4

ELECTROQUASISTATIC FIELDS: THE SUPERPOSITION INTEGRAL POINT OF VIEW

4.0 INTRODUCTION

The reason for taking up electroquasistatic fields first is the relative ease with which such a vector field can be represented. The EQS form of Faraday's law requires that the electric field intensity $\mathbf{E}$ be irrotational.

$$\nabla \times \mathbf{E} = 0 \tag{1}$$

The electric field intensity is related to the charge density ρ by Gauss' law.

$$\nabla \cdot \epsilon_o \mathbf{E} = \rho \tag{2}$$

Thus, the source of an electroquasistatic field is a scalar, the charge density ρ. In free space, the source of a magnetoquasistatic field is a vector, the current density. Scalar sources, are simpler than vector sources and this is why electroquasistatic fields are taken up first.

Most of this chapter is concerned with finding the distribution of $\mathbf{E}$ predicted by these laws, given the distribution of ρ. But before the chapter ends, we will be finding fields in limited regions bounded by conductors. In these more practical situations, the distribution of charge on the boundary surfaces is not known until after the fields have been determined. Thus, this chapter sets the stage for the solving of boundary value problems in Chap. 5.

We start by establishing the electric potential as a scalar function that uniquely represents an irrotational electric field intensity. Byproducts of the derivation are the gradient operator and gradient theorem.

The scalar form of Poisson's equation then results from combining (1) and (2). This equation will be shown to be linear. It follows that the field due to a superposition of charges is the superposition of the fields associated with the individual charge components. The resulting superposition integral specifies how the potential, and hence the electric field intensity, can be determined from the given charge distribution. Thus, by the end of Sec. 4.5, a general approach to finding solutions to (1) and (2) is achieved.

The art of arranging the charge so that, in a restricted region, the resulting fields satisfy boundary conditions, is illustrated in Secs. 4.6 and 4.7. Finally, more general techniques for using the superposition integral to solve boundary value problems are illustrated in Sec. 4.8.

For those having a background in circuit theory, it is helpful to recognize that the approaches used in this and the next chapter are familiar. The solution of (1) and (2) in three dimensions is like the solution of circuit equations, except that for the latter, there is only the one dimension of time. In the field problem, the driving function is the charge density.

One approach to finding a circuit response is based on first finding the response to an impulse. Then the response to an arbitrary drive is determined by superimposing responses to impulses, the superposition of which represents the drive. This response takes the form of a superposition integral, the convolution integral. The impulse response of Poisson's equation that is our starting point is the field of a point charge. Thus, the theme of this chapter is a convolution approach to solving (1) and (2).

In the boundary value approach of the next chapter, concepts familiar from circuit theory are again exploited. There, solutions will be divided into a particular part, caused by the drive, and a homogeneous part, required to satisfy boundary conditions. It will be found that the superposition integral is one way of finding the particular solution.

4.1 IRROTATIONAL FIELD REPRESENTED BY SCALAR POTENTIAL: THE GRADIENT OPERATOR AND GRADIENT INTEGRAL THEOREM

The integral of an irrotational electric field from some reference point $\mathbf{r}_{\text{ref}}$ to the position $\mathbf{r}$ is independent of the integration path. This follows from an integration of (1) over the surface S spanning the contour defined by alternative paths I and II, shown in Fig. 4.1.1. Stokes' theorem, (2.5.4), gives

$$\int_S \nabla \times \mathbf{E} \cdot d\mathbf{a} = \oint_C \mathbf{E} \cdot d\mathbf{s} = 0 \tag{1}$$

Stokes' theorem employs a contour running around the surface in a single direction, whereas the line integrals of the electric field from $\mathbf{r}$ to $\mathbf{r}_{\text{ref}}$, from point a to point b, run along the contour in opposite directions. Taking the directions of the path increments into account, (1) is equivalent to

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = \int_{a_{\text{path I}}}^{b} \mathbf{E} \cdot d\mathbf{s} - \int_{a_{\text{path II}}}^{b} \mathbf{E} \cdot d\mathbf{s}' = 0 \tag{2}$$

and thus, for an irrotational field, the EMF between two points is independent of path.

$$\int_{a_{\text{path I}}}^{b} \mathbf{E} \cdot d\mathbf{s} = \int_{a_{\text{path II}}}^{b} \mathbf{E} \cdot d\mathbf{s}' \tag{3}$$

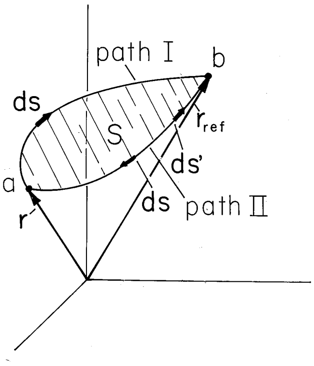


Figure 4.1
positions r
S.

Stokes' theorem employs a contour running around th whereas the line integrals of the electric field from r to $\mathbf{r}_{\text{ref}}$

Sec. 4.1 Irrotational Field

Fig. 4.1.1 Paths I and II between positions $\mathbf{r}$ and $\mathbf{r}_{\text{ref}}$ are spanned by surface S.

A field that assigns a unique value of the line integral between two points independent of path of integration is said to be *conservative.*

With the understanding that the reference point is kept fixed, the integral is a scalar function of the integration endpoint $\mathbf{r}$. We use the symbol $\Phi(\mathbf{r})$ to define this scalar function

$$\Phi(\mathbf{r}) - \Phi(\mathbf{r}_{\text{ref}}) = \int_{\mathbf{r}}^{\mathbf{r}_{\text{ref}}} \mathbf{E} \cdot d\mathbf{s} \tag{4}$$

and call $\Phi(\mathbf{r})$ the *electric potential* of the point $\mathbf{r}$ with respect to the reference point. With the endpoints consisting of "nodes" where wires could be attached, the potential difference of (1) would be the *voltage* at $\mathbf{r}$ relative to that at the reference. Typically, the latter would be the "ground" potential. Thus, for an irrotational field, the EMF defined in Sec. 1.6 becomes the voltage at the point a relative to point b.

We shall show that specification of the scalar function $\Phi(\mathbf{r})$ contains the same information as specification of the field $\mathbf{E}(\mathbf{r})$. This is a remarkable fact because a vector function of $\mathbf{r}$ requires, in general, the specification of three scalar functions of $\mathbf{r}$, say the three Cartesian components of the vector function. On the other hand, specification of $\Phi(\mathbf{r})$ requires one scalar function of $\mathbf{r}$.

Note that the expression $\Phi(\mathbf{r})$ = constant represents a surface in three dimensions. A familiar example of such an expression describes a spherical surface having radius R.

$$x^2 + y^2 + z^2 = R^2 \tag{5}$$

Surfaces of constant potential are called *equipotentials.*

Shown in Fig. 4.1.2 are the cross-sections of two equipotential surfaces, one passing through the point $\mathbf{r}$, the other through the point $\mathbf{r} + \Delta\mathbf{r}$. With $\Delta\mathbf{r}$ taken as a differential vector, the potential at the point $\mathbf{r} + \Delta\mathbf{r}$ differs by the differential

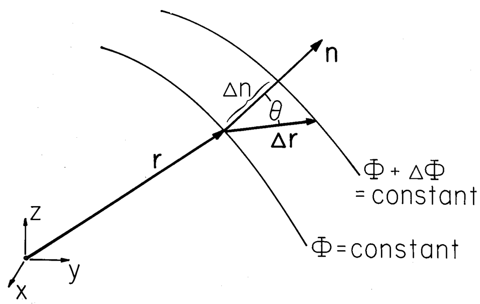


Fig. 4.1.2 Two equipotential surfaces shown cut by a plane containing their normal, $\mathbf{n}$.

amount $\Delta\Phi$ from that at $\mathbf{r}$. The two equipotential surfaces cannot intersect. Indeed, if they intersected, both points $\mathbf{r}$ and $\mathbf{r}+\Delta\mathbf{r}$ would have the same potential, which is contrary to our assumption.

Illustrated in Fig. 4.1.2 is the shortest distance Δn from the point $\mathbf{r}$ to the equipotential at $\mathbf{r}+\Delta\mathbf{r}$. Because of the differential geometry assumed, the length element Δn is perpendicular to both equipotential surfaces. From Fig. 4.1.2, $\Delta n = \cos\theta\Delta r$, and we have

$$\Delta\Phi = \frac{\Delta\Phi}{\Delta n}\cos\theta\Delta r = \frac{\Delta\Phi}{\Delta n}\mathbf{n}\cdot\Delta\mathbf{r} \tag{6}$$

The vector $\Delta\mathbf{r}$ in (6) is of arbitrary direction. It is also of arbitrary differential length. Indeed, if we double the distance Δn, we double $\Delta\Phi$ and Δr; $\Delta\Phi/\Delta n$ remains unchanged and thus (6) holds for any Δr (of differential length). We conclude that (6) assigns to every differential vector length element $\Delta\mathbf{r}$, originating from $\mathbf{r}$, a scalar of magnitude proportional to the magnitude of $\Delta\mathbf{r}$ and to the cosine of the angle between $\Delta\mathbf{r}$ and the unit vector $\mathbf{n}$. This assignment of a scalar to a vector is representable as the scalar product of the vector length element $\Delta\mathbf{r}$ with a vector of magnitude $\Delta\Phi/\Delta n$ and direction $\mathbf{n}$. That is, (6) is equivalent to

$$\Delta\Phi = \mathit{grad}\,\Phi\cdot\Delta\mathbf{r} \tag{7}$$

where the gradient of the potential is defined as

$$\mathit{grad}\,\Phi \equiv \frac{\Delta\Phi}{\Delta n}\mathbf{n} \tag{8}$$

Because it is independent of any particular coordinate system, (8) provides the best way to conceptualize the gradient operator. The same equation provides the algorithm for expressing *grad* Φ in any particular coordinate system. Consider, as an example, Cartesian coordinates. Thus,

$$\mathbf{r} = x\mathbf{i_x} + y\mathbf{i_y} + z\mathbf{i_z}; \qquad \Delta\mathbf{r} = \Delta x\mathbf{i_x} + \Delta y\mathbf{i_y} + \Delta z\mathbf{i_z} \tag{9}$$

and an alternative to (6) for expressing the differential change in Φ is

$$\begin{aligned}\Delta\Phi &= \Phi(x+\Delta x, y+\Delta y, z+\Delta z) - \Phi(x,y,z)\\ &= \frac{\partial\Phi}{\partial x}\Delta x + \frac{\partial\Phi}{\partial y}\Delta y + \frac{\partial\Phi}{\partial z}\Delta z.\end{aligned} \tag{10}$$

In view of (9), this expression is

$$\Delta\Phi = \left(\mathbf{i_x}\frac{\partial\Phi}{\partial x} + \mathbf{i_y}\frac{\partial\Phi}{\partial y} + \mathbf{i_z}\frac{\partial\Phi}{\partial z}\right)\cdot\Delta\mathbf{r} = \nabla\Phi\cdot\Delta\mathbf{r} \tag{11}$$

and it follows that in Cartesian coordinates the gradient operation, as defined by (7), is

$$grad\ \Phi \equiv \nabla\Phi = \frac{\partial\Phi}{\partial x}\mathbf{i_x} + \frac{\partial\Phi}{\partial y}\mathbf{i_y} + \frac{\partial\Phi}{\partial z}\mathbf{i_z} \tag{12}$$

Here, the del operator defined by (2.1.6) is introduced as an alternative way of writing the gradient operator.

Problems at the end of this chapter serve to illustrate how the gradient is similarly determined in other coordinates, with results summarized in Table I at the end of the text.

We are now ready to show that the potential function $\Phi(\mathbf{r})$ defines $\mathbf{E}(\mathbf{r})$ uniquely. According to (4), the potential changes from the point $\mathbf{r}$ to the point $\mathbf{r}+\Delta\mathbf{r}$ by

$$\begin{aligned}\Delta\Phi &= \Phi(\mathbf{r}+\Delta\mathbf{r}) - \Phi(\mathbf{r})\\ &= -\int_{\mathbf{r}_{\text{ref}}}^{\mathbf{r}+\Delta\mathbf{r}}\mathbf{E}\cdot d\mathbf{s} + \int_{\mathbf{r}_{\text{ref}}}^{\mathbf{r}}\mathbf{E}\cdot d\mathbf{s}\\ &= -\int_{\mathbf{r}}^{\mathbf{r}+\Delta\mathbf{r}}\mathbf{E}\cdot d\mathbf{s}\end{aligned} \tag{13}$$

The first two integrals in (13) follow from the definition of Φ, (4). By recognizing that $d\mathbf{s}$ is $\Delta\mathbf{r}$ and that $\Delta\mathbf{r}$ is of differential length, so that $\mathbf{E}(\mathbf{r})$ can be considered constant over the length of the vector $\Delta\mathbf{r}$, it can be seen that the last integral in (13) becomes

$$\Delta\Phi = -\mathbf{E}\cdot\Delta\mathbf{r} \tag{14}$$

The vector element $\Delta\mathbf{r}$ is arbitrary. Therefore, comparison of (14) to (7) shows that

$$\boxed{\mathbf{E} = -\nabla\Phi} \tag{15}$$

Given the potential function $\Phi(\mathbf{r})$, the associated electric field intensity is the negative gradient of Φ.

