Introduction to Simulation - Lecture 25

Model-Order Reduction II

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Thanks to Luca Daniel, Jing Li, Joel Phillips, Michal Rewienski,

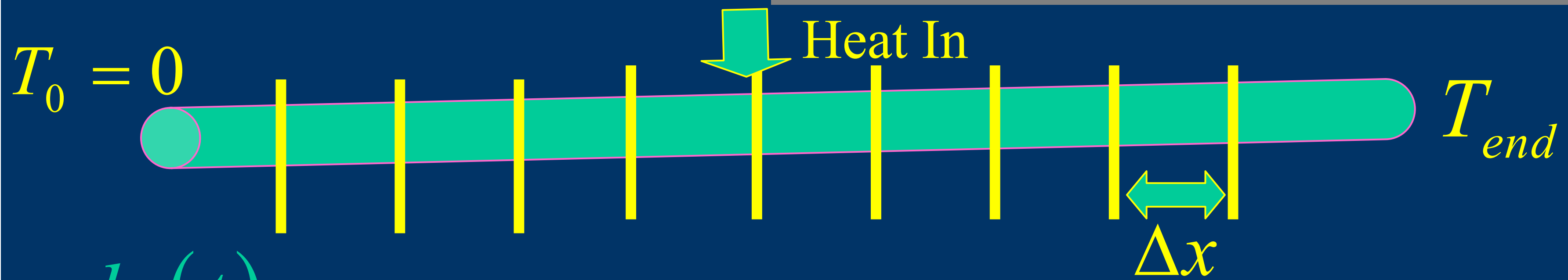
MOR Outline

- Dynamic Linear Case
  - Rational Functions
  - Projection Framework
  - Krylov Methods
- Hankel Reduction and TBR
  - Mention a few issues

Demonstration Example

Heat Conducting Bar

State-Space Description



$$\frac{dx(t)}{dt} = \underbrace{A}_{NxN} x(t) + \underbrace{b}_{Nx1} \underbrace{u(t)}_{\substack{scalar \\ input}}$$

$$\underbrace{y(t)}_{\substack{scalar \\ output}} = \underbrace{c}_{Nx1}{}^{T} x(t)$$

Given the right scaling

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} h(x_1) \\ h(x_2) \\ \vdots \\ \vdots \\ h(x_N) \end{bmatrix} \quad c = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

State-Space Description

Dynamic Linear case

- Original Dynamical System - Single Input/Output

$$\frac{dx(t)}{dt} = \underbrace{A}_{NxN} x(t) + \underbrace{b}_{Nx1} \underbrace{u(t)}_{\substack{scalar\\input}} \qquad \underbrace{y(t)}_{\substack{scalar\\output}} = \underbrace{c}_{Nx1}{}^{T} x(t)$$

- Reduced Dynamical System

$$\frac{dx_r(t)}{dt} = \underbrace{A_r}_{qxq} x(t) + \underbrace{b_r}_{qx1} \underbrace{u(t)}_{\substack{scalar\\input}} \qquad \underbrace{y_r(t)}_{\substack{scalar\\output}} = \underbrace{c_r}_{qx1}{}^{T} x_r(t)$$

- q << N, but input/output behavior preserved

An Aside on Transfer Functions – Laplace Transform

Consider an ODE: $\frac{dx(t)}{dt} = Ax(t) + bu(t)$

Bilateral Laplace Transform: $X(s) = \int_{-\infty}^{\infty} \mathrm{x}(\mathrm{t}) e^{-st} dt$

Key Transform Property: $sX(s) = \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$

Rewrite the ODE in transformed variables

$$sX(s) = AX(s) + bU(s) \quad Y(s) = c^T X(s)$$

$$\Rightarrow Y(s) = \underbrace{c^T (sI - A)^{-1} b}_{H(s)} U(s)$$

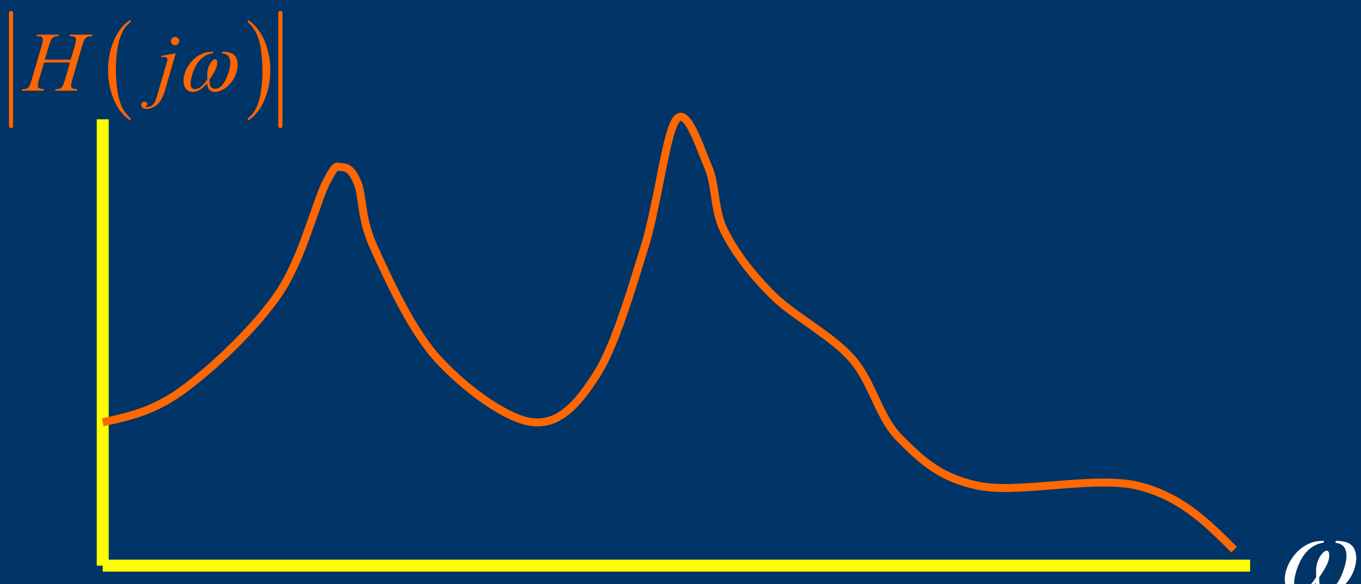
$H(s)$ ← Transfer Function

An Aside on Transfer Functions – Meaning of H(s)

For Stable Systems, H(*jw*) is the frequency response

$$\text{If } u(t) = e^{j\omega t}$$ ←Sinusoid

$$\text{then } y(t) = H(j\omega)e^{j\omega t}$$ Sinusoid with shifted phase and amplitude



An Aside on Transfer Functions – EigenAnalysis

Transfer Function

$$H(s) = c^T (sI - A)^{-1} b$$

Apply Eigendecomposition

$$H(s) = c^T E (sI - \lambda)^{-1} E^{-1} b$$

$$= \tilde{c}^T \begin{bmatrix} \frac{1}{s-\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{s-\lambda_N} \end{bmatrix} \tilde{b} \Rightarrow H(s) = \sum_{i=1}^{N} \frac{\tilde{c}_i \tilde{b}_i}{s - \lambda_i}$$

Rational Transfer Function Representation

Original System Transfer Function

$$H(s) = \frac{\tilde{c}_1 \tilde{b}_1}{(s-\lambda_1)} + \cdots + \frac{\tilde{c}_N \tilde{b}_N}{(s-\lambda_N)} = \underbrace{\frac{b_0 + b_1 s + \cdots + b_{N-1} s^{N-1}}{1 + a_1 s + \cdots + a_N s^N}}_{\text{Rational Function}}$$

Reduced Model Transfer Function

$$H_r(s) = \underbrace{\frac{b_0^r + b_1^r s + \cdots + b_{q-1}^r s^{q-1}}{1 + a_1^r s + \cdots + a_q^r s^q}}_{\text{Lower Order Rational Function}}$$

Model Reduction = Find a low order rational function matching H(s)

