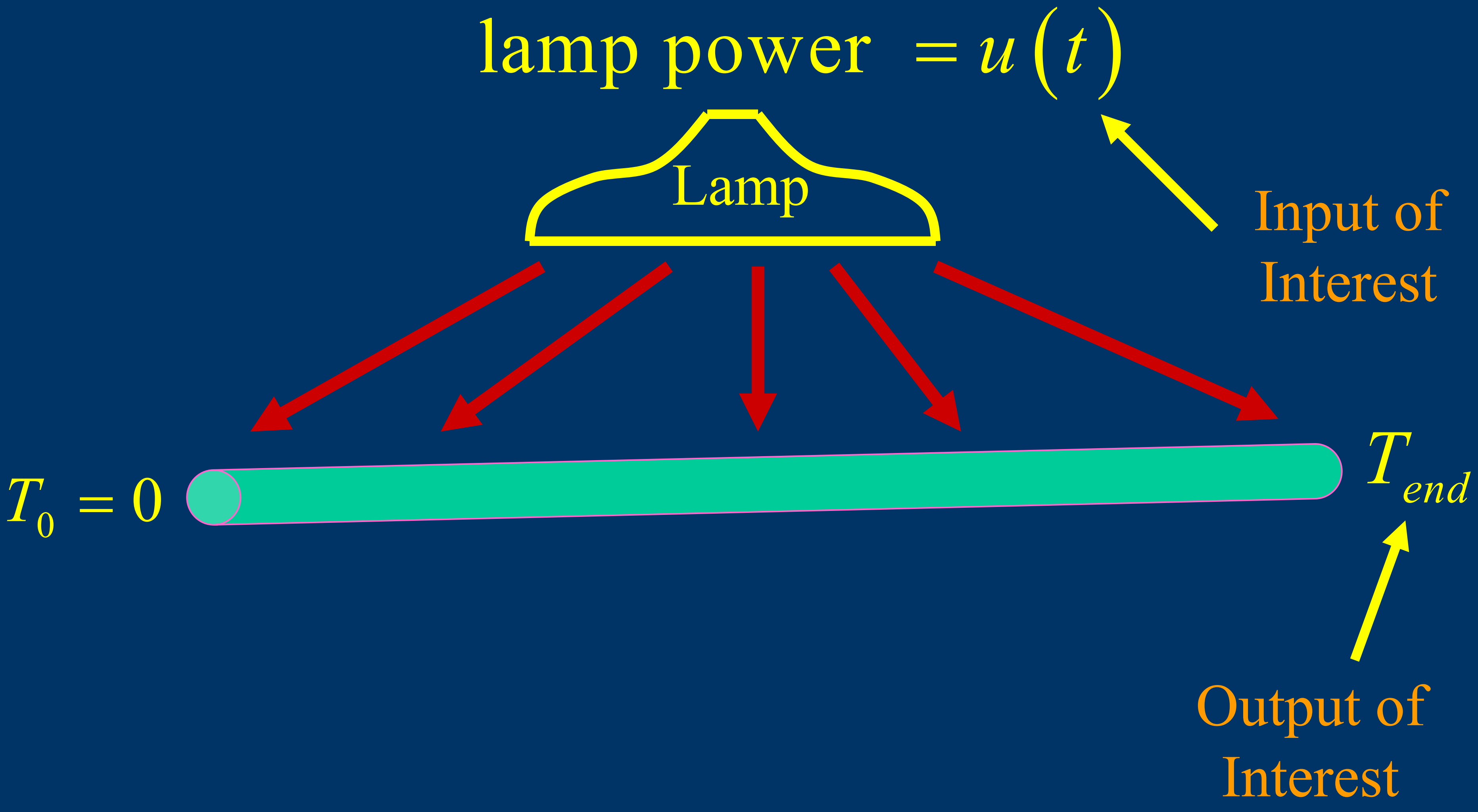
$$\frac{dx_r(t)}{dt} = F(x_r(t)) + b_r u(t)$$

$$y(t) = c_r^T x_r(t)$$

- Fast Solvers for complicated 3-D geometries
  - (Fast enough to solve ENTIRE devices)
  - for fluids, electrostatics, mechanics, …
- Approaches for coupled domain analysis
  - Multilevel-Newton methods
- Automatic extraction of reduced order models

Demonstration Example

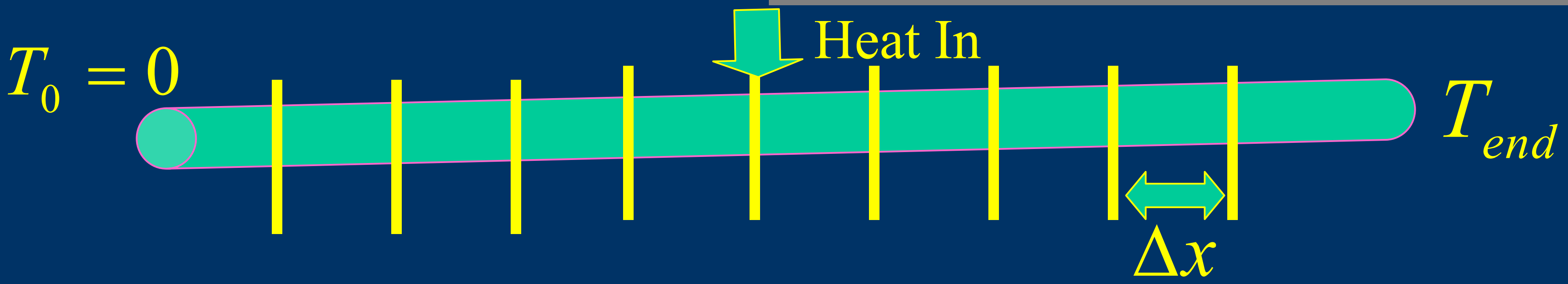
Heat Conducting Bar



Demonstration Example

Heat Conducting Bar

Basic Equations



- Temperature Differential Equation

$$\underbrace{\gamma}_{\text{specific heat}} \frac{\partial T(x,t)}{\partial t} - \underbrace{\kappa}_{\text{thermal conductivity}} \frac{\partial^2 T(x,t)}{\partial x^2} = h(x)\underbrace{u(t)}_{\text{scalar input}}$$