Note that we also obtained a useful integral theorem, for if (15) is substituted into (4), it follows that

$$\boxed{\int_{\mathbf{r}_{\text{ref}}}^{\mathbf{r}}\nabla\Phi\cdot d\mathbf{s} = \Phi(\mathbf{r}) - \Phi(\mathbf{r}_{\text{ref}})} \tag{16}$$

That is, the line integration of the gradient of Φ is simply the difference in potential between the endpoints. Of course, Φ can be any scalar function.

In retrospect, we can observe that the representation of $\mathbf{E}$ by (15) guarantees that it is irrotational, for the vector identity holds

$$\nabla\times(\nabla\Phi) = 0 \tag{17}$$

The curl of the gradient of a scalar potential Φ vanishes. Therefore, given an electric field represented by a potential in accordance with (15), (4.0.1) is automatically satisfied.

Because the preceding discussion shows that the potential Φ contains full information about the field $\mathbf{E}$, the replacement of $\mathbf{E}$ by *grad* (Φ) constitutes a general solution, or integral, of (4.0.1). Integration of a first-order ordinary differential equation leads to one arbitrary integration constant. Integration of the first-order vector differential equation *curl* $\mathbf{E} = 0$ yields a scalar function of integration, $\Phi(\mathbf{r})$.

Thus far, we have not made any specific assignment for the reference point $\mathbf{r}_{\text{ref}}$. Provided that the potential behaves properly at infinity, it is often convenient to let the reference point be at infinity. There are some exceptional cases for which such a choice is not possible. All such cases involve problems with infinite amounts of charge. One such example is the field set up by a charge distribution that extends to infinity in the $\pm z$ directions, as in the second Illustration in Sec. 1.3. The field decays like $1/r$ with radial distance r from the charged region. Thus, the line integral of $\mathbf{E}$, (4), from a finite distance out to infinity involves the difference of $ln\ r$ evaluated at the two endpoints and becomes infinite if one endpoint moves to infinity. In problems that extend to infinity but are not of this singular nature, we shall assume that the reference is at infinity.

Example 4.1.1. Equipotential Surfaces

Consider the potential function $\Phi(x, y)$, which is independent of z:

$$\Phi(x, y) = V_o \frac{xy}{a^2} \tag{18}$$

Surfaces of constant potential can be represented by a cross-sectional view in any $x - y$ plane in which they appear as lines, as shown in Fig. 4.1.3. For the potential given by (18), the equipotentials appear in the $x - y$ plane as hyperbolae. The contours passing through the points (a, a) and $(-a, -a)$ have the potential V_o, while those at $(a, -a)$ and $(-a, a)$ have potential $-V_o$.

The magnitude of $\mathbf{E}$ is proportional to the spatial rate of change of Φ in a direction perpendicular to the constant potential surface. Thus, if the surfaces of constant potential are sketched at equal increments in potential, as is done in Fig. 4.1.3, where the increments are $V_o/4$, the magnitude of $\mathbf{E}$ is inversely proportional to the spacing between surfaces. The closer the spacing of potential lines, the higher the field intensity. Field lines, sketched in Fig. 4.1.3, have arrows that point from high to low potentials. Note that because they are always perpendicular to the equipotentials, they naturally are most closely spaced where the field intensity is largest.

Example 4.1.2. Evaluation of Gradient and Line Integral

Our objective is to exemplify by direct evaluation the fact that the line integration of an irrotational field between two given points is independent of the integration path. In particular, consider the potential given by (18), which, in view of (12), implies the electric field intensity

$$\mathbf{E} = -\nabla\Phi = -\frac{V_o}{a^2}(y\mathbf{i_x} + x\mathbf{i_y}) \tag{19}$$

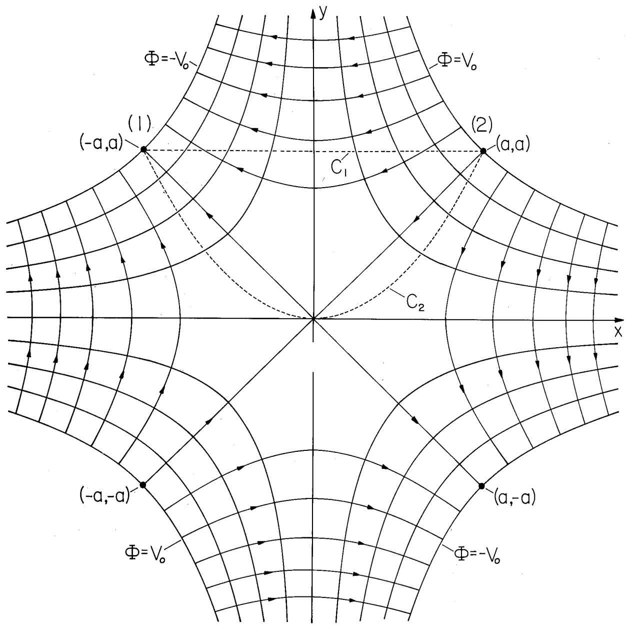


Fig. 4.1.3 Cross-sectional view of surfaces of constant potential for two-dimensional potential given by (18).

We integrate this vector function along two paths, shown in Fig. 4.1.3, which join points (1) and (2). For the first path, C_1, y is held fixed at $y = a$ and hence $d\mathbf{s} = dx\mathbf{i_x}$. Thus, the integral becomes

$$\int_{C_1} \mathbf{E} \cdot d\mathbf{s} = \int_{-a}^{a} E_x(x, a)dx = -\int_{-a}^{a} \frac{V_o}{a^2} a dx = -2V_o \tag{20}$$

For path C_2, $y - x^2/a = 0$ and in general, $d\mathbf{s} = dx\mathbf{i_x} + dy\mathbf{i_y}$, so the required integral is

$$\int_{C_2} \mathbf{E} \cdot d\mathbf{s} = \int_{C_2} (E_x dx + E_y dy) \tag{21}$$

However, for the path C_2 we have $dy - (2x)dx/a = 0$, and hence (21) becomes

$$\begin{aligned} \int_{C_2} \mathbf{E} \cdot d\mathbf{s} &= \int_{-a}^{a} \big(E_x + \frac{2x}{a} E_y\big) dx \\ &= \int_{-a}^{a} -\frac{V_o}{a^2} \left(\frac{x^2}{a} + \frac{2x^2}{a}\right) dx = -2V_o \end{aligned} \tag{22}$$

Because $\mathbf{E}$ is found by taking the negative gradient of Φ, and is therefore irrotational, it is no surprise that (20) and (22) give the same result.

Example 4.1.3. Potential of Spherical Cloud of Charge

A uniform static charge distribution ρ_o occupies a spherical region of radius R. The remaining space is charge free (except, of course, for the balancing charge at infinity). The following illustrates the determination of a piece-wise continuous potential function.

The spherical symmetry of the charge distribution imposes a spherical symmetry on the electric field that makes possible its determination from Gauss' integral law. Following the approach used in Example 1.3.1, the field is found to be

$$E_r = \begin{cases} \frac{r\rho_o}{3\epsilon_o}; & r < R \\ \frac{R^3\rho_o}{3\epsilon_o r^2}; & r > R \end{cases} \tag{23}$$

The potential is obtained by evaluating the line integral of (4) with the reference point taken at infinity, $r = \infty$. The contour follows part of a straight line through the origin. In the exterior region, integration gives

$$\Phi(r) = \int_r^\infty E_r dr = \frac{4\pi R^3}{3}\rho_o\Big(\frac{1}{4\pi\epsilon_o r}\Big); \qquad r > R \tag{24}$$

To find Φ in the interior region, the integration is carried through the outer region, (which gives (24) evaluated at $r = R$) and then into the radius r in the interior region.

$$\Phi(r) = \frac{4\pi R^3}{3}\rho_o\Big(\frac{1}{4\pi\epsilon_o R}\Big) + \frac{\rho_o}{6\epsilon_o}(R^2 - r^2) \tag{25}$$

Outside the charge distribution, where $r \geq R$, the potential acquires the form of the coulomb potential of a point charge.

$$\Phi = \frac{q}{4\pi\epsilon_o r}; \qquad q \equiv \frac{4\pi R^3}{3}\rho_o \tag{26}$$

Note that q is the net charge of the distribution.

Visualization of Two-Dimensional Irrotational Fields. In general, equipotentials are three-dimensional surfaces. Thus, any two-dimensional plot of the contours of constant potential is the intersection of these surfaces with some given plane. If the potential is two-dimensional in its dependence, then the equipotential surfaces have a cylindrical shape. For example, the two-dimensional potential of (18) has equipotential surfaces that are cylinders having the hyperbolic cross-sections shown in Fig. 4.1.3.

We review these geometric concepts because we now introduce a different point of view that is useful in picturing *two-dimensional* fields. A three-dimensional picture is now made in which the third dimension represents the amplitude of the potential Φ. Such a picture is shown in Fig. 4.1.4, where the potential of (18) is used as an example. The floor of the three-dimensional plot is the $x - y$ plane, while the vertical dimension is the potential. Thus, contours of constant potential are represented by lines of constant altitude.

The surface of Fig. 4.1.4 can be regarded as a membrane stretched between supports on the periphery of the region of interest that are elevated or depressed in proportion to the boundary potential. By the definition of the gradient, (8), the lines of electric field intensity follow contours of steepest descent on this surface.

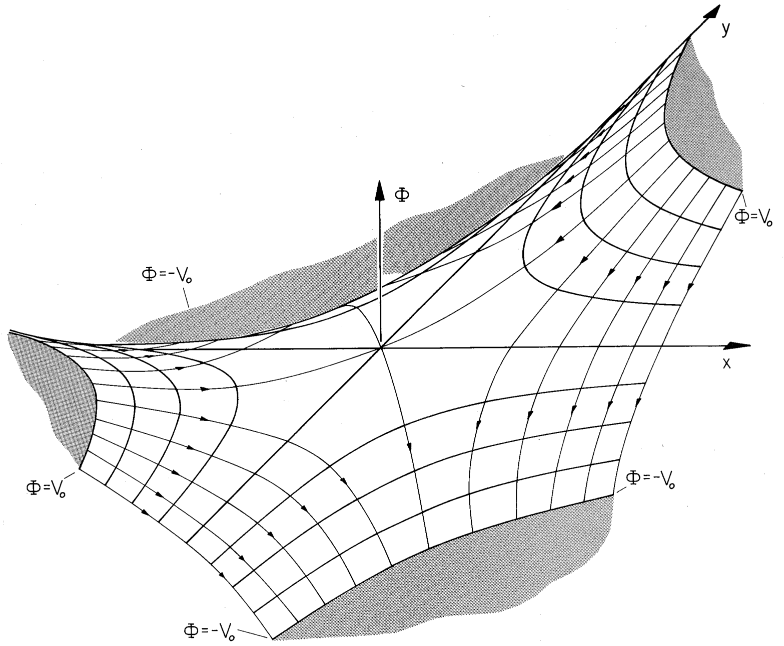


Fig. 4.1.4 Two-dimensional potential of (18) and Fig. 4.1.3 represented in three dimensions. The vertical coordinate, the potential, is analogous to the vertical deflection of a taut membrane. The equipotentials are then contours of constant altitude on the membrane surface.

Potential surfaces have their greatest value in the mind's eye, which pictures a two-dimensional potential as a contour map and the lines of electric field intensity as the flow lines of water streaming down the hill.

4.2 POISSON'S EQUATION

Given that **E** is irrotational, (4.0.1), and given the charge density in Gauss' law, (4.0.2), what is the distribution of electric field intensity? It was shown in Sec. 4.1 that we can satisfy the first of these equations identically by representing the vector **E** by the scalar electric potential Φ.

$$\mathbf{E} = -\nabla\Phi \tag{1}$$

That is, with the introduction of this relation, (4.0.1) has been integrated.

Having integrated (4.0.1), we now discard it and concentrate on the second equation of electroquasistatics, Gauss' law. Introduction of (1) into Gauss' law, (1.0.2), gives

$$\nabla \cdot \nabla\Phi = -\frac{\rho}{\epsilon_o}$$

which is identically

$$\boxed{\nabla^2\Phi = -\frac{\rho}{\epsilon_o}} \tag{2}$$

Integration of this *scalar Poisson's equation*, given the charge density on the right, is the objective in the remainder of this chapter.