Dynamic Linear Case

Rational Transfer Function Representation

Degrees of Freedom

Reduced Model Dynamical System

$$\frac{dx_r(t)}{dt} = \underbrace{A_r}_{qxq} x(t) + \underbrace{b_r}_{qx1} \underbrace{u(t)}_{\substack{scalar\\input}} \qquad \underbrace{y_r(t)}_{\substack{scalar\\output}} = \underbrace{c_r}_{qx1}{}^T x_r(t)$$

$2q + q^2$ coefficients

Reduced Model Transfer Function

$$H_r(s) = \frac{b_0^r + b_1^r s + \cdots + b_{q-1}^r s^{q-1}}{1 + a_1^r s + \cdots + a_q^r s^q}$$

$2q$ coefficients

Dynamic Linear Case

Rational Transfer Function Representation

Variable Changes Do not change transfer functions

Reduced Model Transfer Function

$$\frac{dx_r(t)}{dt} = A_r x(t) + b_r u(t) \qquad y_r(t) = c_r^T x_r(t)$$

$$\Rightarrow H(s) = c_r^T (sI - A_r)^{-1} b_r$$

Similarity (x = Sw) Transformed Transfer Function

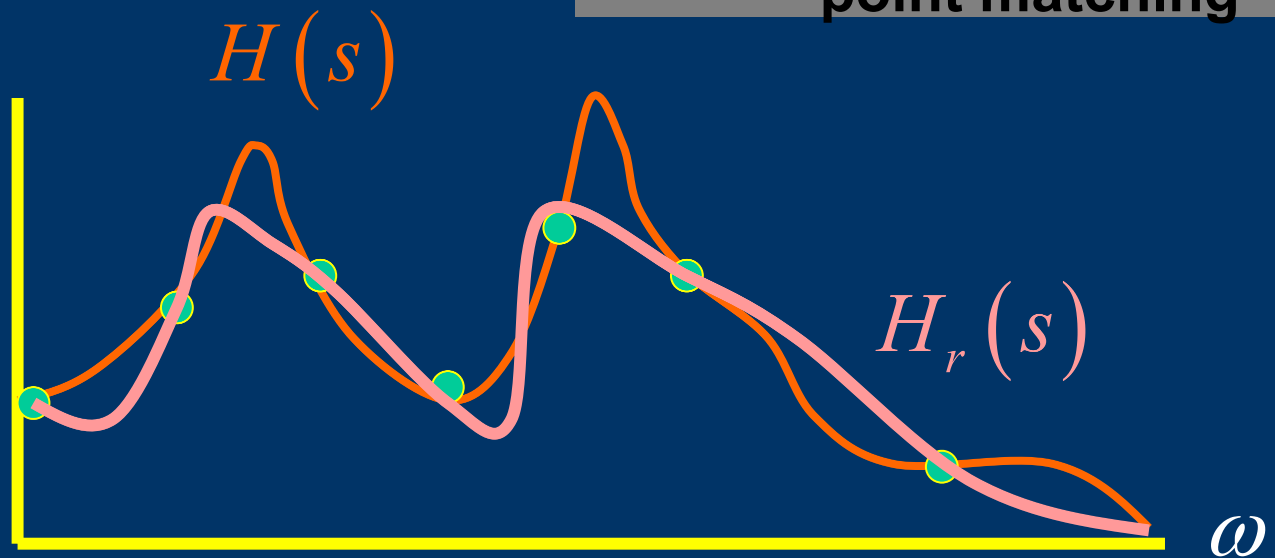
$$\frac{dw_r(t)}{dt} = S^{-1} A_r S w(t) + S^{-1} b_r u(t) \qquad y_r(t) = c_r^T S w_r(t)$$

$$\Rightarrow H(s) = c_r^T S \left(sI - S^{-1} A_r S\right)^{-1} S^{-1} b_r = c_r^T (sI - A_r)^{-1} b_r$$

Many Dynamical Systems have the same transfer function!!

Rational Transfer Function Representation

Rational Function Fitting by point matching



- Can match 2q points
- cross multiplying generates a linear system

For i = 1 to 2q

$$\left(1 + a_1^r s_i + \cdots + a_q^r s_i^{\,q}\right) H(s_i) - \left(b_0^r + b_1^r s + \cdots + b_{q-1}^r s^{q-1}\right) = 0$$

Dynamic Linear Case

Rational Transfer Function Representation

Point Matching Matrix can be ill-conditioned

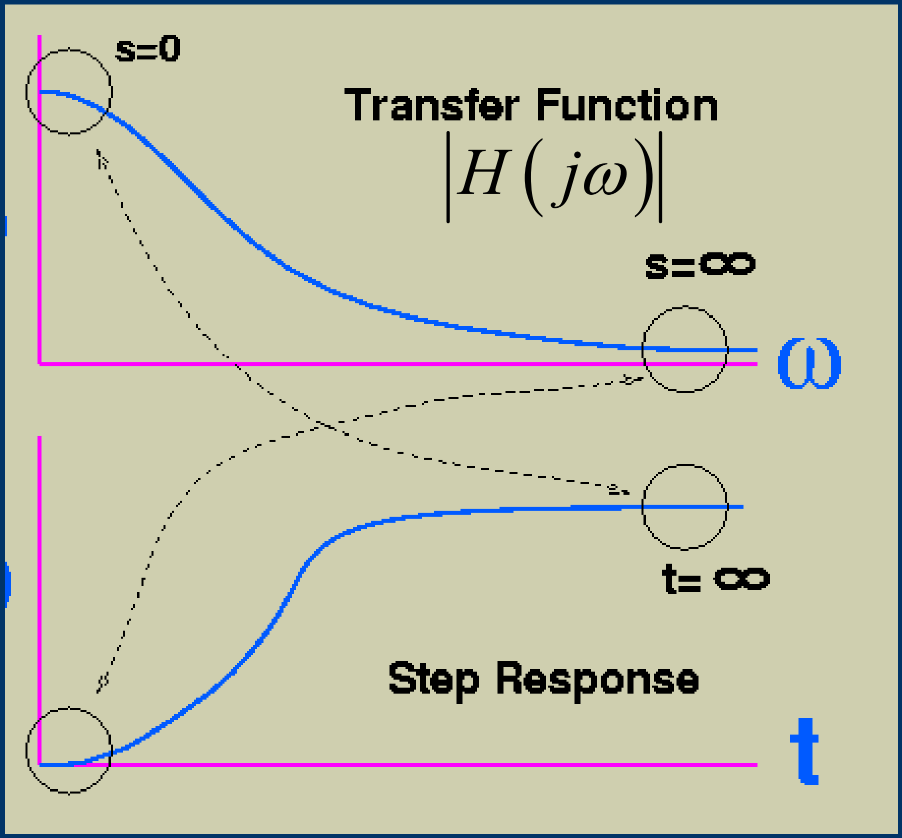
$$\begin{bmatrix} s_1 H(s_1) & s_1^2 H(s_1) & \cdots & -s_1^{q-1} \\ \vdots & \vdots & \cdots & -s_2^{q-1} \\ \vdots & \vdots & \cdots & \vdots \\ s_{2q} H(s_{2q}) & s_{2q}^2 H(s_{2q}) & \cdots & -s_{2q}^{q-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ b_{q-1} \end{bmatrix} = \begin{bmatrix} H(s_1) \\ H(s_2) \\ \vdots \\ H(s_{2q}) \end{bmatrix}$$

- Columns contain progressively higher powers of the test frequencies
- Must orthogonalize columns during construction

Dynamic Linear Case

Rational Transfer Function Representation

Importance of Fitting at low frequency



Correct Steady State behavior requires accurate match at low frequencies

Original System Transfer Function Moments

$$H(s) = c^T(sI - A)^{-1}b = -c^T \underbrace{(I - sA^{-1})^{-1}}_{\text{Taylor Expand with respect to s}} A^{-1}b$$

$$H(s) = -c^T(I - sA^{-1})^{-1}A^{-1}b = \sum_{k=0}^{\infty} c^T A^{-(k+1)} b s^k$$

$$H(s) = \underbrace{c^T A^{-1} b}_{m_0} + \underbrace{c^T A^{-2} b}_{m_1} s + \underbrace{c^T A^{-3} b}_{m_2} s^2 + \cdots = \sum_{k=0}^{\infty} m_k s^k$$