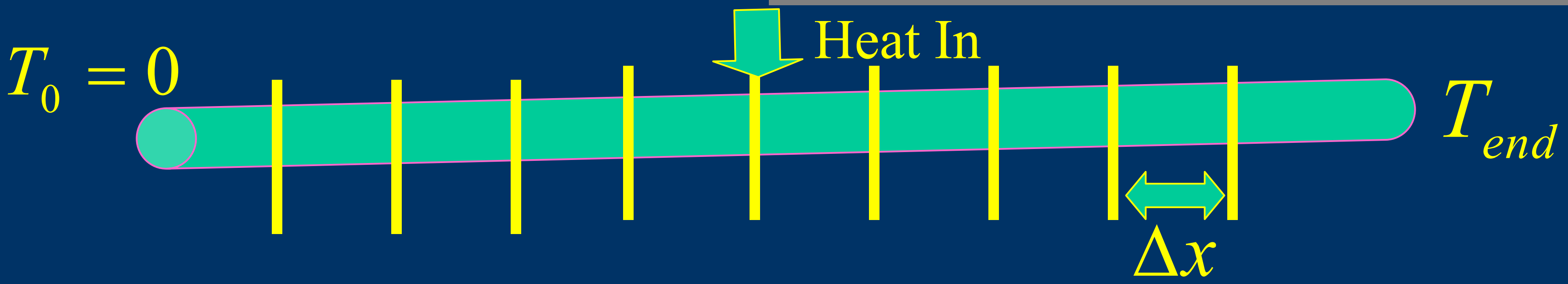
- Spatial Discretization (except at end)

$$\gamma \frac{d\hat{T}_i}{dt} - \frac{\kappa}{(\Delta x)^2}\left(\hat{T}_{i+1} - 2\hat{T}_i + \hat{T}_{i-1}\right) = h(x_i)u(t)$$

Demonstration Example

Heat Conducting Bar

Input-Output Discrete Equations



$$\gamma \frac{d\hat{T}_i}{dt} - \frac{\kappa}{(\Delta x)^2}\left(\hat{T}_{i+1} - 2\hat{T}_i + \hat{T}_{i-1}\right) = h(x_i)u(t) \qquad i \in [1,\ldots,N-1]$$

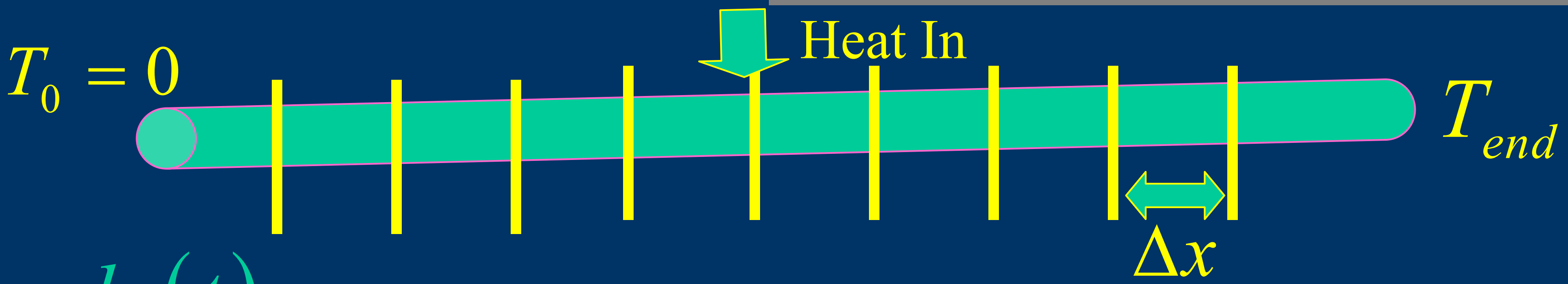
$$\gamma \frac{d\hat{T}_i}{dt} - \frac{\kappa}{(\Delta x)^2}\left(\hat{T}_N - \hat{T}_{N-1}\right) = h(x_N)u(t)$$

$$T_{end} = \hat{T}_N$$

Demonstration Example

Heat Conducting Bar

State-Space Description



$$\frac{dx(t)}{dt} = \underbrace{A}_{NxN} x(t) + \underbrace{b}_{Nx1} \underbrace{u(t)}_{\substack{scalar\\input}} \qquad \underbrace{y(t)}_{\substack{scalar\\output}} = \underbrace{c}_{Nx1}{}^{T} x(t)$$