By analogy to the ordinary differential equations of circuit theory, the charge density on the right is a "driving function." What is on the left is the operator ∇^2, denoted by the second form of (2) and called the *Laplacian* of Φ. In Cartesian coordinates, it follows from the expressions for the divergence and gradient operators, (2.1.5) and (4.1.12), that

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = \frac{-\rho}{\epsilon_o} \tag{3}$$

The Laplacian operator in cylindrical and spherical coordinates is determined in the problems and summarized in Table I at the end of the text. In Cartesian coordinates, the derivatives in this operator have constant coefficients. In these other two coordinate systems, some of the coefficients are space varying.

Note that in (3), time does not appear explicitly as an independent variable. Hence, the mathematical problem of finding a quasistatic electric field at the time t_o for a time-varying charge distribution $\rho(\mathbf{r}, t)$ is the same as finding the static field for the time-independent charge distribution $\rho(\mathbf{r})$ equal to $\rho(\mathbf{r}, t = t_o)$, the charge distribution of the time-varying problem at the particular instant t_o.

In problems where the charge distribution is given, the evaluation of a quasistatic field is therefore equivalent to the evaluation of a succession of static fields, each with a different charge distribution, at the time of interest. We emphasize this here to make it understood that the solution of a static electric field has wider applicability than one would at first suppose: Every static field solution can represent a "snapshot" at a particular instant of time. Having said that much, we shall not indicate the time dependence of the charge density and field explicitly, but shall do so only when this is required for clarity.

4.3 SUPERPOSITION PRINCIPLE

As illustrated in Cartesian coordinates by (4.2.3), Poisson's equation is a linear second-order differential equation relating the potential $\Phi(\mathbf{r})$ to the charge distribution $\rho(\mathbf{r})$. By "linear" we mean that the coefficients of the derivatives in the differential equation are not functions of the dependent variable Φ. An important consequence of the linearity of Poisson's equation is that $\Phi(\mathbf{r})$ obeys the superposition principle. It is perhaps helpful to recognize the analogy to the superposition principle obeyed by solutions of the linear ordinary differential equations of circuit theory. Here the principle can be shown as follows.

Consider two different spatial distributions of charge density, $\rho_a(\mathbf{r})$ and $\rho_b(\mathbf{r})$. These might be relegated to different regions, or occupy the same region. Suppose we have found the potentials Φ_a and Φ_b which satisfy Poisson's equation, (4.2.3), with the respective charge distributions ρ_a and ρ_b. By definition,

$$\nabla^2 \Phi_a(\mathbf{r}) = -\frac{\rho_a(\mathbf{r})}{\epsilon_o} \tag{1}$$

$$\nabla^2 \Phi_b(\mathbf{r}) = -\frac{\rho_b(\mathbf{r})}{\epsilon_o} \tag{2}$$

Adding these expressions, we obtain

$$\nabla^2 \Phi_a(\mathbf{r}) + \nabla^2 \Phi_b(\mathbf{r}) = -\frac{1}{\epsilon_o}[\rho_a(\mathbf{r}) + \rho_b(\mathbf{r})] \tag{3}$$

Because the derivatives called for in the Laplacian operation– for example, the second derivatives of (4.2.3)– give the same result whether they operate on the potentials and then are summed or operate on the sum of the potentials, (3) can also be written as

$$\nabla^2 [\Phi_a(\mathbf{r}) + \Phi_b(\mathbf{r})] = -\frac{1}{\epsilon_o}[\rho_a(\mathbf{r}) + \rho_b(\mathbf{r})] \tag{4}$$

The mathematical statement of the superposition principle follows from (1) and (2) and (4). That is, if

$$\begin{aligned} \rho_a &\Rightarrow \Phi_a \\ \rho_b &\Rightarrow \Phi_b \end{aligned} \tag{5}$$

then

$$\rho_a + \rho_b \Rightarrow \Phi_a + \Phi_b \tag{6}$$

The potential distribution produced by the superposition of the charge distributions is the sum of the potentials associated with the individual distributions.

4.4 FIELDS ASSOCIATED WITH CHARGE SINGULARITIES

At least three objectives are set in this section. First, the superposition concept from Sec. 4.3 is exemplified. Second, we begin to deal with fields that are not highly symmetric. The potential proves invaluable in picturing such fields, and so we continue to develop ways of picturing the potential and field distribution. Finally, the potential functions developed will reappear many times in the chapters that follow. Solutions to Poisson's equation as pictured here filling all of space will turn out to be solutions to Laplace's equation in subregions that are devoid of charge. Thus, they will be seen from a second point of view in Chap. 5, where Laplace's equation is featured.

First, consider the potential associated with a point charge at the origin of a spherical coordinate system. The electric field was obtained using the integral form of Gauss' law in Sec. 1.3, (1.3.12). It follows from the definition of the potential, (4.1.4), that the potential of a point charge q is

$$\Phi = \frac{q}{4\pi\epsilon_o r} \tag{1}$$

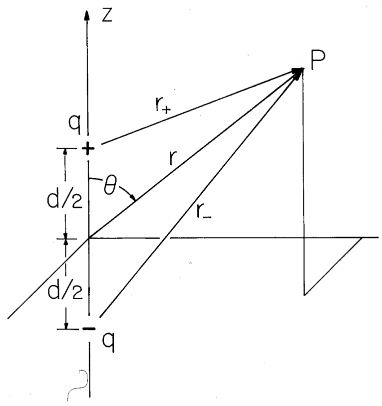


Fig. 4.4.1 Point charges of equal magnitude and opposite sign on the z axis.

This "impulse response" for the three-dimensional Poisson's equation is the starting point in derivations and problem solutions and is worth remembering.

Consider next the field associated with a positive and a negative charge, located on the z axis at $d/2$ and $-d/2$, respectively. The configuration is shown in Fig. 4.4.1. In (1), r is the scalar distance between the point of observation and the charge. With P the observation position, these distances are denoted in Fig. 4.4.1 by r_+ and r_-. It follows from (1) and the superposition principle that the potential distribution for the two charges is

$$\Phi = \frac{q}{4\pi\epsilon_o}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \tag{2}$$

To find the electric field intensity by taking the negative gradient of this function, it is necessary to express r_+ and r_- in Cartesian coordinates.

$$r_+ = \sqrt{x^2 + y^2 + \left(z - \frac{d}{2}\right)^2}; \qquad r_- = \sqrt{x^2 + y^2 + \left(z + \frac{d}{2}\right)^2} \tag{3}$$

Thus, in these coordinates, the potential for the two charges given by (2) is

$$\Phi = \frac{q}{4\pi\epsilon_o}\left(\frac{1}{\sqrt{x^2 + y^2 + \left(z - \frac{d}{2}\right)^2}} - \frac{1}{\sqrt{x^2 + y^2 + \left(z + \frac{d}{2}\right)^2}}\right) \tag{4}$$

Equation (2) shows that in the immediate vicinity of one or the other of the charges, the respective charge dominates the potential. Thus, close to the point charges the equipotentials are spheres enclosing the charge. Also, this expression makes it clear that the plane $z = 0$ is one of zero potential.

One straightforward way to plot the equipotentials in detail is to program a calculator to evaluate (4) at a specified coordinate position. To this end, it is convenient to normalize the potential and the coordinates such that (4) is

$$\underline{\Phi} = \frac{1}{\sqrt{\underline{x}^2 + \underline{y}^2 + \left(\underline{z} - \frac{1}{2}\right)^2}} - \frac{1}{\sqrt{\underline{x}^2 + \underline{y}^2 + \left(\underline{z} + \frac{1}{2}\right)^2}} \tag{5}$$

where

$$\underline{x} = \frac{x}{d}, \quad \underline{y} = \frac{y}{d}, \quad \underline{z} = \frac{z}{d}, \quad \underline{\Phi} = \frac{\Phi}{(q/4\pi d\epsilon_o)}$$

By evaluating Φ for various coordinate positions, it is possible to zero in on the coordinates of a given equipotential in an iterative fashion. The equipotentials shown in Fig. 4.4.2a were plotted in this way with $\underline{x} = 0$. Of course, the equipotentials are actually three-dimensional surfaces obtained by rotating the curves shown about the z axis.

Because $\mathbf{E}$ is the negative gradient of Φ, lines of electric field intensity are perpendicular to the equipotentials. These can therefore be easily sketched and are shown as lines with arrows in Fig. 4.4.2a.

Dipole at the Origin. An important limit of (2) corresponds to a view of the field for an observer far from either of the charges. This is a very important limit because charge pairs of opposite sign are the model for polarized atoms or molecules. The dipole is therefore at center stage in Chap. 6, where we deal with polarizable matter. Formally, the dipole limit is taken by recognizing that rays joining the point of observation with the respective charges are essentially parallel to the r coordinate when $r \gg d$. The approximate geometry shown in Fig. 4.4.3 motivates the approximations.

$$r_+ \simeq r - \frac{d}{2}\cos\theta; \qquad r_- \simeq r + \frac{d}{2}\cos\theta \tag{6}$$

Because the first terms in these expressions are very large compared to the second, powers of r_+ and r_- can be expanded in a binomial expansion.

$$(a+b)^n = a^n + na^{n-1}b + \dots \tag{7}$$

With $n = -1$, (2) becomes approximately

$$\begin{aligned}\Phi &= \frac{q}{4\pi\epsilon_o}\Big[\big(\frac{1}{r} + \frac{d}{2r^2}\cos\theta + \dots\big) \\ &\quad - \big(\frac{1}{r} - \frac{d}{2r^2}\cos\theta + \dots\big)\Big] \\ &= \frac{qd}{4\pi\epsilon_o}\frac{\cos\theta}{r^2}\end{aligned} \tag{8}$$

Remember, the potential is pictured in spherical coordinates.

Suppose the equipotential is to be sketched that passes through the z axis at some specified location. What is the shape of the potential as we move in the positive θ direction? On the left in (8) is a constant. With an increase in θ, the cosine function on the right decreases. Thus, to stay on the surface, the distance r from the origin must decrease. As the angle approaches $\pi/2$, the cosine decreases to zero, making it clear that the equipotential must approach the origin. The equipotentials and associated lines of $\mathbf{E}$ are shown in Fig. 4.4.2b.

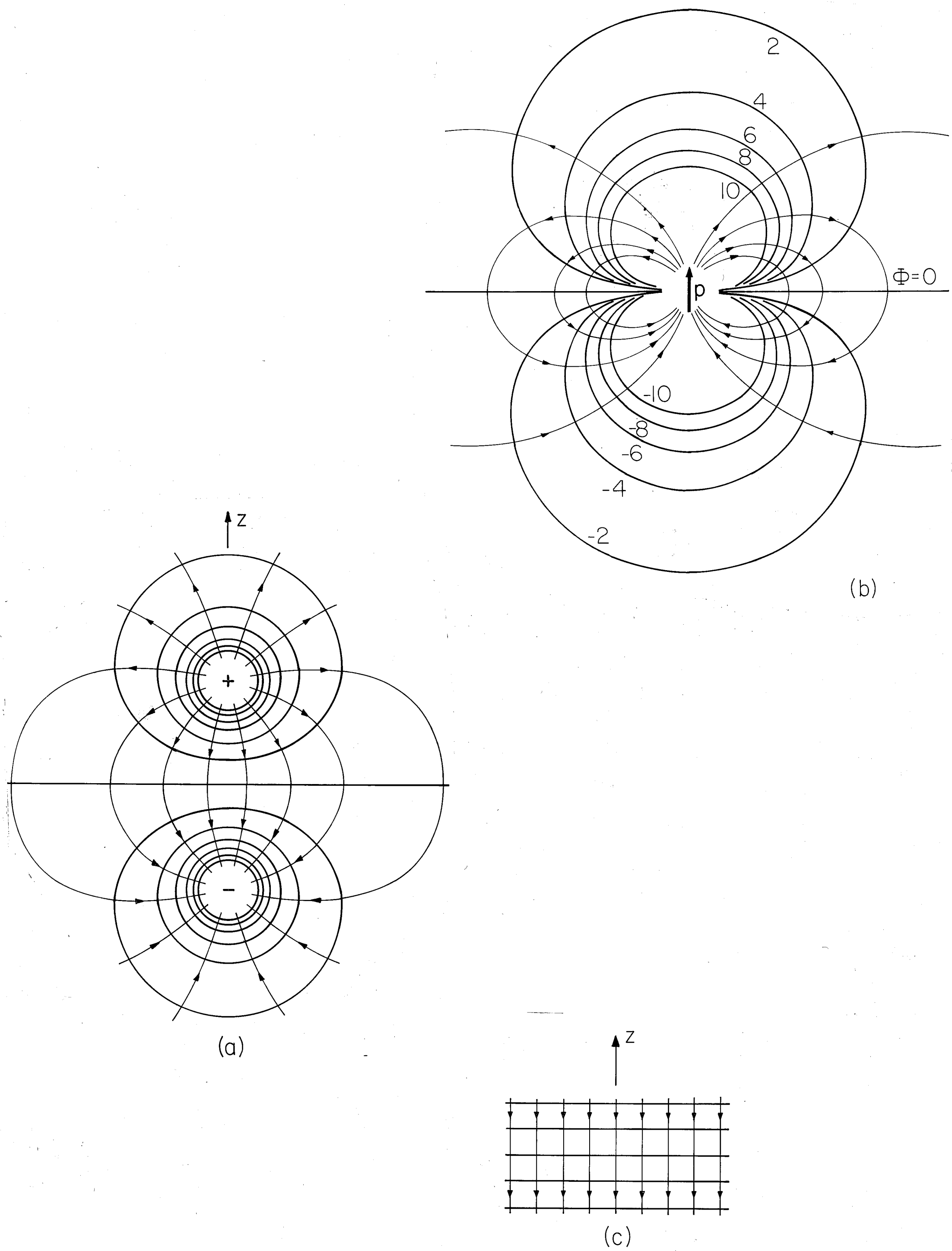


Fig. 4.4.2 (a) Cross-section of equipotentials and lines of electric field intensity for the two charges of Fig. 4.4.1. (b) Limit in which pair of charges form a dipole at the origin. (c) Limit of charges at infinity.