Moments

Rational Transfer Function Representation

Moment Matching for accurate low frequency behavior

Reduced Model Matches Original Systems Moments

$$H_r(s) = \frac{b_0^r + b_1^r s + \cdots + b_{q-1}^r s^{q-1}}{1 + a_1^r s + \cdots + a_q^r s^q} = m_0 + m_1 s + \cdots + m_{2q-1} s + \cdots$$

Cross-Multiplying and Matching Terms

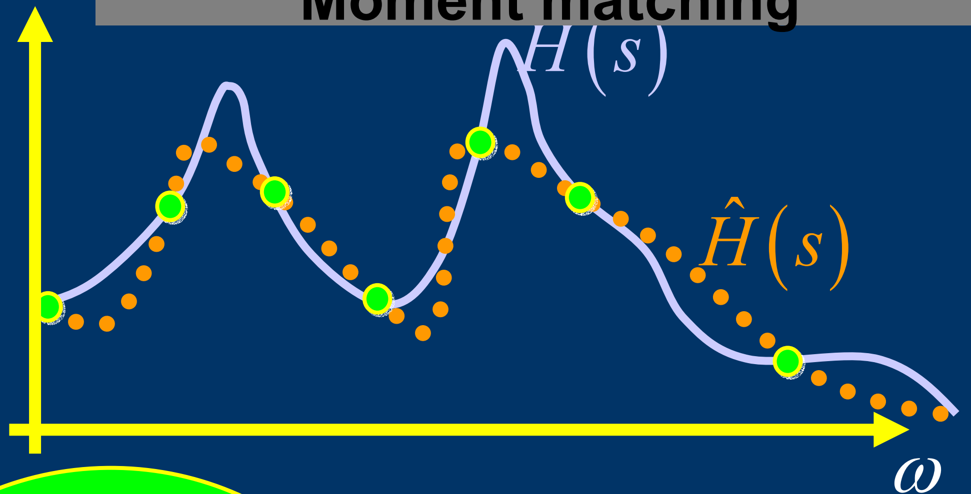
$$\begin{bmatrix} m_0 & m_1 & \cdots & m_{k-1} \\ m_1 & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & m_{2q-3} \\ m_{k-1} & \cdots & m_{2q-3} & m_{2q-2} \end{bmatrix} \begin{bmatrix} a_q \\ a_{q-1} \\ \vdots \\ a_1 \end{bmatrix} = \begin{bmatrix} m_q \\ m_{q+1} \\ \vdots \\ m_{2q-1} \end{bmatrix}$$

Dynamic Linear Case

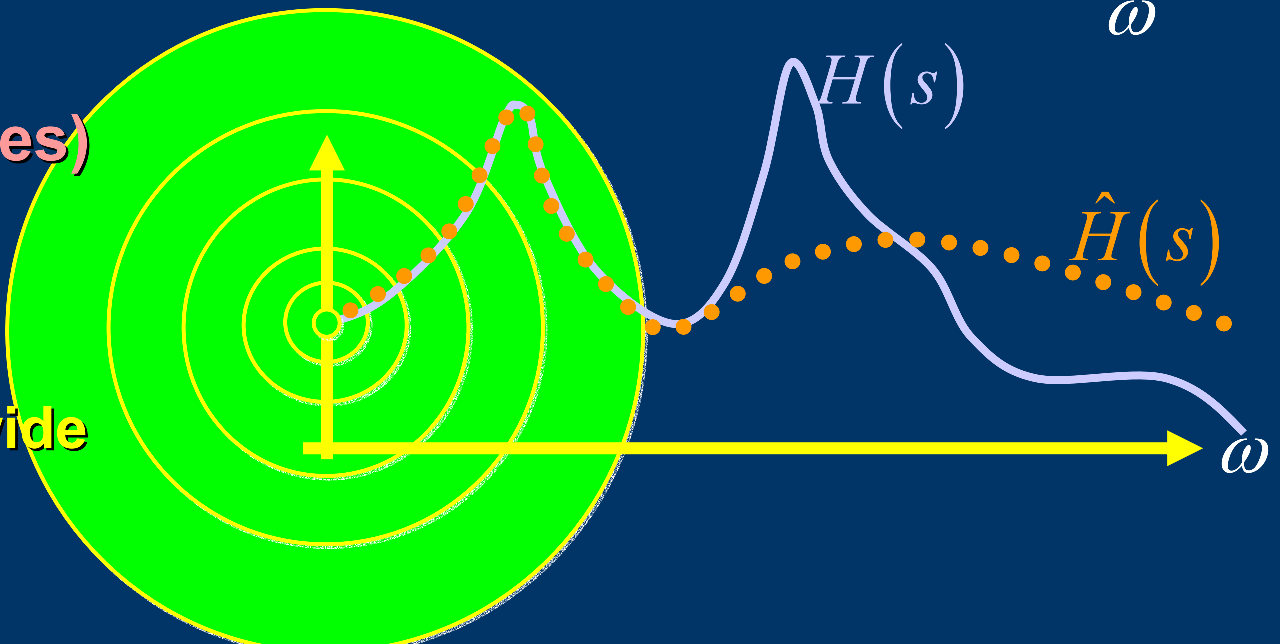
Rational Transfer Function Representation

Point Matching Versus Moment matching

Point matching:
can be very inaccurate in between points



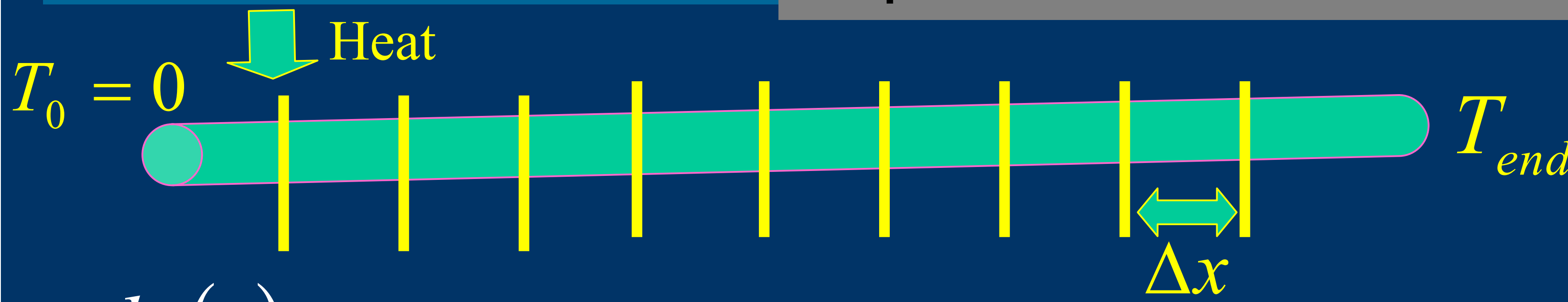
Moment (derivatives) matching:
accurate around expansion point, but inaccurate on wide frequency band