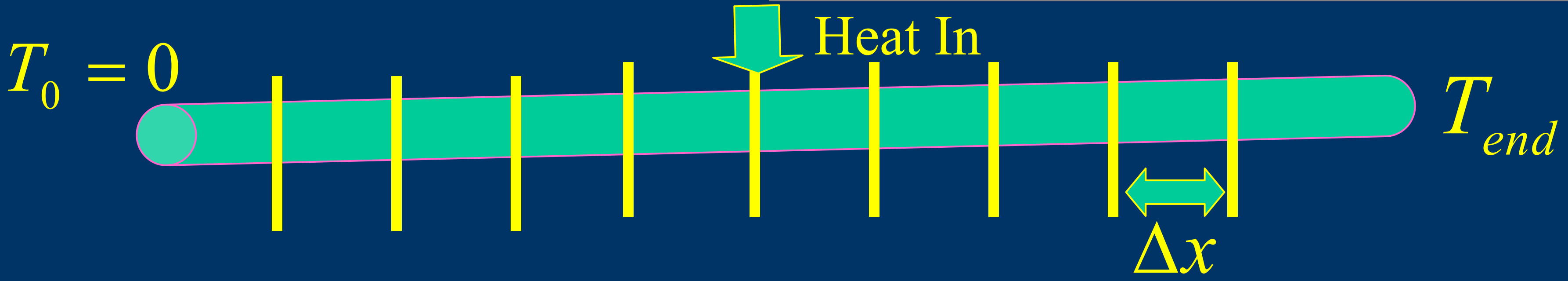
Given the right scaling

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} h(x_1) \\ h(x_2) \\ \vdots \\ \vdots \\ h(x_N) \end{bmatrix} \quad c = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Demonstration Example

Heat Conducting Bar

Temperature Dependent Thermal Conductivity



- Temperature Differential Equation

$$\underbrace{\gamma}_{\substack{specific\\heat}} \frac{\partial T(x,t)}{\partial t} - \underbrace{\kappa\big(T(x,t)\big)}_{\substack{thermal\\conductivity}} \frac{\partial^2 T(x,t)}{\partial x^2} = h(x)$$

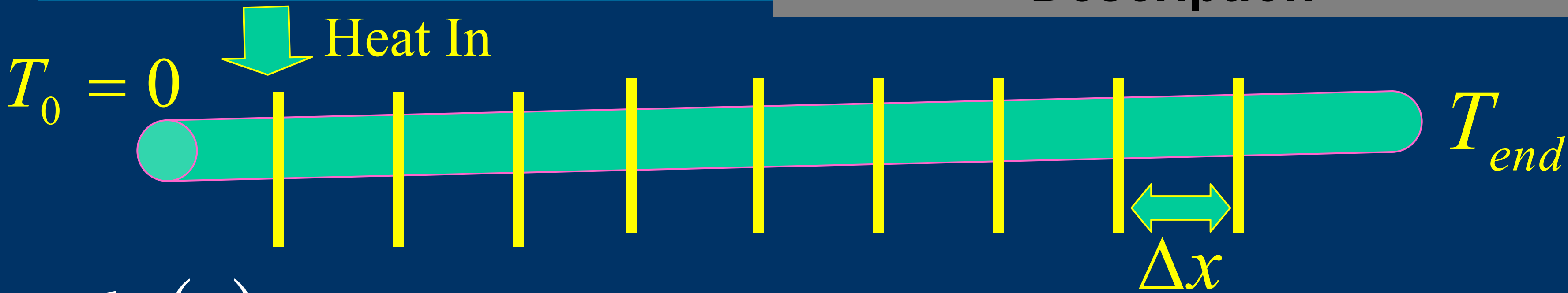
- Simple Spatial Discretization (not at ends)

$$\gamma \frac{d\hat{T}_i}{dt} - \frac{\kappa\left(\hat{T}_i\right)}{(\Delta x)^2}\left(\hat{T}_{i+1} - 2\hat{T}_i + \hat{T}_{i-1}\right) = 0$$

Demonstration Example

Heat Conducting Bar

Nonlinear State-Space Description



$$\frac{dx(t)}{dt} = F\big(x(t)\big) + \underbrace{b}_{Nx1}\ \underbrace{u(t)}_{\substack{scalar\\ input}} \qquad \underbrace{y(t)}_{\substack{scalar\\ output}} = \underbrace{c}_{Nx1}{}^{T}\, x(t)$$

$$F(x) = \begin{bmatrix} 2\kappa(x_1) & -\kappa(x_1) & 0 & \cdots & 0 \\ -\kappa(x_2) & 2\kappa(x_2) & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2\kappa(x_{N-1}) & -\kappa(x_{N-1}) \\ 0 & \cdots & 0 & -\kappa(x_N) & \kappa(x_N) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_N \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Linear example

No Dynamics (Steady-State) Case

- Original System - Single Input/Output

$$0 = \underbrace{A}_{NxN} x + \underbrace{b}_{Nx1} \underbrace{u}_{\substack{scalar \\ input}} \qquad \underbrace{y}_{\substack{scalar \\ output}} = \underbrace{c}_{Nx1}{}^{T} x$$

- Reduced System

$$y = -\underbrace{c^T A^{-1} b}_{1x1} u$$

- Satisfies Reduced Model Criteria
  - Cheap to evaluate
  - Exactly reproduces I/O Behavior