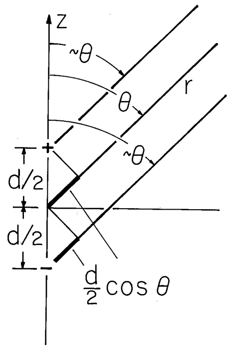


Fig. 4.4.3 Far from the dipole, rays from the charges to the point of observation are essentially parallel to r coordinate.

The dipole model is made mathematically exact by defining it as the limit in which two charges of equal magnitude and opposite sign approach to within an infinitesimal distance of each other while increasing in magnitude. Thus, with the dipole moment p defined as

$$p = \lim_{\substack{d\to 0\\ q\to\infty}} qd \tag{9}$$

the potential for the dipole, (8), becomes

$$\Phi = \frac{p}{4\pi\epsilon_o}\frac{\cos\theta}{r^2} \tag{10}$$

Another more general way of writing (10) with the dipole positioned at an arbitrary point $\mathbf{r}'$ and lying along a general axis is to introduce the dipole moment vector. This vector is defined to be of magnitude p and directed along the axis of the two charges pointing from the $-$ charge to the $+$ charge. With the unit vector $\mathbf{i}_{\mathbf{r}'\mathbf{r}}$ defined as being directed from the point $\mathbf{r}'$ (where the dipole is located) to the point of observation at $\mathbf{r}$, it follows from (10) that the generalized potential is

$$\Phi = \frac{\mathbf{p}\cdot\mathbf{i}_{\mathbf{r}'\mathbf{r}}}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|^2} \tag{11}$$

Pair of Charges at Infinity Having Equal Magnitude and Opposite Sign. Consider next the appearance of the field for an observer located between the charges of Fig. 4.4.2a, in the neighborhood of the origin. We now confine interest to distances from the origin that are small compared to the charge spacing d. Effectively, the charges are at infinity in the $+z$ and $-z$ directions, respectively.

With the help of Fig. 4.4.4 and the three-dimensional Pythagorean theorem, the distances from the charges to the observer point are expressed in spherical coordinates as

$$r_+ = \sqrt{\Big(\frac{d}{2} - r\cos\theta\Big)^2 + (r\sin\theta)^2}; \qquad r_- = \sqrt{\Big(\frac{d}{2} + r\cos\theta\Big)^2 + (r\sin\theta)^2} \tag{12}$$

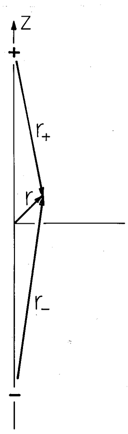


Fig. 4.4.4 Relative displacements with charges going to infinity.

In these expressions, d is large compared to r, so they can be expanded by again using (7) and keeping only linear terms in r.

$$r_+^{-1} \simeq \frac{2}{d} + \frac{4r}{d^2}\cos\theta; \qquad r_-^{-1} \simeq \frac{2}{d} - \frac{4r}{d^2}\cos\theta \tag{13}$$

Introduction of these approximations into (2) results in the desired expression for the potential associated with charges that are at infinity on the z axis.

$$\Phi \to \frac{2(q/d^2)}{\pi\epsilon_o} r\cos\theta \tag{14}$$

Note that $z = r\cos\theta$, so what appears to be a complicated field in spherical coordinates is simply

$$\Phi \to \frac{2q/d^2}{\pi\epsilon_o} z \tag{15}$$

The z coordinate can just as well be regarded as Cartesian, and the electric field evaluated using the gradient operator in Cartesian coordinates. Thus, the surfaces of constant potential, shown in Fig. 4.4.2c, are horizontal planes. It follows that the electric field intensity is uniform and downward directed. Note that the electric field that follows from (15) is what is obtained by direct evaluation of (1.3.12) as the field of point charges q at a distance $d/2$ above and below the point of interest.

Other Charge Singularities. A two-dimensional dipole consists of a pair of oppositely charged parallel lines, rather than a pair of point charges. Pictured in a plane perpendicular to the lines, and in polar coordinates, the equipotentials appear similar to those of Fig. 4.4.2b. However, in three dimensions the surfaces are cylinders of circular cross-section and not at all like the closed surfaces of revolution that are the equipotentials for the three-dimensional dipole. Two-dimensional dipole fields are derived in Probs. 4.4.1 and 4.4.2, where the potentials are given for reference.

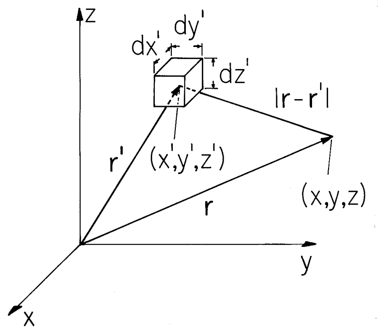


Fig. 4.5.1 An elementary volume of charge at $\mathbf{r}'$ gives rise to a potential at the observer position $\mathbf{r}$.

There is an infinite number of charge singularities. One of the "higher order" singularities is illustrated by the quadrupole fields developed in Probs. 4.4.3 and 4.4.4. We shall see these same potentials again in Chap. 5.

4.5 SOLUTION OF POISSON'S EQUATION FOR SPECIFIED CHARGE DISTRIBUTIONS

The superposition principle is now used to find the solution of Poisson's equation for any given charge distribution $\rho(\mathbf{r})$. The argument presented in the previous section for singular charge distributions suggests the approach.

For the purpose of representing the arbitrary charge density distribution as a sum of "elementary" charge distributions, we subdivide the space occupied by the charge density into elementary volumes of size $dx'dy'dz'$. Each of these elements is denoted by the Cartesian coordinates (x', y', z'), as shown in Fig. 4.5.1. The charge contained in one of these elementary volumes, the one with the coordinates (x', y', z'), is

$$dq = \rho(\mathbf{r}')dx'dy'dz' = \rho(\mathbf{r}')dv' \tag{1}$$

We now express the total potential due to the charge density ρ as the superposition of the potentials $d\Phi$ due to the differential elements of charge, (1), positioned at the points $\mathbf{r}'$. Note that each of these elementary charge distributions has zero charge density at all points outside of the volume element dv' situated at $\mathbf{r}'$. Thus, they represent point charges of magnitudes dq given by (1). Provided that $|\mathbf{r} - \mathbf{r}'|$ is taken as the distance between the point of observation $\mathbf{r}$ and the position of one incremental charge $\mathbf{r}'$, the potential associated with this incremental charge is given by (4.4.1).

$$d\Phi(\mathbf{r}, \mathbf{r}') = \frac{\rho(\mathbf{r}')dv'}{4\pi\epsilon_o|\mathbf{r} - \mathbf{r}'|} \tag{2}$$

where in Cartesian coordinates

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

Note that (2) is a function of two sets of Cartesian coordinates: the (observer) coordinates (x, y, z) of the point $\mathbf{r}$ at which the potential is evaluated and the (source) coordinates (x', y', z') of the point $\mathbf{r}'$ at which the incremental charge is positioned.

According to the superposition principle, we obtain the total potential produced by the sum of the differential charges by adding over all differential potentials, keeping the observation point (x, y, z) fixed. The sum over the differential volume elements becomes a volume integral over the coordinates (x', y', z').

$$\boxed{\Phi(\mathbf{r}) = \int_{V'} \frac{\rho(\mathbf{r}')dv'}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|}} \tag{3}$$

This is the *superposition integral* for the electroquasistatic potential.

The evaluation of the potential requires that a triple integration be carried out. With the help of a computer, or even a programmable calculator, this is a straightforward process. There are few examples where the three successive integrations are carried out analytically without considerable difficulty.

There are special representations of (3), appropriate in cases where the charge distribution is confined to surfaces, lines, or where the distribution is two dimensional. For these, the number of integrations is reduced to two or even one, and the difficulties in obtaining analytical expressions are greatly reduced.

Three-dimensional charge distributions can be represented as the superposition of lines and sheets of charge and, by exploiting the potentials found analytically for these distributions, the numerical integration that might be required to determine the potential for a three-dimensional charge distribution can be reduced to two or even one numerical integration.

Superposition Integral for Surface Charge Density. If the charge density is confined to regions that can be described by surfaces having a very small thickness Δ, then one of the three integrations of (3) can be carried out in general. The situation is as pictured in Fig. 4.5.2, where the distance to the observation point is large compared to the thickness over which the charge is distributed. As the integration of (3) is carried out over this thickness Δ, the distance between source and observer, $|\mathbf{r}-\mathbf{r}'|$, varies little. Thus, with ξ used to denote a coordinate that is locally perpendicular to the surface, the general superposition integral, (3), reduces to

$$\Phi(\mathbf{r}) = \int_{A'} \frac{da'}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|} \int_0^{\Delta} \rho(\mathbf{r}')d\xi \tag{4}$$

The integral on ξ is by definition the surface charge density. Thus, (4) becomes a form of the superposition integral applicable where the charge distribution can be modeled as being on a surface.

$$\boxed{\Phi(\mathbf{r}) = \int_{A'} \frac{\sigma_s(\mathbf{r}')da'}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|}} \tag{5}$$

The following example illustrates the application of this integral.

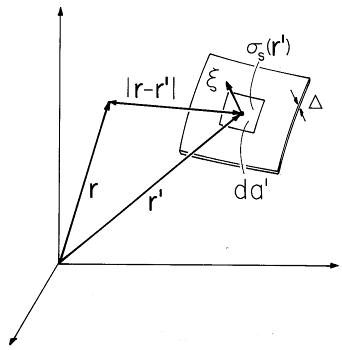


Fig. 4.5.2 An element of surface charge at the location $\mathbf{r}'$ gives rise to a potential at the observer point $\mathbf{r}$.

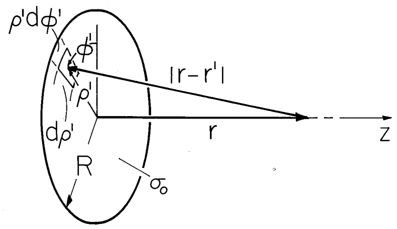


Fig. 4.5.3 A uniformly charged disk with coordinates for finding the potential along the z axis.

Example 4.5.1. Potential of a Uniformly Charged Disk

The disk shown in Fig. 4.5.3 has a radius R and carries a uniform surface charge density σ_o. The following steps lead to the potential and field on the axis of the disk.

The distance $|\mathbf{r}-\mathbf{r}'|$ between the point $\mathbf{r}'$ at radius ρ and angle ϕ (in cylindrical coordinates) and the point $\mathbf{r}$ on the axis of the disk (the z axis) is given by

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{\rho'^2 + z^2} \tag{6}$$

It follows that (5) is expressible in terms of the following double integral

$$\begin{aligned}\Phi &= \frac{\sigma_o}{4\pi\epsilon_o}\int_0^{2\pi}\int_0^R \frac{\rho' d\rho' d\phi'}{\sqrt{\rho'^2 + z^2}} \\ &= \frac{\sigma_o}{4\pi\epsilon_o}2\pi\int_0^R \frac{\rho' d\rho'}{\sqrt{\rho'^2 + z^2}} \\ &= \frac{\sigma_o}{2\epsilon_o}\big(\sqrt{R^2 + z^2} - |z|\big)\end{aligned} \tag{7}$$

where we have allowed for both positive z, the case illustrated in the figure, and negative z. Note that these are points on opposite sides of the disk.