Heat Conducting Bar

Dynamic Linear Case

Heat applied at one end, temperature measured at the other



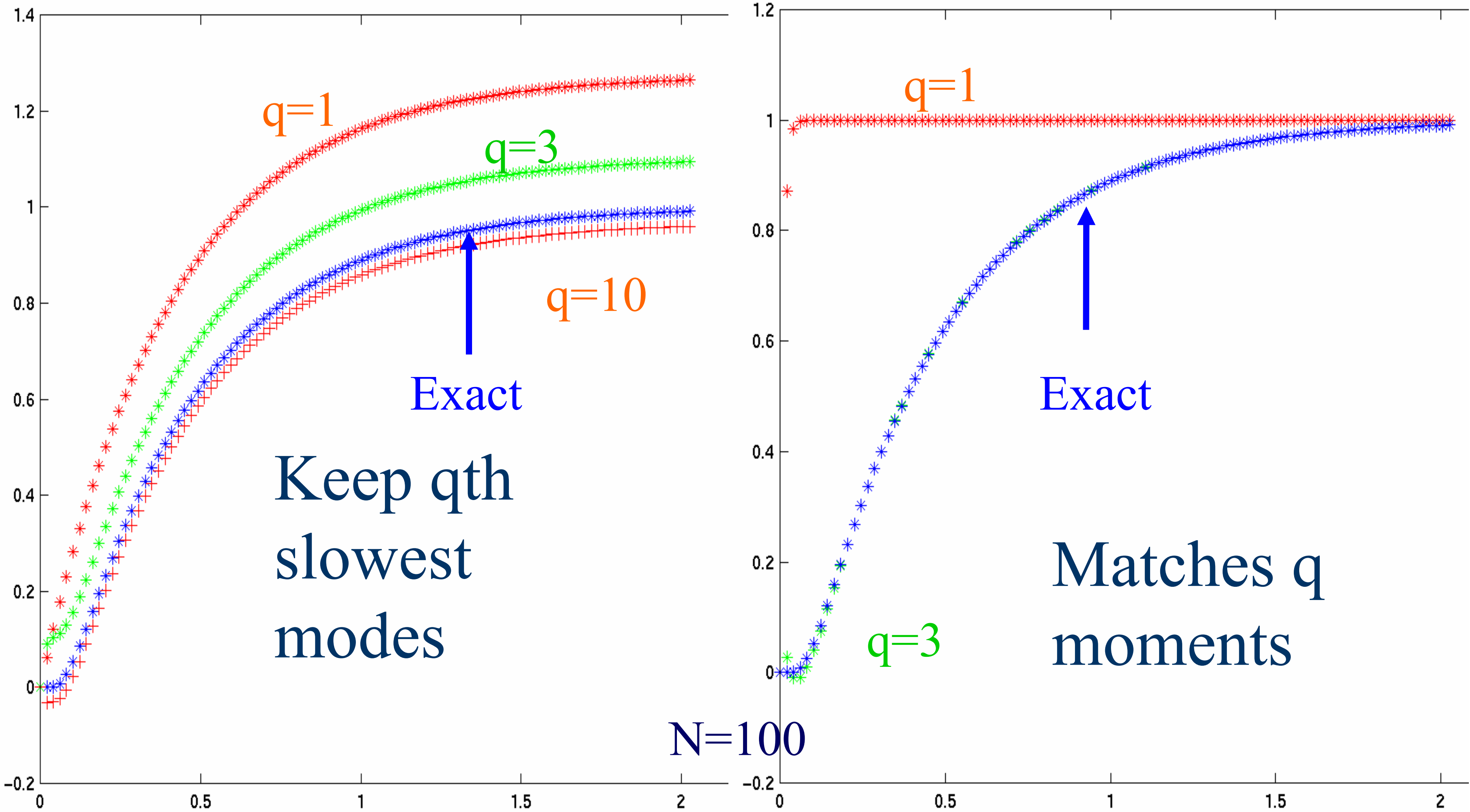
$$\frac{dx(t)}{dt} = \underbrace{A}_{NxN} x(t) + \underbrace{b}_{Nx1} \underbrace{u(t)}_{\substack{scalar \\ input}} \qquad \underbrace{y(t)}_{\substack{scalar \\ output}} = \underbrace{c}_{Nx1}{}^T x(t)$$

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Dynamic Linear Case

Keeping Eigenmodes versus matching moments

Heat Flow Results



Dynamic Linear Case

Rational Transfer Function Representation

Explicit Moment Matching Problem

System of equations extremely ill-conditioned

$$\begin{bmatrix} m_0 & m_1 & \cdots & m_{k-1} \\ m_1 & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & m_{2q-3} \\ m_{k-1} & \cdots & m_{2q-3} & m_{2q-2} \end{bmatrix} \begin{bmatrix} a_q \\ a_{q-1} \\ \vdots \\ a_1 \end{bmatrix} = \begin{bmatrix} m_q \\ m_{q+1} \\ \vdots \\ m_{2q-1} \end{bmatrix}$$

$$m_i = c^T A^{-i} b \approx \lambda_{A_{\max}} m_{i-1}$$

Columns become linearly dependent for large q!

Rational Transfer Function Representation

Problems with explicit fitting methods

- Linear Systems for fitting ill-conditioned
  - Need specialized algorithms which avoid explicit fitting matrix construction
- Rational function must be converted to state-space
  - Needed by most simulation tools
  - Requires root finding procedure, very sensitive to parameter variation

Dynamic Linear Case

Dimension Reducing Change of Variables

$$\begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x_N \end{bmatrix} \approx \underbrace{\begin{bmatrix} \uparrow & \cdots & \uparrow \\ \uparrow & \cdots & \uparrow \\ \uparrow & \cdots & \uparrow \\ \vec{u}_1 & \cdots & \vec{u}_q \\ \downarrow & \cdots & \downarrow \\ \downarrow & \cdots & \downarrow \\ \downarrow & \cdots & \downarrow \end{bmatrix}}_{U_q} \begin{bmatrix} x_{r_1} \\ \vdots \\ x_{r_q} \end{bmatrix}$$

Projection Framework

Dynamic Linear Case

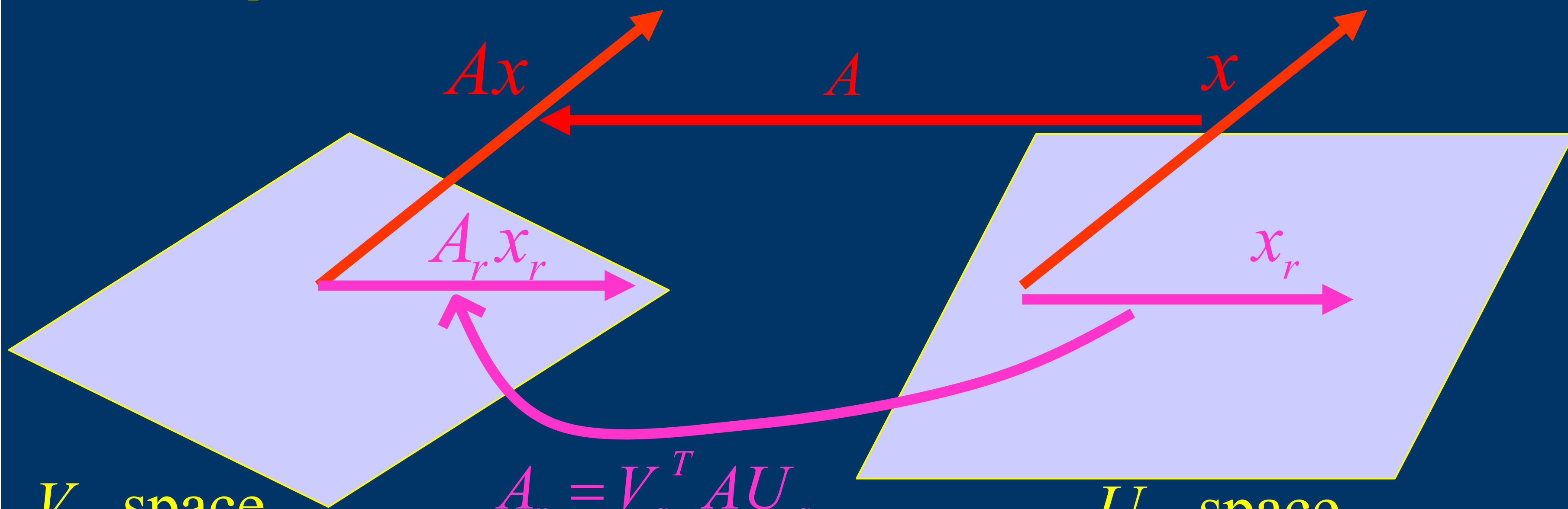
$$\dot{x} = Ax + bu,\ \ y = c^T x \Rightarrow \dot{x}_r = A_r x_r + b_r u,\ \ y_r = c_r^T x$$