Nonlinear Example

No Dynamics Case

- Original System - Single Input/Output

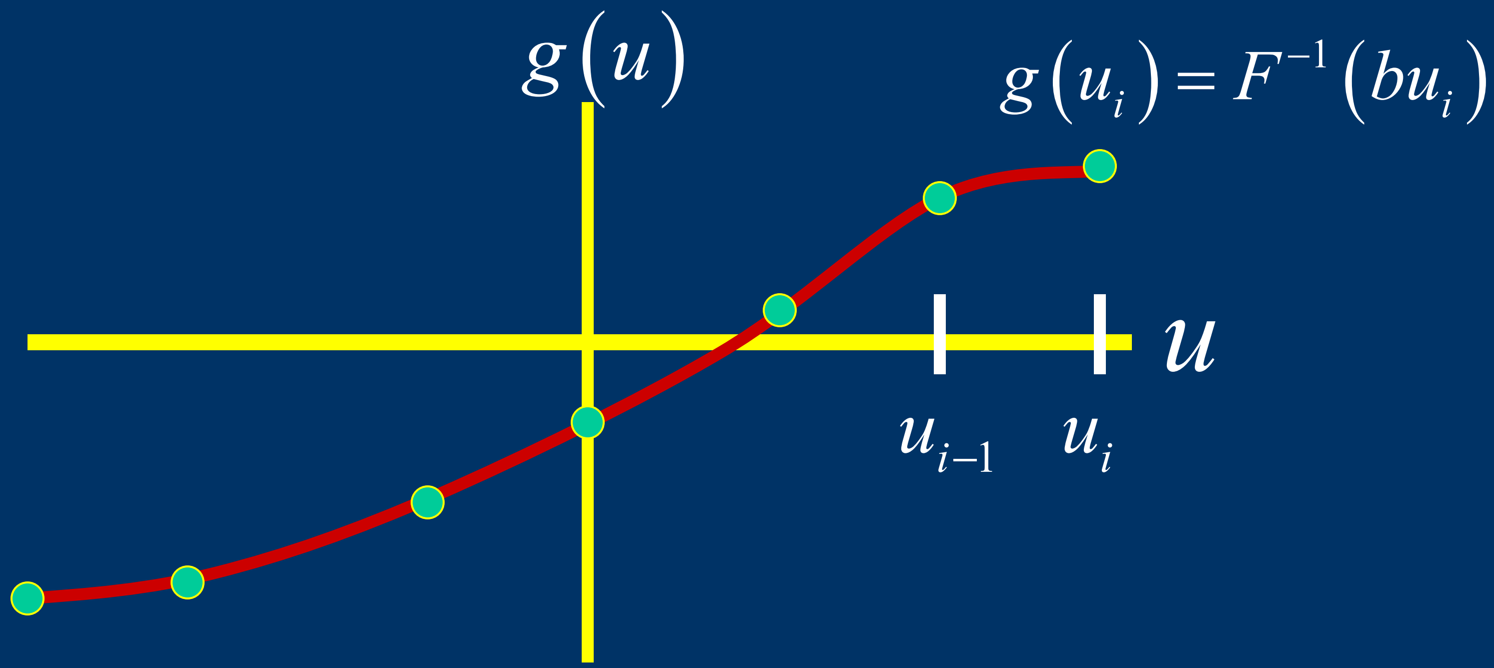
$$0 = F(x) + \underbrace{b}_{Nx1} \underbrace{u}_{\substack{scalar \\ input}} \qquad \underbrace{y}_{\substack{scalar \\ output}} = \underbrace{c}_{Nx1}^{T} x$$

- Reduced System

$$y = g(u)$$

- Is "g(u)" a reduced-order model?
  - Depends how we represent g!

- Could use an interpolated table of data



- Table is a reduced order model
  - Cheap to evaluate
  - Accurate if enough points used

Model Construction Time

No Dynamics Case

- Linear Case, one solve, one inner product
  - Solve $Ax = b \Rightarrow x = A^{-1}b$
  - Form $c^T x = c^T A^{-1} b$
- Nonlinear Case (if an interpolated table is used)
  - Solve

    $$F(x_i) = bu_i \Rightarrow x_i = F^{-1}(bu_i) \quad for\ i = 1, \ldots \# samples$$

  - Form $c^T x_i = c^T F^{-1}(bu_i) \quad for\ i = 1, \ldots \# samples$
- Nonlinear Reduction adds a representation problem to model reduction

State-Space Description

Dynamic Linear case

- Original Dynamical System - Single Input/Output

$$\frac{dx(t)}{dt} = \underbrace{A}_{NxN} x(t) + \underbrace{b}_{Nx1} \underbrace{u(t)}_{\substack{scalar \\ input}} \qquad \underbrace{y(t)}_{\substack{scalar \\ output}} = \underbrace{c}_{Nx1}{}^{T} x(t)$$

- Reduced Dynamical System

$$\frac{dx_r(t)}{dt} = \underbrace{A_r}_{qxq} x(t) + \underbrace{b_r}_{qx1} \underbrace{u(t)}_{\substack{scalar \\ input}} \qquad \underbrace{y_r(t)}_{\substack{scalar \\ output}} = \underbrace{c_r}_{qx1}{}^{T} x_r(t)$$

- q << N, but input/output behavior preserved

Reminder about Eigenanalysis

Consider an ODE: $$\frac{dx(t)}{dt} = Ax(t) + bu(t), \quad x(0) = 0$$

Eigendecomposition: $$A = \underbrace{\begin{bmatrix} \vdots & \vdots & \vdots \\ E_1 & E_2 & E_N \\ \vdots & \vdots & \vdots \end{bmatrix}}_{E} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_N \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots \\ E_1 & E_2 & E_N \\ \vdots & \vdots & \vdots \end{bmatrix}^{-1}$$

Change of variables: $Ew(t) = x(t) \Leftrightarrow w(t) = E^{-1}x(t)$

Substituting: $$\frac{dEw(t)}{dt} = AEw(t) + bu(t), \quad Ew(0) = 0$$

Multiply by E^{-1}: $$\frac{dw(t)}{dt} = E^{-1}AEw(t) + E^{-1}bu(t)$$

Reminder about Eigenanalysis Cont.