The axial field intensity E_z can be found by taking the gradient of (7) in the z direction.

$$E_z = -\frac{\partial \Phi}{\partial z} = -\frac{\sigma_o}{2\epsilon_o}\frac{d}{dz}\left(\sqrt{R^2+z^2}-|z|\right) \\ = -\frac{\sigma_o}{2\epsilon_o}\left(\frac{z}{\sqrt{R^2+z^2}} \mp 1\right) \tag{8}$$

The upper sign applies to positive z, the lower sign to negative z.

The potential distribution of (8) can be checked in two limiting cases for which answers are easily obtained by inspection: the potential at a distance $|z| \gg R$, and the field at $|z| \ll R$.

(a) At a very large distance $|z|$ of the point of observation from the disk, the radius of the disk R is small compared to $|z|$, and the potential of the disk must approach the potential of a point charge of magnitude equal to the total charge of the disk, $\sigma_o \pi R^2$. The potential given by (7) can be expanded in powers of R/z

$$\sqrt{R^2+z^2} - |z| = |z|\left(1 + \frac{1}{2}\frac{R^2}{z^2}\right) \tag{9}$$

to find that Φ indeed approaches the potential function

$$\Phi \simeq \frac{\sigma_o}{4\pi\epsilon_o}\pi R^2 \frac{1}{|z|} \tag{10}$$

of a point charge at distance $|z|$ from the observation point.

(b) At $|z| \ll R$, on either side of the disk, the field of the disk must approach that of a charge sheet of very large (infinite) extent. But that field is $\pm\sigma_o/2\epsilon_o$. We find, indeed, that in the limit $|z| \rightarrow 0$, (8) yields this limiting result.

Superposition Integral for Line Charge Density. Another special case of the general superposition integral, (3), pertains to fields from charge distributions that are confined to the neighborhoods of lines. In practice, dimensions of interest are large compared to the cross-sectional dimensions of the area A' of the charge distribution. In that case, the situation is as depicted in Fig. 4.5.4, and in the integration over the cross-section the distance from source to observer is essentially constant. Thus, the superposition integral, (3), becomes

$$\Phi(\mathbf{r}) = \int_{L'} \frac{dl'}{4\pi\epsilon_o|r-r'|}\int_{A'} \rho(\mathbf{r}')da' \tag{11}$$

In view of the definition of the line charge density, (1.3.10), this expression becomes

$$\boxed{\Phi(\mathbf{r}) = \int_{L'} \frac{\lambda_l(\mathbf{r}')dl'}{4\pi\epsilon_o|r-r'|}} \tag{12}$$

Example 4.5.2. Field of Collinear Line Charges of Opposite Polarity

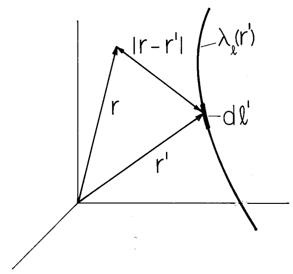


Fig. 4.5.4 An element of line charge at the position $\mathbf{r}'$ gives rise to a potential at the observer location $\mathbf{r}$.

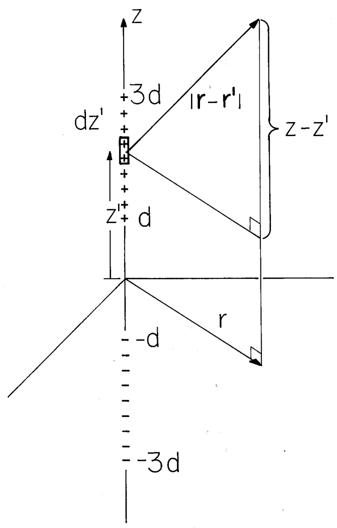


Fig. 4.5.5 Collinear positive and negative line elements of charge symmetrically located on the z axis.

A positive line charge density of magnitude λ_o is uniformly distributed along the z axis between the points $z = d$ and $z = 3d$. Negative charge of the same magnitude is distributed between $z = -d$ and $z = -3d$. The axial symmetry suggests the use of the cylindrical coordinates defined in Fig. 4.5.5.

The distance from an element of charge $\lambda_o dz'$ to an arbitrary observer point (r, z) is

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 + (z - z')^2} \tag{13}$$

Thus, the line charge form of the superposition integral, (12), becomes

$$\Phi = \frac{\lambda_o}{4\pi\epsilon_o}\left(\int_d^{3d} \frac{dz'}{\sqrt{(z - z')^2 + r^2}} - \int_{-3d}^{-d} \frac{dz'}{\sqrt{(z - z')^2 + r^2}}\right) \tag{14}$$

These integrations are carried out to obtain the desired potential distribution

$$\underline{\Phi} = ln \frac{\left(3 - \underline{z} + \sqrt{(3 - \underline{z})^2 + \underline{r}^2}\right)\left(\underline{z} + 1 + \sqrt{(\underline{z} + 1)^2 + \underline{r}^2}\right)}{\left(1 - \underline{z} + \sqrt{(1 - \underline{z})^2 + \underline{r}^2}\right)\left(\underline{z} + 3 + \sqrt{(\underline{z} + 3)^2 + \underline{r}^2}\right)} \tag{15}$$

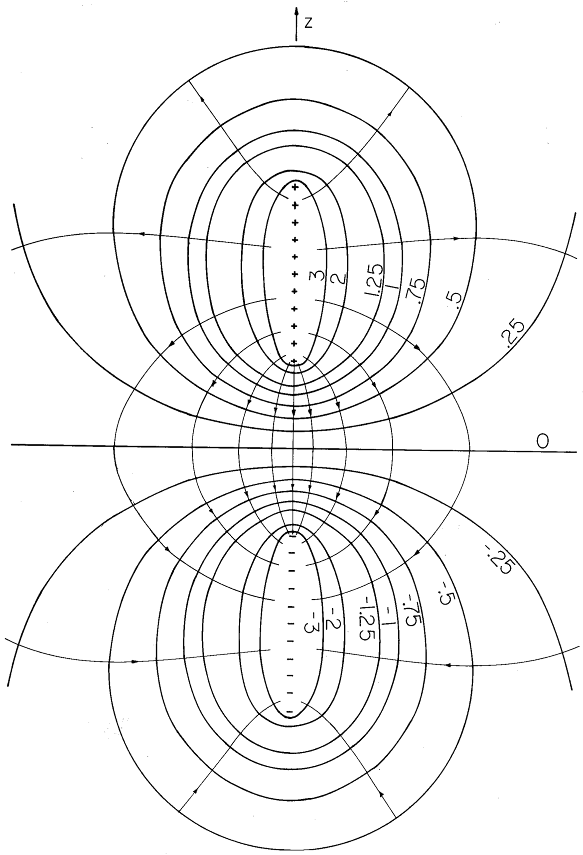


Fig. 4.5.6 Cross-section of equipotential surfaces and lines of electric field intensity for the configuration of Fig. 4.5.5.

Here, lengths have been normalized to d, so that $\underline{z} = z/d$ and $\underline{r} = r/d$. Also, the potential has been normalized such that

$$\underline{\Phi} \equiv \frac{\Phi}{(\lambda_o/4\pi\epsilon_o)} \tag{16}$$

A programmable calculator can be used to evaluate (15), given values of $(\underline{r}, \underline{z})$. The equipotentials in Fig. 4.5.6 were, in fact, obtained in this way, making it possible to sketch the lines of field intensity shown. Remember, the configuration is axisymmetric, so the equipotentials are surfaces generated by rotating the cross-section shown about the z axis.

Two-Dimensional Charge and Field Distributions. In two-dimensional configurations, where the charge distribution uniformly extends from $z = -\infty$ to $z = +\infty$, one of the three integrations of the general superposition integral is carried out by representing the charge by a superposition of line charges, each extending from $z = -\infty$ to $z = +\infty$. The fundamental element of charge, shown in

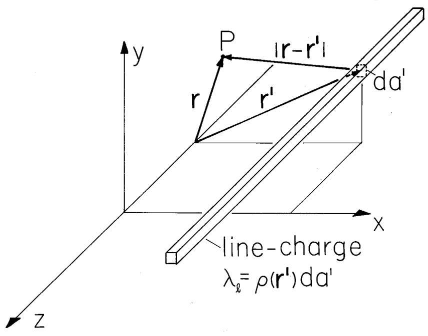


Fig. 4.5.7 For two-dimensional charge distributions, the elementary charge takes the form of a line charge of infinite length. The observer and source position vectors, **r** and **r**′, are two-dimensional vectors.

Fig. 4.5.7, is not the point charge of (1) but rather an infinitely long line charge. The associated potential is not that of a point charge but rather of a line charge.

With the line charge distributed along the z axis, the electric field is given by (1.3.13) as

$$E_r = -\frac{\partial \Phi}{\partial r} = \frac{\lambda_l}{2\pi\epsilon_o r} \tag{17}$$

and integration of this expression gives the potential

$$\Phi = \frac{-\lambda_l}{2\pi\epsilon_o} ln\,\Big(\frac{r}{r_o}\Big) \tag{18}$$

where r_o is a reference radius brought in as a constant of integration. Thus, with da denoting an area element in the plane upon which the source and field depend and $\mathbf{r}$ and $\mathbf{r}'$ the vector positions of the observer and source respectively in that plane, the potential for the incremental line charge of Fig. 4.5.7 is written by making the identifications

$$\lambda_l \rightarrow \rho(\mathbf{r}')da'; \qquad r \rightarrow |\mathbf{r} - \mathbf{r}'| \tag{19}$$

Integration over the given two-dimensional source distribution then gives as the two-dimensional superposition integral

$$\boxed{\Phi = -\int_{S'} \frac{\rho(\mathbf{r}')da'\, ln|\mathbf{r} - \mathbf{r}'|}{2\pi\epsilon_o}} \tag{20}$$

In dealing with charge distributions that extend to infinity in the z direction, the potential at infinity can not be taken as a reference. The potential at an arbitrary finite position can be defined as zero by adding an integration constant to (20).

The following example leads to a result that will be found useful in solving boundary value problems in Sec. 4.8.

Example 4.5.3. Two-Dimensional Potential of Uniformly Charged Sheet

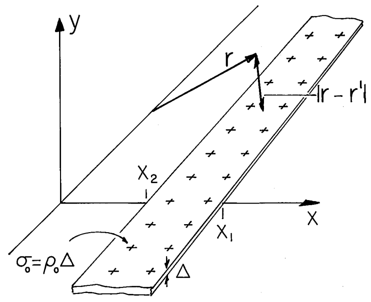


Fig. 4.5.8 Strip of uniformly charged material stretches to infinity in the $\pm z$ directions, giving rise to two-dimensional potential distribution.

A uniformly charged strip lying in the $y = 0$ plane between $x = x_2$ and $x = x_1$ extends from $z = +\infty$ to $z = -\infty$, as shown in Fig. 4.5.8. Because the thickness of the sheet in the y direction is very small compared to other dimensions of interest, the integrand of (20) is essentially constant as the integration is carried out in the y direction. Thus, the y integration amounts to a multiplication by the thickness Δ of the sheet

$$\rho(\mathbf{r}')da' = \rho(\mathbf{r}')\Delta dx = \sigma_s dx \tag{21}$$

and (20) is written in terms of the surface charge density σ_s as

$$\Phi = -\int \frac{\sigma_s(x')dx' \; ln|\mathbf{r} - \mathbf{r}'|}{2\pi\epsilon_o} \tag{22}$$

If the distance between source and observer is written in terms of the Cartesian coordinates of Fig. 4.5.8, and it is recognized that the surface charge density is uniform so that $\sigma_s = \sigma_o$ is a constant, (22) becomes

$$\Phi = -\frac{\sigma_o}{2\pi\epsilon_o}\int_{x_2}^{x_1} ln\sqrt{(x - x')^2 + y^2}dx' \tag{23}$$

Introduction of the integration variable $u = x - x'$ converts this integral to an expression that is readily integrated.

$$\begin{aligned}
\Phi &= \frac{\sigma_o}{2\pi\epsilon_o}\int_{x-x_2}^{x-x_1} ln\sqrt{u^2 + y^2}du \\
&= \frac{\sigma_o}{2\pi\epsilon_o}\Bigg[(x - x_1)ln\sqrt{(x - x_1)^2 + y^2} \\
&\quad - (x - x_2)ln\sqrt{(x - x_2)^2 + y^2} + y\tan^{-1}\Big(\frac{x - x_1}{y}\Big) \\
&\quad - y\tan^{-1}\Big(\frac{x - x_2}{y}\Big) + (x_1 - x_2)\Bigg]
\end{aligned} \tag{24}$$

Two-dimensional distributions of surface charge can be piece-wise approximated by uniformly charged planar segments. The associated potentials are then represented by superpositions of the potential given by (24).