<u>Equation Testing</u>

$$V_q^T A x \approx A_r x_r$$

<u>Change of variables</u>

$$x \approx U_q x_r$$



V_q space $\quad A_r = V_q^T A U_q \quad$ U_q space

Galerkin $\rightarrow V_q$ space $= U_q$ space

Projection Framework

Dynamic Linear Case

Assumed Biorthogonal Relationship between V and U

- Original System

$$\dot{x} = Ax + bu, \;\; y = c^T x$$

- Substitute $x = U_q x_r$

$$U_q \dot{x}_r = AU_q x_r + b_r u, \;\; y_r = c^T U_q x_r$$

- Test by multiplying by V

$$V_q^T U_q \dot{x}_r = V_q^T AU_q x_r + V_q^T b_r u, \;\; y_r = c^T U_q x_r$$

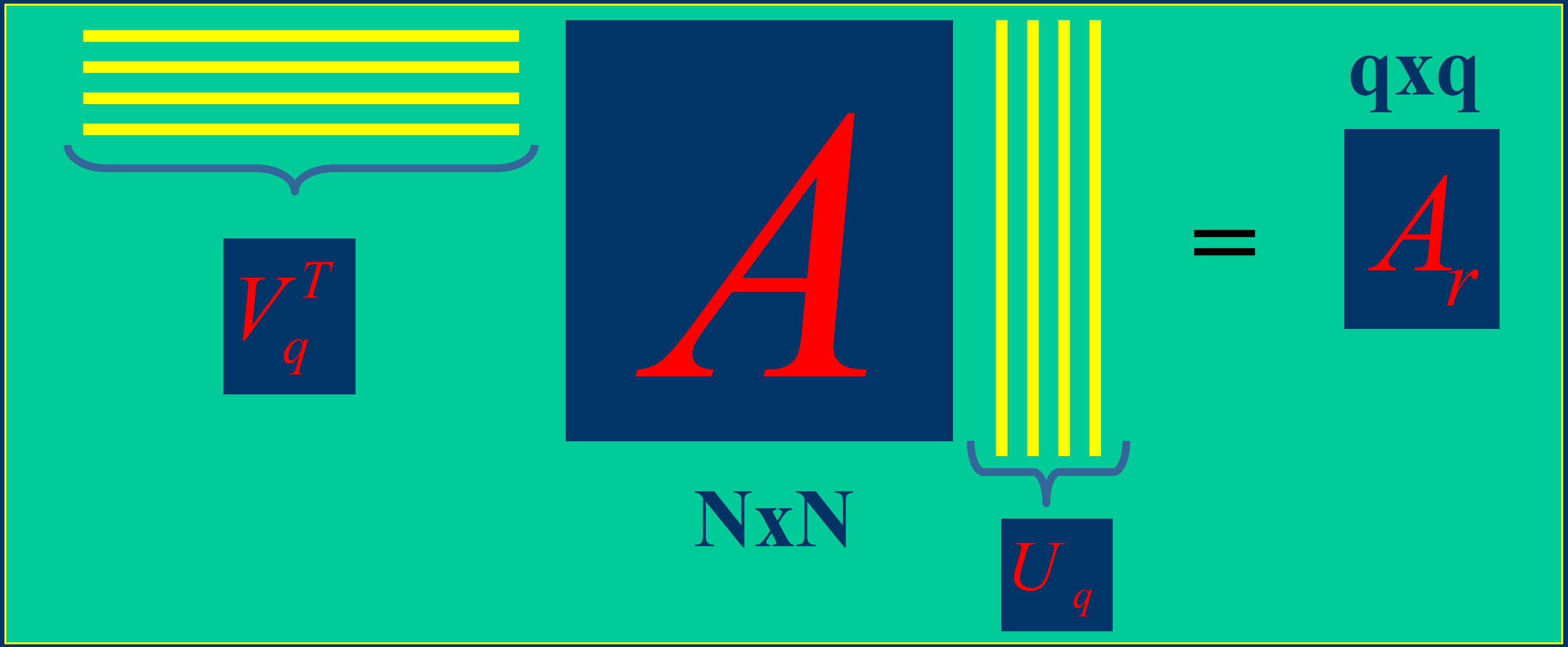
- Previous Slide Assumed that V and U biorthogonal

$$V_q^T U_q = I \Rightarrow \dot{x}_r = A_r x_r + b_r u, \;\; y_r = c_r^T x_r$$

Dynamic Linear Case

Projection Framework

Forming the reduced system matrix



- No explicit A need, Only Matrix-vector products

  For each column of U_q

  Multiply by A, then dot result with columns of V_q

Dynamic Linear Case

Projection Framework

V=U can preserve definiteness properties

- Original System

$$\frac{dx}{dt} = Ax + bu, \;\; y = c^T x$$

- Reduced System

$$\frac{dx_r}{dt} = \underbrace{U_q^T A U_q}_{A_r} x_r + U_q^T b_r u, \;\; y_r = c^T U_q x_r$$

- If A is (+ or -) definite, so is A_r
  - Preserves stability in the definite case
  - Can also preserve passivity

- Use Eigenvectors
- Use Time Series Data
  - Compute $x(t_0), x(t_1), \cdots, x(t_k)$
  - Use the SVD to pick $q < k$ important vectors
- Use Frequency Domain Data
  - Compute $X(s_1), X(s_2), \cdots, X(s_k)$
  - Use the SVD to pick $q < k$ important vectors
- Use Krylov Subspace Vectors?
- Use Singular Vectors of System Grammians?

Aside on Krylov Subspaces - Definition

The order k Krylov subspace generated from matrix A and vector b is defined as

$$\kappa_k(A,b) \equiv span\{b, Ab, A^2b, \ldots A^{k-1}b\}$$

Dynamic Linear Case

Projection Framework

Moment Matching Theorem

If

$$span\left\{\vec{u}_1,\ldots,\vec{u}_q\right\} \supseteq \bigcup_{j=1}^{J} \kappa_{k_j^b}\left(\left(A-s_jI\right)^{-1},\left(A-s_jI\right)^{-1}b\right)$$

And

$$span\left\{\vec{v}_1,\ldots,\vec{v}_q\right\} \supseteq \bigcup_{j=1}^{J} \kappa_{k_j^c}\left(\left(A-s_jI\right)^{-T},\left(A-s_jI\right)^{-T}c\right)$$

Then

$$\frac{\partial^l H\left(s_j\right)}{\partial s^l} = \frac{\partial^l H_r\left(s_j\right)}{\partial s^l} \quad \text{for } l = 0,\ldots,k_j^b + k_j^c - 1$$

Dynamic Linear Case

Projection Framework

Special Case Moment Matching Theorem

If U and V are such that

$$U = V = \left\{ \vec{u}_1, \ldots, \vec{u}_q \right\} \text{ and } U^T U = I$$

$$span\left\{ \vec{u}_1, \ldots, \vec{u}_q \right\} = span\left\{ A^{-1}b, A^{-2}b, \ldots, A^{-q}b \right\}$$

Then the first q moments of reduced system match

$$H(s) = -c^T \left(I - sA^{-1} \right)^{-1} A^{-1}b = \sum_{k=0}^{\infty} c^T A^{-(k+1)} b s^k$$

$$H_r(s) = -c_r^T \left(I - sA_r^{-1} \right)^{-1} A_r^{-1} b_r = \sum_{k=0}^{\infty} c_r^T A_r^{-(k+1)} b_r s^k$$

$$c^T A^{-(k+1)} b = c^T U_q \left(U_q^T A U_q \right)^{-(k+1)} U_q^T b \qquad k = \left\{ 0, \ldots, q-1 \right\}$$

A Projection Alternative

First Invert A before applying reduction

$$A^{-1}\dot{x} = x + A^{-1}bu,\ \ y = c^T x \ \Rightarrow H(s) = \underbrace{-c^T\left(I - sA^{-1}\right)A^{-1}b}_{\text{Unchanged Tranfer function}}$$