Decoupled Equations

$$\frac{d}{dt}\begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_N \end{bmatrix}\begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} + \underbrace{\begin{bmatrix} \left(E^{-1}b\right)_1 \\ \vdots \\ \left(E^{-1}b\right)_N \end{bmatrix}}_{\tilde{b}} u(t)$$

Output Equation

$$y(t) = c^T x(t) = c^T E w(t) = \underbrace{\left(E^T c\right)^T}_{\tilde{c}} w(t)$$

Reminder about Eigenanalysis Cont.

Solving Decoupled Equations

$$w_i(t) = \int_0^t e^{\lambda_i (t-\tau)} \tilde{b}_i \, u(\tau) \, d\tau$$

Assuming Zero Initial Conditions

Output Equation

$$y(t) = \sum_{i=1}^{N} \tilde{c}_i \, w_i(t)$$

Dynamic Linear Case

Reduced models via mode truncation

$$\begin{bmatrix} \dot{w}_1 \\ \vdots \\ \dot{w}_q \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_q \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_q \end{bmatrix} + \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_q \end{bmatrix} u(t)$$

Output Equation

$$y(t) = \sum_{i=1}^{q} \tilde{c}_i \, w_i(t)$$

Dynamic Linear Case

Reduced models via mode Truncation

Why?

- Certain modes are not affected by the input

  $\tilde{b}_{k+1}, \ldots, \tilde{b}_N$ are all small

- Certain modes do not affect the output

  $\tilde{c}_{k+1}, \ldots, \tilde{c}_N$ are all small

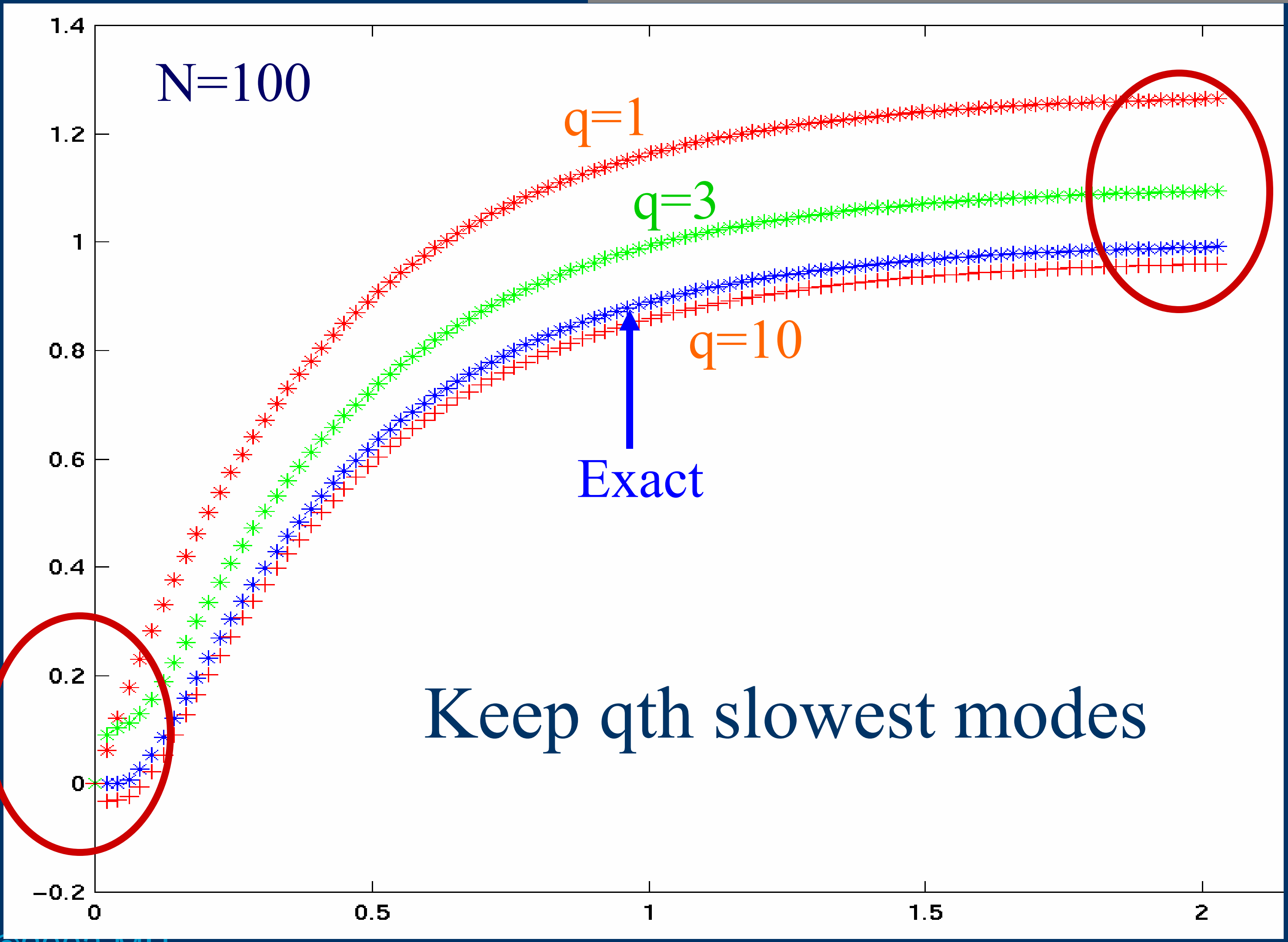
- Keep least negative evals (slowest modes)
  - Look at response to a constant input

$$w_i(t) = \int_0^t e^{\lambda_i (t-\tau)} \tilde{b}_i u \, d\tau = \underbrace{\frac{1}{\lambda_i}\left(\tilde{b}_i u - \tilde{b}_i u e^{\lambda_i t}\right)}_{\text{Small if } |\lambda_i| \text{ large}}$$