Potential of Uniform Dipole Layer. The potential produced by a dipole of charges $\pm q$ spaced a vector distance $\mathbf{d}$ apart has been found to be given by (4.4.11)

$$\Phi = \frac{\mathbf{p} \cdot \mathbf{i}_{\mathbf{r}'\mathbf{r}}}{4\pi\epsilon_o} \frac{1}{|\mathbf{r} - \mathbf{r}'|^2} \tag{25}$$

where

$$\mathbf{p} \equiv q\mathbf{d}$$

A *dipole layer*, shown in Fig. 4.5.9, consists of a pair of surface charge distributions $\pm\sigma_s$ spaced a distance $\mathbf{d}$ apart. An area element $d\mathbf{a}$ of such a layer, with the direction of $d\mathbf{a}$ (pointing from the negative charge density to the positive one), can be regarded as a differential dipole producing a (differential) potential $d\Phi$

$$d\Phi = \frac{(\sigma_s d) d\mathbf{a} \cdot \mathbf{i}_{\mathbf{r}'\mathbf{r}}}{4\pi\epsilon_o} \frac{1}{|\mathbf{r} - \mathbf{r}'|^2} \tag{26}$$

Denote the surface dipole density by π_s where

$$\pi_s \equiv \sigma_s d \tag{27}$$

and the potential produced by a surface dipole distribution over the surface S is given by

$$\boxed{\Phi = \frac{1}{4\pi\epsilon_o} \int_S \frac{\pi_s \mathbf{i}_{\mathbf{r}'\mathbf{r}}}{|\mathbf{r} - \mathbf{r}'|^2} \cdot d\mathbf{a}} \tag{28}$$

This potential can be interpreted particularly simply if the dipole density is constant. Then π_s can be pulled out from under the integral, and there Φ is equal to $\pi_s/(4\pi\epsilon_o)$ times the integral

$$\Omega \equiv \int_S \frac{\mathbf{i}_{\mathbf{r}'\mathbf{r}} \cdot d\mathbf{a}'}{|\mathbf{r} - \mathbf{r}'|^2} \tag{29}$$

This integral is dimensionless and has a simple geometric interpretation. As shown in Fig. 4.5.9, $\mathbf{i}_{\mathbf{r}'\mathbf{r}} \cdot d\mathbf{a}$ is the area element projected into the direction connecting the source point to the point of observation. Division by $|\mathbf{r} - \mathbf{r}'|^2$ reduces this projected area element onto the unit sphere. Thus, the integrand is the differential solid angle subtended by $d\mathbf{a}$ as seen by an observer at $\mathbf{r}$. The integral, (29), is equal to the solid angle subtended by the surface S when viewed from the point of observation $\mathbf{r}$. In terms of this solid angle,

$$\boxed{\Phi = \frac{\pi_s}{4\pi\epsilon_o} \Omega} \tag{30}$$

Next consider the discontinuity of potential in passing through the surface S containing the dipole layer. Suppose that the surface S is approached from the $+$ side; then, from Fig. 4.5.10, the surface is viewed under the solid angle Ω_o.

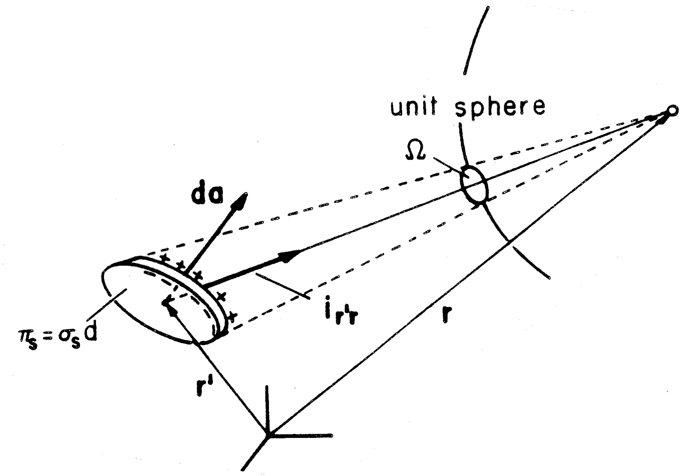


Fig. 4.5.9 The differential solid angle subtended by dipole layer of area da.

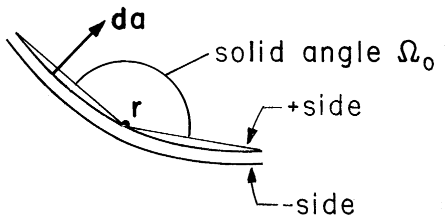


Fig. 4.5.10 The solid angle from opposite sides of dipole layer.

Approached from the other side, the surface subtends the solid angle $-(4\pi - \Omega_o)$. Thus, there is a discontinuity of potential across the surface of

$$\Delta\Phi = \frac{\pi_s}{4\pi\epsilon_o}\Omega_o - \frac{\pi_s}{4\pi\epsilon_o}(\Omega_s - 4\pi) = \frac{\pi_s}{\epsilon_o} \tag{31}$$

Because the dipole layer contains an infinite surface charge density σ_s, the field within the layer is infinite. The "fringing" field, i.e., the external field of the dipole layer, is finite and hence negligible in the evaluation of the internal field of the dipole layer. Thus, the internal field follows directly from Gauss' law under the assumption that the field exists solely between the two layers of opposite charge density (see Prob. 4.5.12). Because contributions to (28) are dominated by π_s in the immediate vicinity of a point $\mathbf{r}$ as it approaches the surface, the discontinuity of potential is given by (31) even if π_s is a function of position. In this case, the tangential $\mathbf{E}$ is not continuous across the interface (Prob. 4.5.12).

4.6 ELECTROQUASISTATIC FIELDS IN THE PRESENCE OF PERFECT CONDUCTORS

In most electroquasistatic situations, the surfaces of metals are equipotentials. In fact, if surrounded by insulators, the surfaces of many other conducting materials

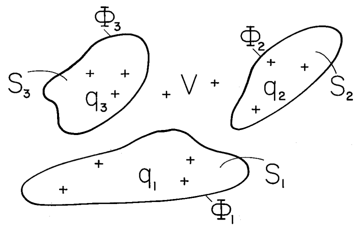


Fig. 4.6.1 Once the superposition principle has been used to determine the potential, the field in a volume V confined by equipotentials is just as well induced by perfectly conducting electrodes having the shapes and potentials of the equipotentials they replace.

also tend to form equipotential surfaces. The electrical properties and dynamical conditions required for representing a boundary surface of a material by an equipotential will be identified in Chap. 7.

Consider the situation shown in Fig. 4.6.l, where three surfaces $S_i,\ i = 1, 2, 3$ are held at the potentials Φ_1, Φ_2, and Φ_3, respectively. These are presumably the surfaces of conducting electrodes. The field in the volume V surrounding the surfaces S_i and extending to infinity is not only due to the charge in that volume but due to charges outside that region as well. Fields normal to the boundaries terminate on surface charges. Thus, as far as the fields in the region of interest are concerned, the sources are the charge density in the volume V (if any) and the surface charges on the surrounding electrodes.

The superposition integral, which is a solution to Poisson's equation, gives the potential when the volume and surface charges are known. In the present statement of the problem, the volume charge densities are known in V, but the surface charge densities are not. The only fact known about the latter is that they must be so distributed as to make the S_i's into equipotential surfaces at the potentials Φ_i.

The determination of the charge distribution for the set of specified equipotential surfaces is not a simple matter and will occupy us in Chap. 5. But many interesting physical situations are uncovered by a different approach. Suppose we are given a potential function $\Phi(\mathbf{r})$. Then any equipotential surface of that potential can be replaced by an electrode at the corresponding potential. Some of the electrode configurations and associated fields obtained in this manner are of great practical interest.

Suppose such a procedure has been followed. To determine the charge on the i-th electrode, it is necessary to integrate the surface charge density over the surface of the electrode.

$$q_i = \int_{S_i} \sigma_s da = \int_{S_i} \epsilon_o \mathbf{E} \cdot d\mathbf{a} \tag{1}$$

In the volume V, the contributions of the surface charges on the equipotential surfaces are exactly equivalent to those of the charge distribution inside the regions enclosed by the surface S_i causing the original potential function. Thus, an alternative to the use of (1) for finding the total charge on the electrode is

$$q_i = \int_{V_i} \rho dv \tag{2}$$

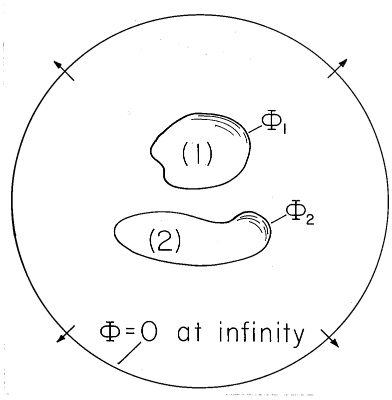


Fig. 4.6.2 Pair of electrodes used to define capacitance.

where V_i is the volume *enclosed* by the surface S_i and ρ is the charge density *inside* S_i associated with the *original* potential.

Capacitance. Suppose the system consists of only two electrodes, as shown in Fig. 4.6.2. The charges on the surfaces of conductors (1) and (2) can be evaluated from the assumedly known solution by using (1).

$$q_1 = \oint_{S_1} \epsilon_o \mathbf{E} \cdot d\mathbf{a}; \qquad q_2 = \oint_{S_2} \epsilon_o \mathbf{E} \cdot d\mathbf{a} \tag{3}$$

Further, there is a charge at infinity of

$$q_\infty = \oint_{S_\infty} \epsilon_o \mathbf{E} \cdot d\mathbf{a} = -q_1 - q_2 \tag{4}$$

The charge at infinity is the negative of the sum of the charges on the two electrodes. This follows from the fact that the field is divergence free, and all field lines originating from q_1 and q_2 must terminate at infinity. Instead of the charges, one could specify the potentials of the two electrodes with respect to infinity. If the charge on electrode 1 is brought to it by a voltage source (battery) that takes charge away from electrode 2 and deposits it on electrode 1, the normal process of charging up two electrodes, then $q_1 = -q_2$. A capacitance C between the two electrodes can be defined as the ratio of charge on electrode 1 divided by the voltage between the two electrodes. In terms of the fields, this definition becomes

$$C = \frac{\oint_{S_1} \epsilon_o \mathbf{E} \cdot d\mathbf{a}}{\int_{(1)}^{(2)} \mathbf{E} \cdot d\mathbf{s}} \tag{5}$$

In order to relate this definition to the capacitance concept used in circuit theory, one further observation must be made. The capacitance relates the charge of one electrode to the voltage between the two electrodes. In general, there may also exist a voltage between electrode 1 and infinity. In this case, capacitances must also be assigned to relate the voltage with regard to infinity to the charges on the electrodes. If the electrodes are to behave as the single terminal-pair element of circuit theory, these capacitances must be negligible. Returning to (5), note that C is independent of the magnitude of the field variables. That is, if the magnitude of the charge distribution is doubled everywhere, it follows from the superposition integral that the potential doubles as well. Thus, the electric field in the numerator and denominator of (3) is doubled everywhere. Each of the integrals therefore also doubles, their ratio remaining constant.

Example 4.6.1. Capacitance of Isolated Spherical Electrodes

A spherical electrode having radius a has a well-defined capacitance C relative to an electrode at infinity. To determine C, note that the equipotentials of a point charge q at the origin

$$\Phi = \frac{q}{4\pi\epsilon_o r} \tag{6}$$

are spherical. In fact, the equipotential having radius $r = a$ has a voltage with respect to infinity of

$$\Phi = v = \frac{q}{4\pi\epsilon_o a} \tag{7}$$

The capacitance is defined as the the net charge on the surface of the electrode per unit voltage, (5). But the net charge found by integrating the surface charge density over the surface of the sphere is simply q, and so the capacitance follows from (7) as

$$C = \frac{q}{v} = 4\pi\epsilon_o a \tag{8}$$

By way of illustrating the conditions necessary for the capacitance to be well defined, consider a pair of spherical electrodes. Electrode (1) has radius a while electrode (2) has radius R. If these are separated by many times the larger of these radii, the potentials in their vicinities will again take the form of (6). Thus, with the voltages v_1 and v_2 defined relative to infinity, the charges on the respective spheres are

$$q_1 = 4\pi\epsilon_o a v_1; \qquad q_2 = 4\pi\epsilon_o R v_2 \tag{9}$$

With all of the charge on sphere (1) taken from sphere (2),

$$q_1 = -q_2 \Rightarrow a v_1 = -R v_2 \tag{10}$$

Under this condition, all of the field lines from sphere (1) terminate on sphere (2). To determine the capacitance of the electrode pair, it is necessary to relate the charge q_1 to the voltage difference between the spheres. To this end, (9) is used to write

$$\frac{q_1}{4\pi\epsilon_o a} - \frac{q_2}{4\pi\epsilon_o R} = v_1 - v_2 \equiv v \tag{11}$$

and because $q_1 = -q_2$, it follows that

$$q_1 = vC; \qquad C \equiv \frac{4\pi\epsilon_o}{\left(\frac{1}{a} + \frac{1}{R}\right)} \tag{12}$$

where C is now the capacitance of one sphere relative to the other.