Form reduced model by projecting inverse of A

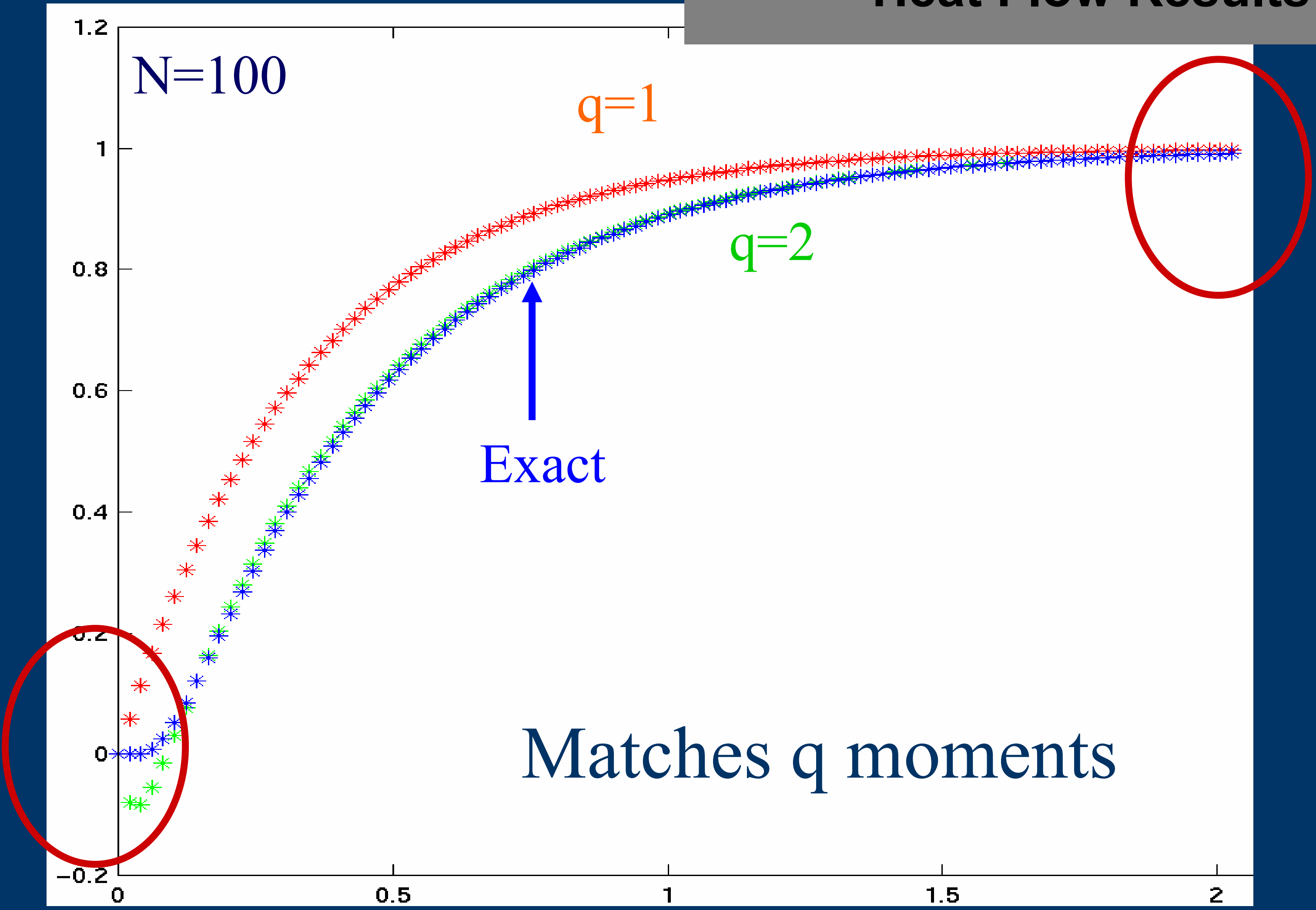
$$\underbrace{V_q^T A^{-1} U_q}_{A_r^{-1}}\dot{x}_r = x_r + V_q^T A^{-1}bu,\ \ y_r = c_r^{\ T} x_r$$

The Projection Theorem Still Holds!!