Dynamic Linear Case

Reduced models via mode truncation

Heat Conducting bar Results



An Aside on Transfer Functions – Laplace Transform

Consider an ODE: $\frac{dx(t)}{dt} = Ax(t) + bu(t)$

Bilateral Laplace Transform: $X(s) = \int_{-\infty}^{\infty} \mathrm{x}(\mathrm{t}) e^{-st} dt$

Key Transform Property: $sX(s) = \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$

Rewrite the ODE in transformed variables

$$sX(s) = AX(s) + bU(s) \qquad Y(s) = c^T X(s)$$

$$\Rightarrow Y(s) = \underbrace{c^T (sI - A)^{-1} b}_{H(s)} U(s)$$

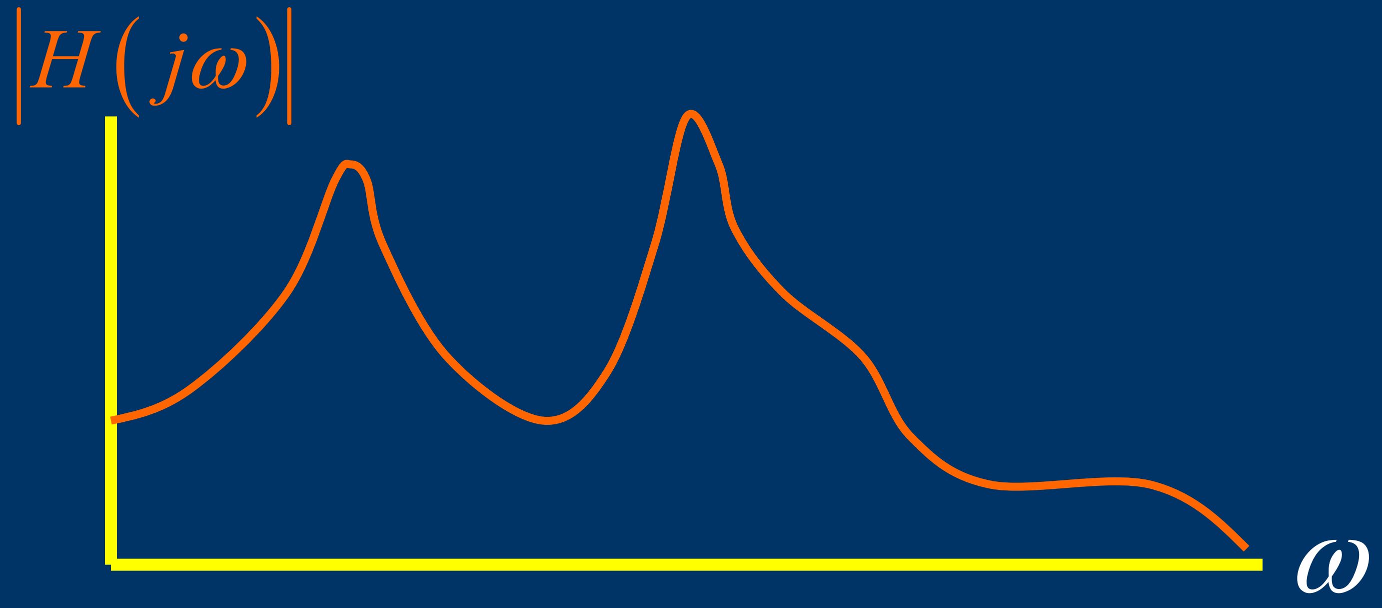
← Transfer Function

An Aside on Transfer Functions – Meaning of H(s)

For Stable Systems, H(j*w*) is the frequency response

$$\text{If } u(t) = e^{j\omega t}$$ ← Sinusoid

$$\text{then } y(t) = H\left(j\omega\right)e^{j\omega t}$$ Sinusoid with shifted phase and amplitude



An Aside on Transfer Functions – EigenAnalysis

Transfer Function

$$H(s) = c^T (sI - A)^{-1} b$$

Apply Eigendecomposition

$$H(s) = c^T E (sI - \lambda)^{-1} E^{-1} b$$

$$= \tilde{c}^T \begin{bmatrix} \frac{1}{s-\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{s-\lambda_N} \end{bmatrix} \tilde{b} \Rightarrow H(s) = \sum_{i=1}^{N} \frac{\tilde{c}_i \tilde{b}_i}{s - \lambda_i}$$

Rational Transfer Function Representation

Original System Transfer Function

$$H(s) = \frac{\tilde{c}_1 \tilde{b}_1}{(s-\lambda_1)} + \cdots + \frac{\tilde{c}_N \tilde{b}_N}{(s-\lambda_N)} = \underbrace{\frac{b_0 + b_1 s + \cdots + b_{N-1} s^{N-1}}{1 + a_1 s + \cdots + a_N s^N}}_{\text{Rational Function}}$$