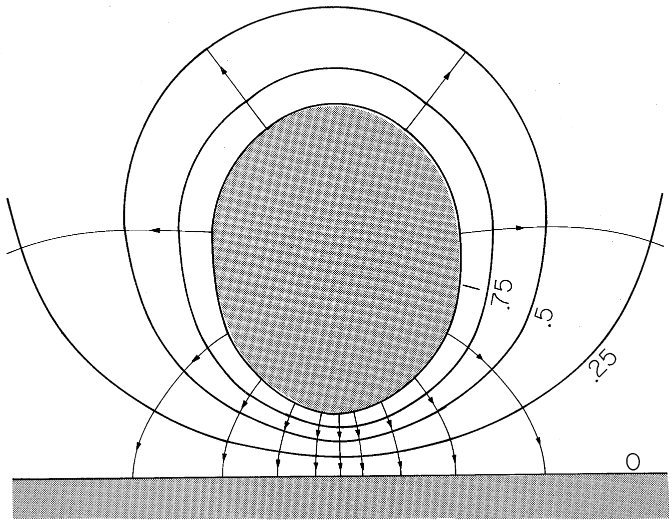


Fig. 4.6.3 The $\underline{\Phi} = 1$ and $\underline{\Phi} = 0$ equipotentials of Fig. 4.5.6 are turned into perfectly conducting electrodes having the capacitance of (4.6.16).

Note that in order to maintain no net charge on the two spheres, it follows from (9), (10), and (12) that the average of the voltages relative to infinity must be retained at

$$\frac{1}{2}(v_1 + v_2) = \frac{1}{2}\left(\frac{q_1}{4\pi\epsilon_o a} + \frac{q_2}{4\pi\epsilon_o R}\right) = \frac{1}{2}v\frac{\left(\frac{1}{a} - \frac{1}{R}\right)}{\left(\frac{1}{a} + \frac{1}{R}\right)} \tag{13}$$

Thus, the average potential must be raised in proportion to the potential difference v.

Example 4.6.2. Field and Capacitance of Shaped Electrodes

The field due to oppositely charged collinear line charges was found to be (4.5.15) in Example 4.5.2. The equipotential surfaces, shown in cross-section in Fig. 4.5.6, are melon shaped and tend to enclose one or the other of the line charge elements.

Suppose that the surfaces on which the normalized potentials are equal to 1 and to 0, respectively, are turned into electrodes, as shown in Fig. 4.6.3. Now the field lines originate on positive surface charges on the upper electrode and terminate on negative charges on the ground plane. By contrast with the original field from the line charges, the field in the region now inside the electrodes is zero.

One way to determine the net charge on one of the electrodes requires that the electric field be found by taking the gradient of the potential, that the unit normal vector to the surface of the electrode be determined, and hence that the surface charge be determined by evaluating $\epsilon_o \mathbf{E} \cdot d\mathbf{a}$ on the electrode surface. Integration of this quantity over the electrode surface then gives the net charge. A far easier way to determine this net charge is to recognize that it is the same as the net charge enclosed by this surface for the original line charge configuration. Thus, the net charge is simply $2d\lambda_l$, and if the potentials of the respective electrodes are taken as $\pm V$, the capacitance is

$$C \equiv \frac{q}{v} = \frac{2d\lambda_l}{V} \tag{14}$$

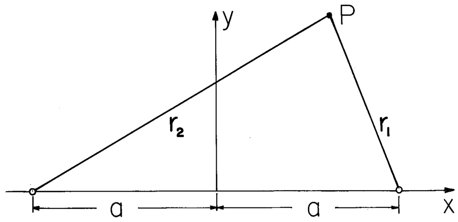


Fig. 4.6.4 Definition of coordinates for finding field from line charges of opposite sign at $x = \pm a$. The displacement vectors are two dimensional and hence in the $x - y$ plane.

For the surface of the electrode in Fig. 4.6.3,

$$\frac{V}{\lambda_l/4\pi\epsilon_o} = 1 \Rightarrow \frac{\lambda_l}{V} = 4\pi\epsilon_o \tag{15}$$

It follows from these relations that the desired capacitance is simply

$$C = 8\pi\epsilon_o d \tag{16}$$

In these two examples, the charge density is zero everywhere between the electrodes. Thus, throughout the region of interest, Poisson's equation reduces to Laplace's equation.

$$\nabla^2\Phi = 0 \tag{17}$$

The solution to Poisson's equation throughout all space is tantamount to solving Laplace's equation in a limited region, subject to certain boundary conditions. A more direct approach to finding such solutions is taken in the next chapter. Even then, it is well to keep in mind that solutions to Laplace's equation in a limited region are solutions to Poisson's equation throughout the entire space, including those regions that contain the charges.

The next example leads to an often-used result, the capacitance per unit length of a two-wire transmission line.

Example 4.6.3. Potential of Two Oppositely Charged Conducting Cylinders

The potential distribution between two equal and opposite parallel line charges has circular cylinders for its equipotential surfaces. Any pair of these cylinders can be replaced by perfectly conducting surfaces so as to obtain the solution to the potential set up between two perfectly conducting parallel cylinders of circular cross-section.

We proceed in the following ways: (a) The potentials produced by two oppositely charged parallel lines positioned at $x = +a$ and $x = -a$, respectively, as shown in Fig. 4.6.4, are superimposed. (b) The intersections of the equipotential surfaces with the $x - y$ plane are circles. The above results are used to find the potential distribution produced by two parallel circular cylinders of radius R with their centers spaced by a distance $2l$. (c) The cylinders carry a charge per unit length λ_l and have a potential difference V, and so their capacitance per unit length is determined.

(a) The potential associated with a single line charge on the z axis is most easily obtained by integrating the electric field, (1.3.13), found from Gauss' integral

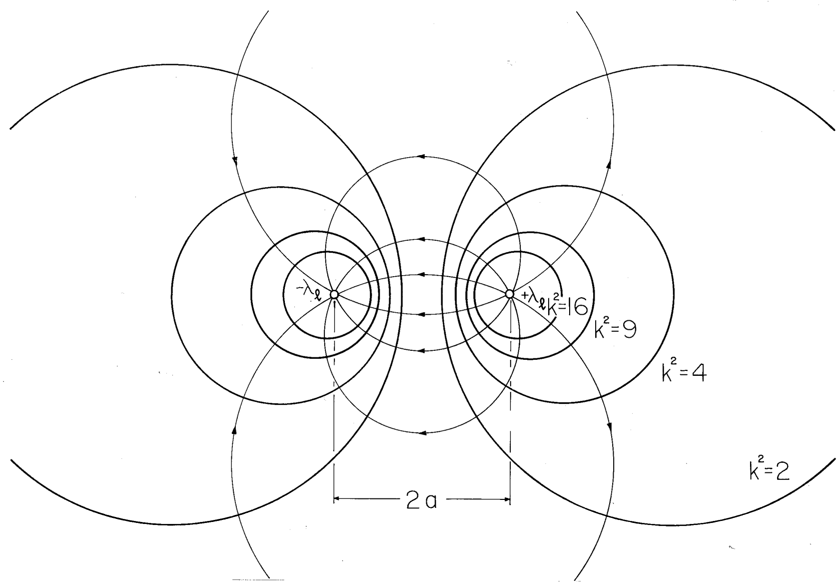


Fig. 4.6.5 Cross-section of equipotentials and electric field lines for line charges.

law. It follows by superposition that the potential for two parallel line charges of charge per unit length $+\lambda_l$ and $-\lambda_l$, positioned at $x = +a$ and $x = -a$, respectively, is

$$\Phi = \frac{-\lambda_l}{2\pi\epsilon_o} ln\ r_1 + \frac{\lambda_l}{2\pi\epsilon_o} ln\ r_2 = \frac{-\lambda_l}{2\pi\epsilon_o} ln\ \frac{r_1}{r_2} \tag{18}$$

Here r_1 and r_2 are the distances of the field point P from the + and − line charges, respectively, as shown in Fig. 4.6.4.

(b) On an equipotential surface, $\Phi = U$ is a constant and the equation for that surface, (18), is

$$\frac{r_2}{r_1} = \exp\Big(\frac{2\pi\epsilon_o U}{\lambda_l}\Big) = \text{const} \tag{19}$$

where in Cartesian coordinates

$$r_2^2 = (a + x)^2 + y^2; \qquad r_1^2 = (a - x)^2 + y^2$$

With the help of Fig. 4.6.4, (19) is seen to represent cylinders of circular cross-section with centers on the x axis. This becomes apparent when the equation is expressed in Cartesian coordinates. The equipotential circles are shown in Fig. 4.6.5 for different values of

$$k \equiv \exp\left(\frac{2\pi\epsilon_o U}{\lambda_l}\right) \tag{20}$$

(c) Given two conducting cylinders whose centers are a distance $2l$ apart, as shown in Fig. 4.6.6, what is the location of the two line charges such that their field

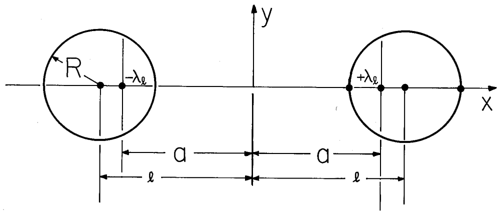


Fig. 4.6.6 Cross-section of parallel circular cylinders with centers at $x = \pm l$ and line charges at $x = \pm a$, having equivalent field.

has equipotentials coincident with these two cylinders? In terms of k as defined by (20), (19) becomes

$$k^2 = \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} \tag{21}$$

This expression can be written as a quadratic function of x and y.

$$x^2 - 2xa\frac{(k^2+1)}{(k^2-1)} + a^2 + y^2 = 0 \tag{22}$$

Equation (22) confirms that the loci of constant potential in the $x - y$ plane are indeed circles. In order to relate the radius and location of these circles to the parameters a and k, note that the expression for a circle having radius R and center on the x axis at $x = l$ is

$$(x-l)^2 + y^2 - R^2 = 0 \Rightarrow x^2 - 2xl + (l^2 - R^2) + y^2 = 0 \tag{23}$$

We can make (22) identical to this expression by setting

$$-2l = -2a\frac{(k^2+1)}{(k^2-1)} \tag{24}$$

and

$$a^2 = l^2 - R^2 \tag{25}$$

Given the spacing $2l$ and radius R of parallel conductors, this last expression can be used to locate the positions of the line charges. It also can be used to see that $(l-a) = R^2/(l+a)$, which can be used with (24) solved for k^2 to deduce that

$$k = \frac{l+a}{R} \tag{26}$$

Introduction of this expression into (20) then relates the potential of the cylinder on the right to the line charge density. The net charge per unit length that is actually on the surface of the right conductor is equal to the line charge density λ_l. With the voltage difference between the cylinders defined as $V = 2U$, we can therefore solve for the capacitance per unit length.

$$C = \frac{\lambda_l}{V} = \frac{\pi\epsilon_o}{ln\left[\frac{l}{R} + \sqrt{(l/R)^2 - 1}\right]} \tag{27}$$

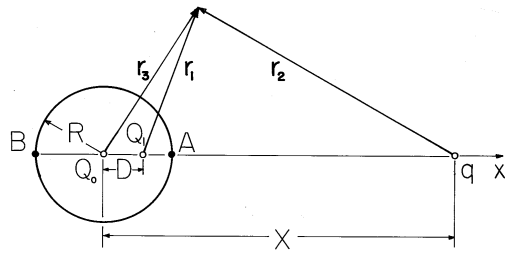


Fig. 4.6.7 Cross-section of spherical electrode having radius R and center at the origin of x axis, showing charge q at $x = X$. Charge Q_1 at $x = D$ makes spherical surface an equipotential, while Q_o at origin makes the net charge on the sphere zero without disturbing the equipotential condition.