Model-Order Reduction

Projection Alternative

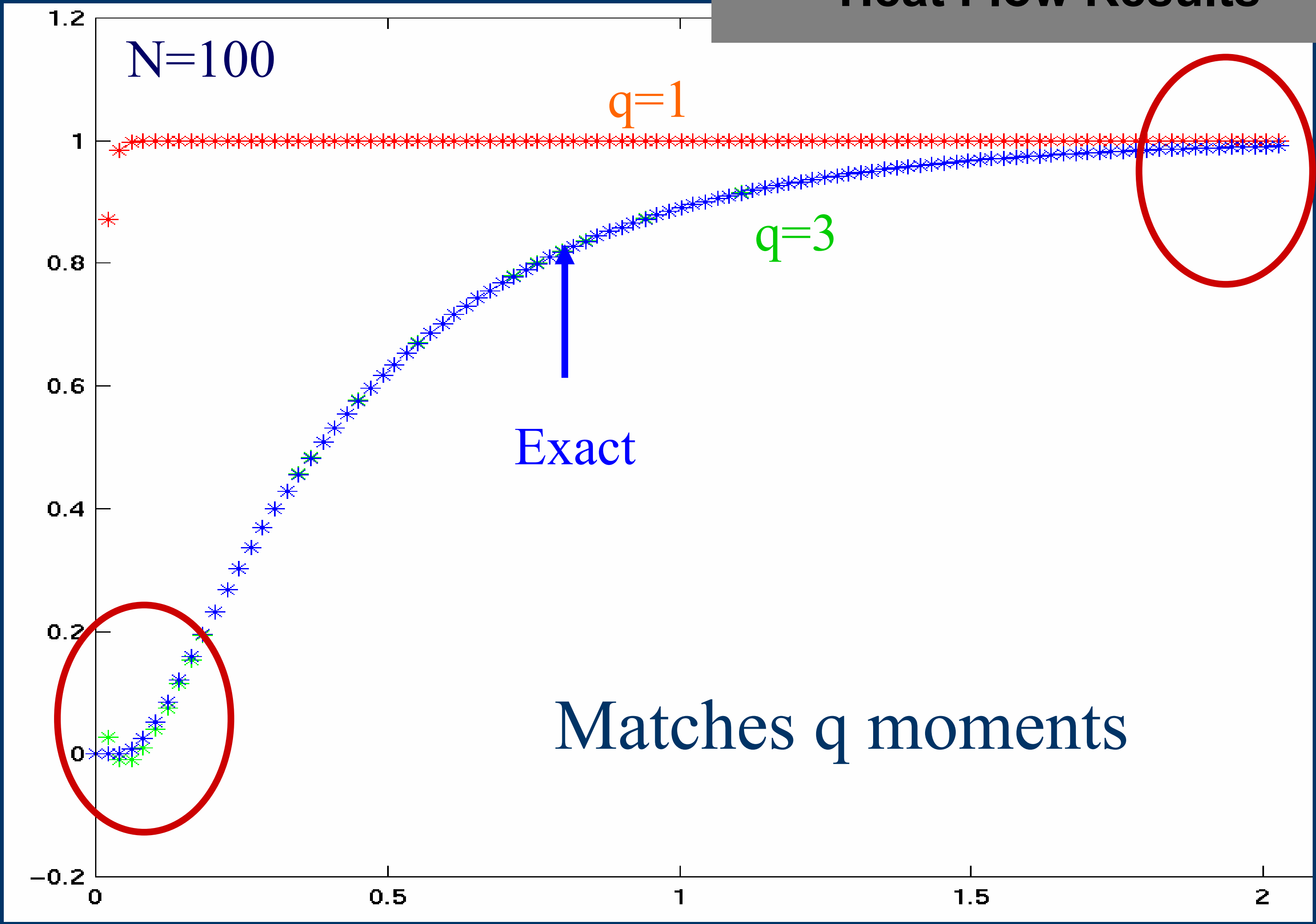
Heat Flow Results



Model-Order Reduction

Noninverse Formulation

Heat Flow Results



Dynamic Linear Case

Computing U

Need for Orthogonalization

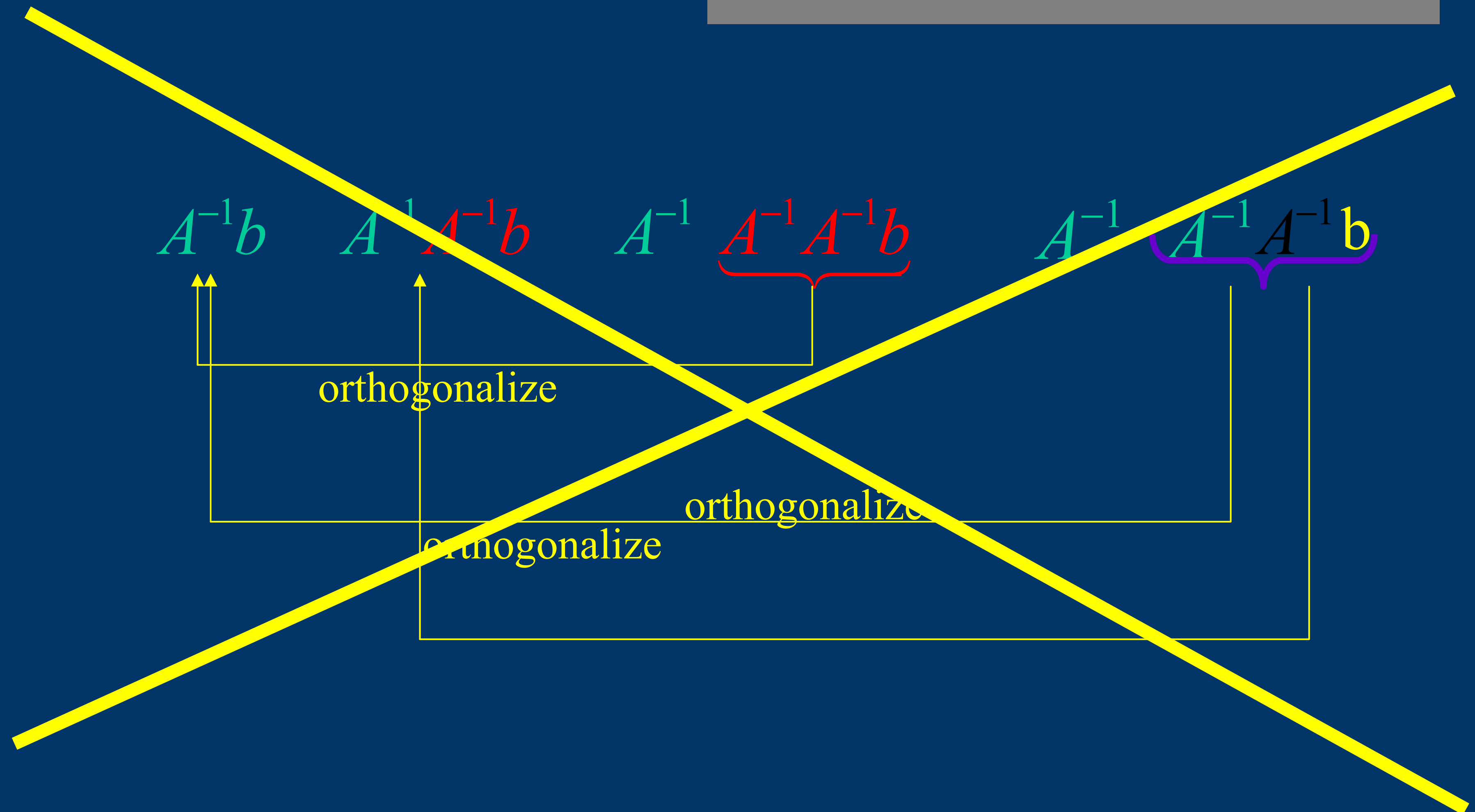
$\left\{ A^{-1}b, A^{-2}b, \ldots, A^{-k}b \right\}$ can not be computed directly

Vectors will line up with dominant eigenspace!

Dynamic Linear Case

Computing U

Need for Orthogonalization



- Only requires solves with A and vector inner products

Dynamic Linear Case

Computing orthogonal U

Arnoldi Algorithm

$$\vec{u}_1 = A^{-1}b / \left\| A^{-1}b \right\|$$

For i = 1 to q

$$\vec{u}_{i+1} = A^{-1}\vec{u}_i$$

Generates q+1 vectors!

For j = 1 to i

$$\vec{u}_{i+1} \leftarrow \vec{u}_{i+1} - \underbrace{\left(\vec{u}_{i+1}^T \vec{u}_j \right)}_{H_{ji}} \vec{u}_j$$

Orthogonalize New Vector

$$\vec{u}_{i+1} \leftarrow \frac{1}{\underbrace{\left\| \vec{u}_{i+1} \right\|}_{H_{i+1,i}}} \vec{u}_{i+1}$$

Normalize

Dynamic Linear Case

Computing U

Arnoldi Identity

$$A^{-1}\underbrace{\begin{bmatrix} \uparrow & \cdots & \uparrow \\ \uparrow & \cdots & \uparrow \\ \uparrow & \cdots & \uparrow \\ \vec{u}_1 & \cdots & \vec{u}_q \\ \downarrow & \cdots & \downarrow \\ \downarrow & \cdots & \downarrow \\ \downarrow & \cdots & \downarrow \end{bmatrix}}_{U_q} = \underbrace{\begin{bmatrix} \uparrow & \cdots & \uparrow \\ \uparrow & \cdots & \uparrow \\ \uparrow & \cdots & \uparrow \\ \vec{u}_1 & \cdots & \vec{u}_q \\ \downarrow & \cdots & \downarrow \\ \downarrow & \cdots & \downarrow \\ \downarrow & \cdots & \downarrow \end{bmatrix}}_{U_q} \underbrace{\begin{bmatrix} H_{11} & \cdots & \cdots & \cdots & H_{1q} \\ H_{21} & H_{22} & \cdots & \cdots & \vdots \\ 0 & H_{32} & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & H_{q,q-1} & H_{qq} \end{bmatrix}}_{H_q}$$

Rank 1 matrix → $+ \; H_{q+1,q}\vec{u}_{q+1}e_q^T$

Computing U

Arnoldi Identities Continued

Multiplying U by the inverse of A yeilds

$$A^{-1}U_q = U_q H_q + H_{q+1,q}\vec{u}_{q+1}e_q^T$$

Multiplying by the transpose of U

$$U_q^T A^{-1}U_q = U_q^T U_q H_q + U_q^T H_{q+1,q}\vec{u}_{q+1}e_q^T$$

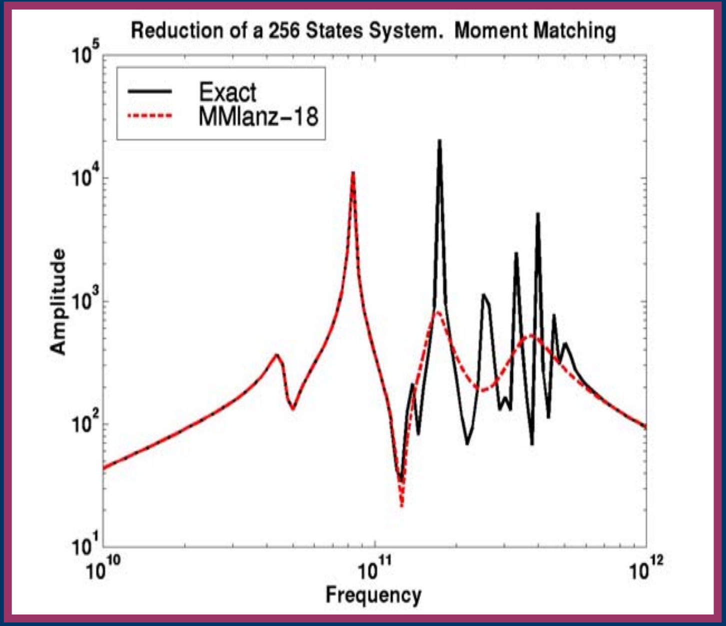
By orthogonality

$$U_q^T A^{-1}U_q = H_q = A_r^{-1}$$

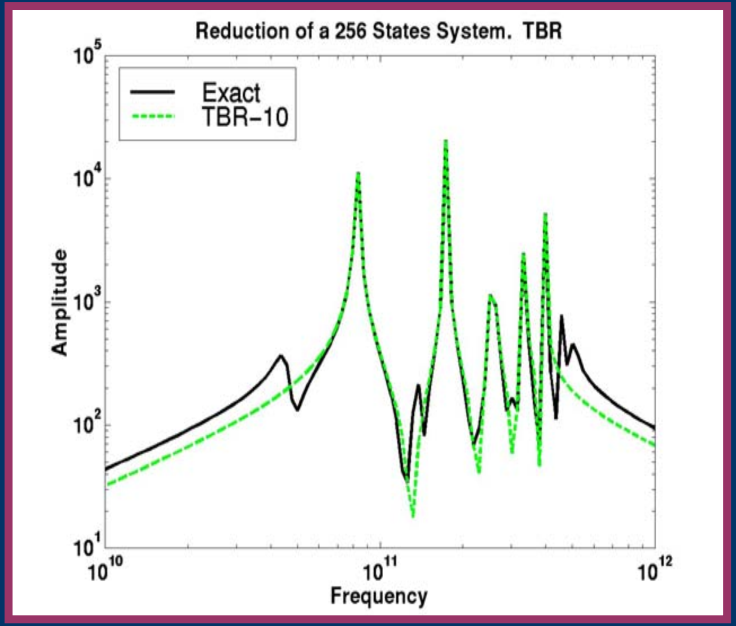
← The Projection Alternative Reduced Model

Two Existing Approaches

- Moment Matching
  - Accurate over a narrow band.
    - Matching function value and derivatives.
  - Cheap:
    - O(n) if A is very sparse.