Reduced Model Transfer Function

$$H_r(s) = \underbrace{\frac{b_0^r + b_1^r s + \cdots + b_{q-1}^r s^{q-1}}{1 + a_1^r s + \cdots + a_q^r s^q}}_{\text{Lower Order Rational Function}}$$

Model Reduction = Find a low order rational function matching H(s)

Dynamic Linear Case

Rational Transfer Function Representation

Degrees of Freedom

Reduced Model Dynamical System

$$\frac{dx_r(t)}{dt} = \underbrace{A_r}_{qxq} x(t) + \underbrace{b_r}_{qx1} \underbrace{u(t)}_{\substack{scalar\\input}} \qquad \underbrace{y_r(t)}_{\substack{scalar\\output}} = \underbrace{c_r}_{qx1}{}^T x_r(t)$$

$2q + q^2$ coefficients

Reduced Model Transfer Function

$$H_r(s) = \frac{b_0^r + b_1^r s + \cdots + b_{q-1}^r s^{q-1}}{1 + a_1^r s + \cdots + a_q^r s^q}$$

$2q$ coefficients

Dynamic Linear Case

Rational Transfer Function Representation

Variable Changes Do not change transfer functions

Reduced Model Transfer Function

$$\frac{dx_r(t)}{dt} = A_r x(t) + b_r u(t) \quad y_r(t) = c_r^T x_r(t)$$

$$\Rightarrow H(s) = c_r^T (sI - A_r)^{-1} b_r$$

Similarity (x = Sw) Transformed Transfer Function

$$\frac{dw_r(t)}{dt} = S^{-1} A_r S w(t) + S^{-1} b_r u(t) \quad y_r(t) = c_r^T S w_r(t)$$

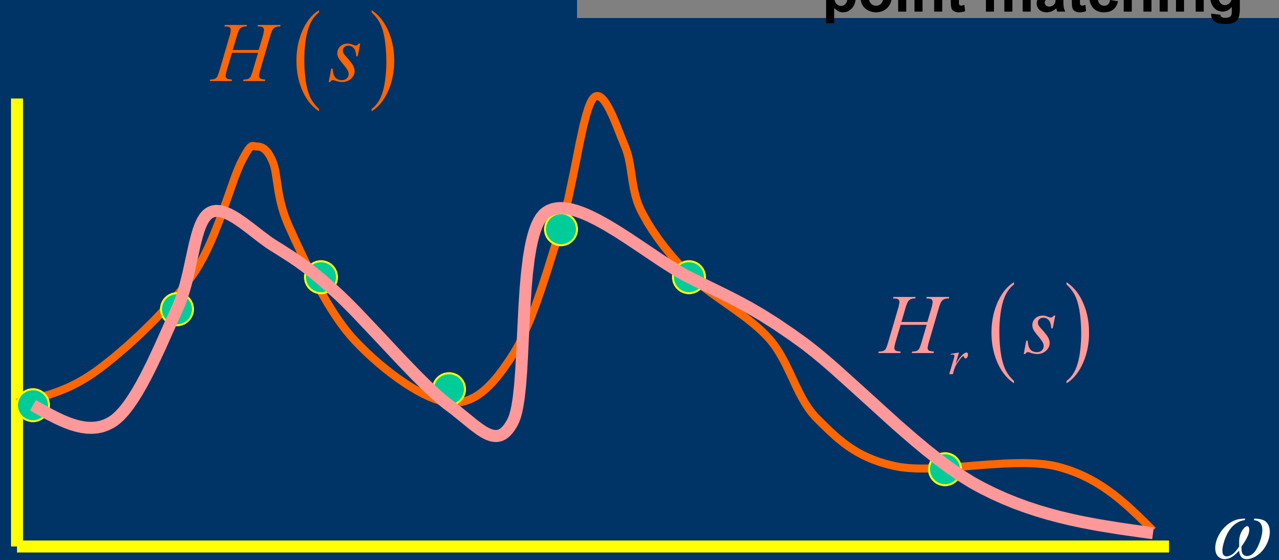
$$\Rightarrow H(s) = c_r^T S \left(sI - S^{-1} A_r S\right)^{-1} S^{-1} b_r = c_r^T (sI - A_r)^{-1} b_r$$

Many Dynamical Systems have the same transfer function!!

Dynamic Linear Case

Rational Transfer Function Representation

Rational Function Fitting by point matching



- Can match 2q points
- cross multiplying generates a linear system

For i = 1 to 2q

$$\left(1 + a_1^r s_i + \cdots + a_q^r s_i^{\,q}\right) H(s_i) - \left(b_0^r + b_1^r s + \cdots + b_{q-1}^r s^{q-1}\right) = 0$$

Rational Transfer Function Representation

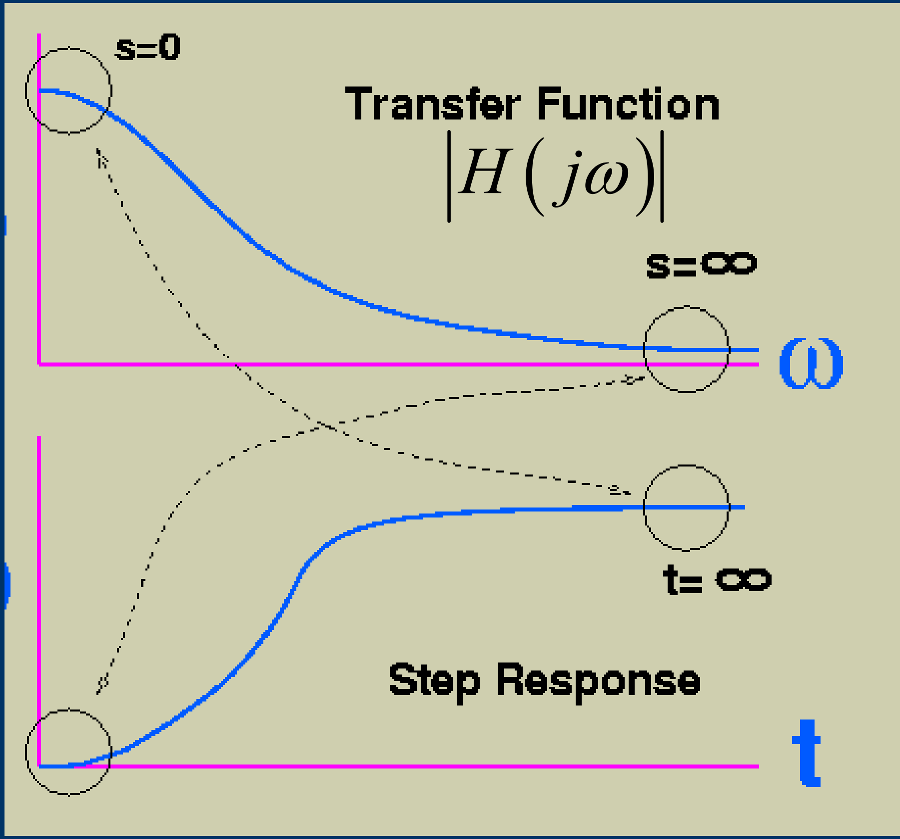
Point Matching Matrix can be ill-conditioned

$$\begin{bmatrix} s_1 H(s_1) & s_1^2 H(s_1) & \cdots & -s_1^{q-1} \\ \vdots & \vdots & \cdots & -s_2^{q-1} \\ \vdots & \vdots & \cdots & \vdots \\ s_{2q} H(s_{2q}) & s_{2q}^2 H(s_{2q}) & \cdots & -s_{2q}^{q-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ b_{q-1} \end{bmatrix} = \begin{bmatrix} H(s_1) \\ H(s_2) \\ \vdots \\ H(s_{2q}) \end{bmatrix}$$

- Columns contain progressively higher powers of the test frequencies
- Must orthogonalize columns during construction

Rational Transfer Function Representation

Importance of Fitting at low frequency



Correct Steady State behavior requires accurate match at low frequencies

Original System Transfer Function Moments

$$H(s) = c^T (sI - A)^{-1} b = -c^T \underbrace{(I - sA^{-1})^{-1}}_{\text{Taylor Expand with respect to s}} A^{-1} b$$

$$H(s) = -c^T (I - sA^{-1})^{-1} A^{-1} b = \sum_{k=0}^{\infty} c^T A^{-(k+1)} b s^k$$

$$H(s) = \underbrace{c^T A^{-1} b}_{m_0} + \underbrace{c^T A^{-2} b}_{m_1} s + \underbrace{c^T A^{-3} b}_{m_2} s^2 + \cdots = \sum_{k=0}^{\infty} m_k s^k$$

Moments

Rational Transfer Function Representation

Moment Matching for accurate low frequency behavior

Reduced Model Matches Original Systems Moments

$$H_r(s) = \frac{b_0^r + b_1^r s + \cdots + b_{q-1}^r s^{q-1}}{1 + a_1^r s + \cdots + a_q^r s^q} = m_0 + m_1 s + \cdots + m_{2q-1} s + \cdots$$

Cross-Multiplying and Matching Terms

$$\begin{bmatrix} m_0 & m_1 & \cdots & m_{k-1} \\ m_1 & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & m_{2q-3} \\ m_{k-1} & \cdots & m_{2q-3} & m_{2q-2} \end{bmatrix} \begin{bmatrix} a_q \\ a_{q-1} \\ \vdots \\ a_1 \end{bmatrix} = \begin{bmatrix} m_q \\ m_{q+1} \\ \vdots \\ m_{2q-1} \end{bmatrix}$$

Dynamic Linear Case

Rational Transfer Function Representation

Explicit Moment Matching Problem

System of equations extremely ill-conditioned

$$\begin{bmatrix} m_0 & m_1 & \cdots & m_{k-1} \\ m_1 & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & m_{2q-3} \\ m_{k-1} & \cdots & m_{2q-3} & m_{2q-2} \end{bmatrix} \begin{bmatrix} a_q \\ a_{q-1} \\ \vdots \\ a_1 \end{bmatrix} = \begin{bmatrix} m_q \\ m_{q+1} \\ \vdots \\ m_{2q-1} \end{bmatrix}$$

$$m_i = c^T A^{-i} b \approx \lambda_{A_{\max}} m_{i-1}$$

Columns become linearly dependent for large q!

Rational Transfer Function Representation

Problems with explicit fitting methods

- Linear Systems for fitting ill-conditioned
  - Need specialized algorithms which avoid explicit fitting matrix construction
- Rational function must be converted to state-space
  - Needed by most simulation tools
  - Requires root finding procedure, very sensitive to parameter variation

Summary

- Need For Model Reduction
  - Circuits, MEMS, Optics, Jet Engines
- Simple Example Problem
  - Heat Conducting bar example
- Steady-State Case (linear and nonlinear)
- Dynamic Linear Case
  - Truncating Eigenmodes
    - Loss correct steady state values
    - Select modes to delete
  - Rational Function Fitting
    - Generates ill-conditioned matrices