- Truncated Balanced Realization
  - Wide-band accuracy.
    - Does not follow all details.
  - Theoretical error bound.
  - Expensive: $O(n^3)$



Reminder about Eigenanalysis

Transfer Function $H(s) = c^T (sI - A)^{-1} b$

Apply Eigendecomposition

$$H(s) = c^T E (sI - \lambda)^{-1} E^{-1} b$$

$$= \tilde{c}^T \begin{bmatrix} \frac{1}{s-\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{s-\lambda_N} \end{bmatrix} \tilde{b} \Rightarrow H(s) = \sum_{i=1}^{N} \frac{\tilde{c}_i \tilde{b}_i}{s - \lambda_i}$$

Should keep controllable and observable "modes", but should they be the eigenmodes?

Truncated Balanced Realization: Non-symmetric Systems

- 1. Calculate controllability gramian, P, and observability gramian, Q, by solving two Lyapunov equations,

$$\mathbf{AP} + \mathbf{PA}^T + \mathbf{bb}^T = \mathbf{0},$$

$$\mathbf{A}^T\mathbf{Q} + \mathbf{QA} + \mathbf{c}^T\mathbf{c} = \mathbf{0}.$$

- 2. If have Cholesky factors, $\mathbf{P} = \mathbf{Z}_b\mathbf{Z}_b^T, \mathbf{Q} = \mathbf{Z}_c\mathbf{Z}_c^T$
- 3. Projection: a. $\mathbf{U}_c\mathbf{D}\mathbf{U}_b^T = \mathbf{Z}_c^T\mathbf{Z}_b,$

b. $\mathbf{S}_b = \mathbf{Z}_b\mathbf{U}_b(:,1:k)\mathbf{D}^{-\frac{1}{2}}(1:k,1:k),$

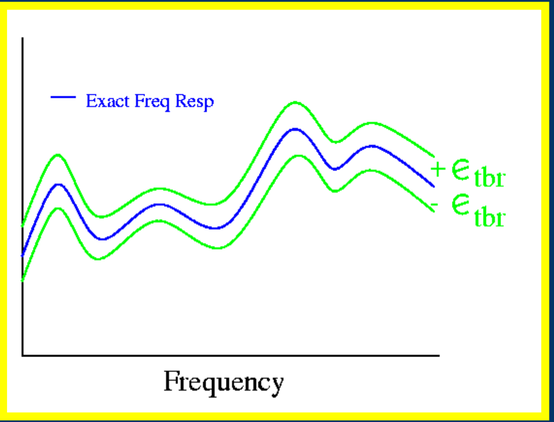
$\mathbf{S}_c = \mathbf{Z}_c\mathbf{U}_c(:,1:k)\mathbf{D}^{-\frac{1}{2}}(1:k,1:k),$

c. $\mathbf{A}_{tbr}^k = \mathbf{S}_c^T\mathbf{A}\mathbf{S}_b, \quad \tilde{\mathbf{b}}_{tbr}^k = \mathbf{S}_c^T\tilde{\mathbf{b}}, \quad \mathbf{c}_{tbr}^k = \mathbf{c}\mathbf{S}_b$

Properties of the TBR Reduction

- Globally accurate reduced model.
- Maximum frequency domain error is bounded by,

$$\left\| G(jw) - G_{tbr}^{k}(jw) \right\|_{L^{\infty}}$$
$$\leq 2(\sigma_{k+1} + \sigma_{k+2} + \cdots + \sigma_{n}) \equiv \varepsilon_{tbr}.$$



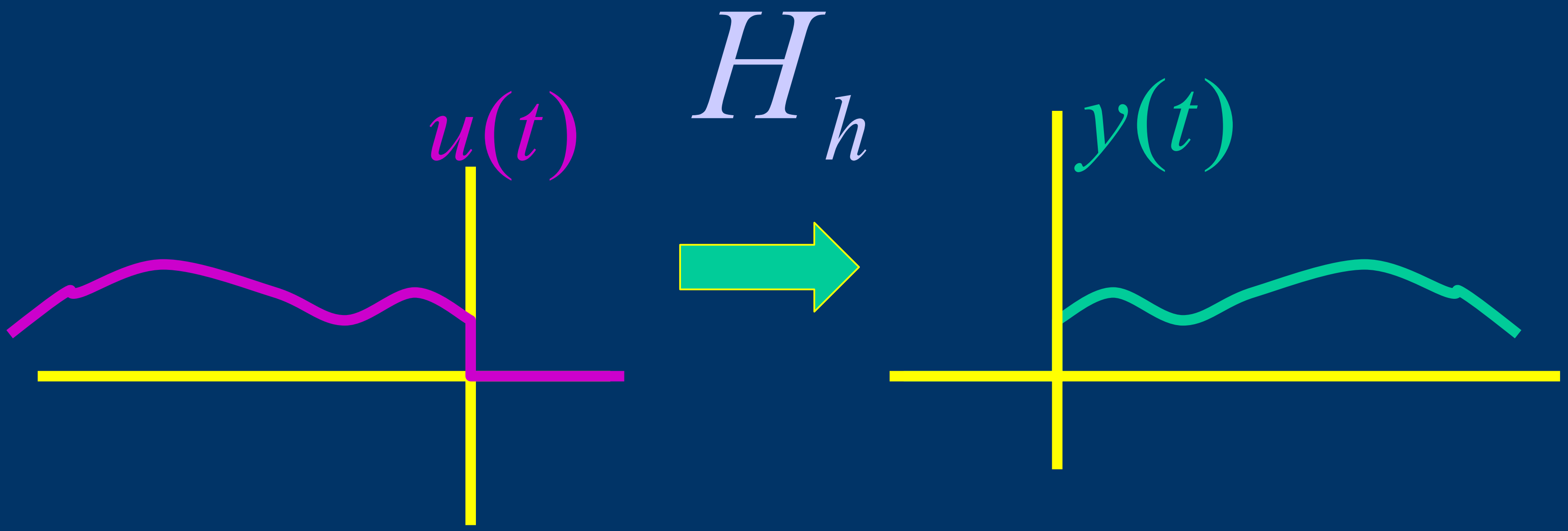
- Guaranteed stable.
- Expensive:
  - Lyapunov equation solve: $O(n^3)$.
  - Singular value decomposition: $O(n^3)$.

Solving Lyapunov Equations

- How to find (approximate) P, Q, or just as good, their factors Z_b, Z_c, efficiently?
- No expensive operations on A: no matrix decompositions.
- Cheap operations: matrix-vector products and solves.
- Low rank approximations.
  - Z_b, and Z_c have only a few columns.
- Recent Approach Cholesky-Free ADI methods

Reduction Based on Hankel Operators

Hankel Operator Maps Past Inputs to Future Outputs



The Hankel Operator has an SVD

- The Singular Values of the Hankel Operator

$$\sigma_i(H) = \sqrt{\lambda_i(PQ)}$$

- P, Q are Observability and Controllability Grammians
  - P and Q are NxN matrices
  - Hankel Operator has a finite set of singular values
- Reduction by ignoring small singular values
  - Just like with any matrix

Summary

- Dynamic Linear Case
  - Rational Functions
  - Projection Framework
  - Krylov Methods
- TBR and Hankel Reduction
  - Optimal Reduced Model
  - Extremely computationally expensive