Often, the cylinders are wires and it is appropriate to approximate this result for large ratios of l/R.

$$\frac{l}{R} + \sqrt{(l/R)^2 - 1} = \frac{l}{R}\big[1 + \sqrt{1 - (R/l)^2}\big] \simeq \frac{2l}{R} \tag{28}$$

Thus, the capacitance per unit length is approximately

$$\frac{\lambda_l}{V} \equiv C = \frac{\pi\epsilon_o}{ln\ \frac{2l}{R}} \tag{29}$$

This same result can be obtained directly from (18) by recognizing that when $a \gg l$, the line charges are essentially at the center of the cylinders. Thus, evaluated on the surface of the right cylinder where the potential is $V/2$, $r_1 \simeq R$ and $r_2 \simeq 2l$, (18) gives (29).

Example 4.6.4. Attraction of a Charged Particle to a Neutral Sphere

A charged particle facing a conducting sphere induces a surface charge distribution on the sphere. This distribution adjusts itself so as to make the spherical surface an equipotential. In this problem, we take advantage of the fact that two charges of opposite sign produce a potential distribution, one equipotential surface of which is a sphere.

First we find the potential distribution set up by a perfectly conducting sphere of radius R, carrying a net charge Q, and a point charge q at a distance X ($X \geq R$) from the center of the sphere. Then the result is used to determine the force on the charge q exerted by a *neutral* sphere ($Q = 0$)! The configuration is shown in Fig. 4.6.7.

Consider first the potential distribution set up by a point charge Q_1 and another point charge q. The construction of the potential is familiar from Sec. 4.4.

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_o r_2} + \frac{Q_1}{4\pi\epsilon_o r_1} \tag{30}$$

In general, the equipotentials are not spherical. However, the surface of zero potential

$$\Phi(\mathbf{r}) = 0 = \frac{q}{4\pi\epsilon_o r_2} + \frac{Q_1}{4\pi\epsilon_o r_1} \tag{31}$$

is described by

$$\frac{r_2}{r_1} = -\frac{q}{Q_1} \tag{32}$$

and if $q/Q_1 \leq 0$, this represents a sphere. This can be proven by expressing (32) in Cartesian coordinates and noting that in the plane of the two charges, the result is the equation of a circle with its center on the axis intersecting the two charges [compare (19)].

Using this fact, we can apply (32) to the points A and B in Fig. 4.6.7 and eliminate q/Q_1. Taking R as the radius of the sphere and D as the distance of the point charge Q_1 from the center of the sphere, it follows that

$$\frac{R-D}{X-R} = \frac{R+D}{X+R} \Rightarrow D = \frac{R^2}{X} \tag{33}$$

This specifies the distance D of the point charge Q_1 from the center of the equipotential sphere. Introduction of this result into (32) applied to point A gives the (fictitious) charge Q_1.

$$-Q_1 = q\frac{R}{X} \tag{34}$$

With this value for Q_1 located in accordance with (33), the surface of the sphere has zero potential. Without altering its equipotential character, the potential of the sphere can be shifted by positioning another fictitious charge at its center. If the net charge of the spherical conductor is to be Q, then a charge $Q_o = Q - Q_1$ is to be positioned at the center of the sphere. The net field retains the sphere as an equipotential surface, now of nonzero potential. The field outside the sphere is the sought-for solution. With r_3 defined as the distance from the center of the sphere to the point of observation, the field outside the sphere is

$$\Phi = \frac{q}{4\pi\epsilon_o r_2} + \frac{Q_1}{4\pi\epsilon_o r_1} + \frac{Q - Q_1}{4\pi\epsilon_o r_3} \tag{35}$$

With $Q = 0$, the force on the charge follows from an evaluation of the electric field intensity directed along an axis passing through the center of the sphere and the charge q. The self-field of the charge is omitted from this calculation. Thus, along the x axis the potential due to the fictitious charges within the sphere is

$$\Phi = \frac{Q_1}{4\pi\epsilon_o(x-D)} - \frac{Q_1}{4\pi\epsilon_o x} \tag{36}$$

The x directed electric field intensity, and hence the required force, follows as

$$f_x = qE_x = -q\frac{\partial\Phi}{\partial x} = \frac{qQ_1}{4\pi\epsilon_o}\left[\frac{1}{(x-D)^2} - \frac{1}{x^2}\right]_{x=X} \tag{37}$$

In view of (33) and (34), this can be written in terms of the actual physical quantities as

$$f_x = -\frac{q^2 R}{4\pi\epsilon_o X^3}\left[\frac{1}{\left[1-(R/X)^2\right]^2} - 1\right] \tag{38}$$

The field implied by (34) with $Q = 0$ is shown in Fig. 4.6.8. As the charge approaches the spherical conductor, images are induced on the nearest parts of the surface. To

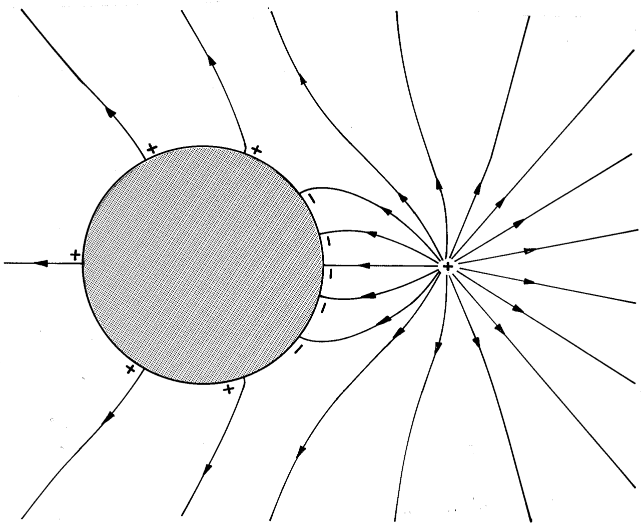


Fig. 4.6.8 Field of point charge in vicinity of neutral perfectly conducting spherical electrode.

keep the net charge zero, charges of opposite sign must be induced on parts of the surface that are more remote from the point charge. The force of attraction results because the charges of opposite sign are closer to the point charge than those of the same sign.

4.7 METHOD OF IMAGES

Given a charge distribution throughout all of space, the superposition integral can be used to determine the potential that satisfies Poisson's equation. However, it is often the case that interest is confined to a limited region, and the potential must satisfy a boundary condition on surfaces bounding this region. In the previous section, we recognized that any equipotential surface could be replaced by a physical electrode, and found solutions to boundary value problems in this way. The art of solving problems in this "backwards" fashion can be remarkably practical but hinges on having a good grasp of the relationship between fields and sources.

Symmetry is often the basis for superimposing fields to satisfy boundary conditions. Consider for example the field of a point charge a distance $d/2$ above a plane conductor, represented by an equipotential. As illustrated in Fig. 4.7.1a, the field $\mathbf{E}_+$ of the charge by itself has a component tangential to the boundary, and hence violates the boundary condition on the surface of the conductor.

To satisfy this condition, forget the conductor and consider the field of two charges of equal magnitude and opposite signs, spaced a distance $2d$ apart. In the symmetry plane, the normal components add while the tangential components cancel. Thus, the composite field is normal to the symmetry plane, as illustrated in the figure. In fact, the configuration is the same as discussed in Sec. 4.4. The

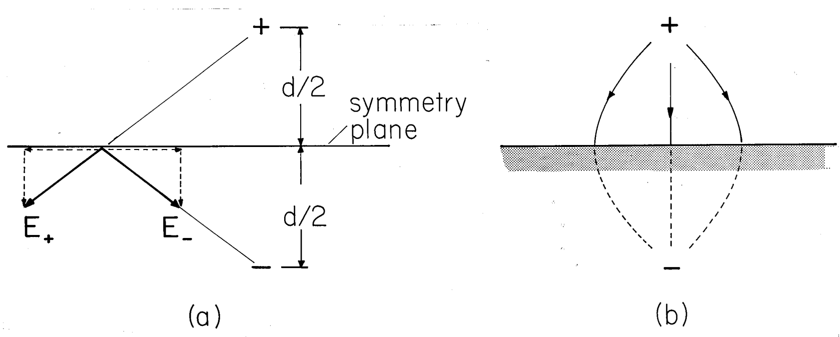


Fig. 4.7.1 (a) Field of positive charge tangential to horizontal plane is canceled by that of symmetrically located image charge of opposite sign. (b) Net field of charge and its image.

fields are as in Fig. 4.4.2a, where now the planar $\Phi = 0$ surface is replaced by a conducting sheet.

This method of satisfying the boundary conditions imposed on the field of a point charge by a plane conductor by using an opposite charge at the mirror image position of the original charge, is called the *method of images*. The charge of opposite sign at the mirror-image position is the "image-charge."

Any superposition of charge pairs of opposite sign placed symmetrically on two sides of a plane results in a field that is normal to the plane. An example is the field of the pair of line charge elements shown in Fig. 4.5.6. With an electrode having the shape of the equipotential enclosing the upper line charge and a ground plane in the plane of symmetry, the field is as shown in Fig. 4.6.3. This identification of a physical situation to go with a known field was used in the previous section. The method of images is only a special case involving planar equipotentials.

To compare the replacement of the symmetry plane by a planar conductor, consider the following demonstration.

Demonstration 4.7.1. Charge Induced in Ground Plane by Overhead Conductor

The circular cylindrical conductor of Fig. 4.7.2, separated by a distance l from an equipotential (grounded) metal surface, has a voltage $U = U_o \cos \omega t$. The field between the conductor and the ground plane is that of a line charge inside the conductor and its image below the ground plane. Thus, the potential is that determined in Example 4.6.3. In the Cartesian coordinates shown, (4.6.18), the definitions of r_1 and r_2 with (4.6.19) and (4.6.25) (where $U = V/2$) provide the potential distribution

$$\Phi = -\frac{\lambda_l}{2\pi\epsilon_o} ln \frac{\sqrt{(a-x)^2 + y^2}}{\sqrt{(a+x)^2 + y^2}} \tag{1}$$

The charge per unit length on the cylinder is [compare (4.6.27)]

$$\lambda_l = CU; \qquad C = \frac{2\pi\epsilon_o}{ln\left[\frac{l}{R} + \sqrt{\left(\frac{l}{R}\right)^2 - 1}\right]} \tag{2}$$

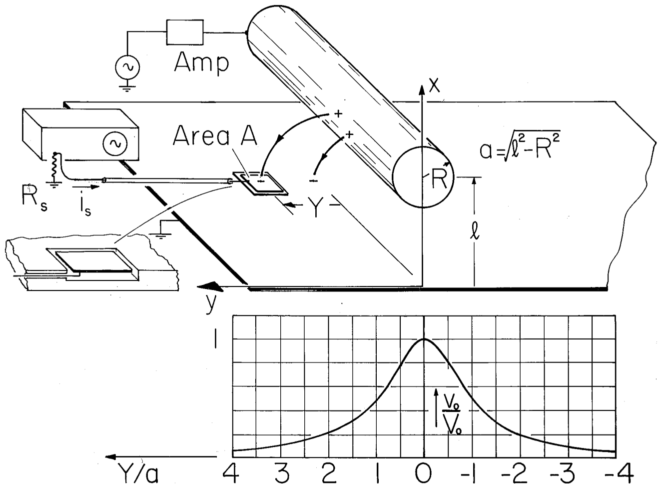


Fig. 4.7.2 Charge induced on ground plane by overhead conductor is measured by probe. Distribution shown is predicted by (4.7.7).

In the actual physical situation, images of this charge are induced on the surface of the ground plane. These can be measured by using a flat probe that is connected through the cable to ground and insulated from the ground plane just below. The input resistance of the oscilloscope is low enough so that the probe surface is at essentially the same (zero) potential as the ground plane. What is the measured current, and hence voltage v_o, as a function of the position Y of the probe?

Given the potential, the surface charge is (1.3.17)

$$\sigma_s = \epsilon_o E_x(x=0) = -\epsilon_o \frac{\partial \Phi}{\partial x}\bigg|_{x=0} \tag{3}$$

Evaluation of this expression using (1) gives

$$\begin{aligned}\sigma_s &= \frac{CU}{2\pi}\left[-\frac{(a-x)}{(a-x)^2+y^2} - \frac{(a+x)}{(a+x)^2+y^2}\right]_{x=0} \\ &= -\frac{CU}{\pi}\frac{a}{a^2+y^2}\end{aligned} \tag{4}$$

Conservation of charge requires that the probe current be the time rate of change of the charge q on the probe surface.

$$i_s = \frac{dq}{dt} \tag{5}$$

Because the probe area is small, the integration of the surface charge over its surface is approximated by the product of the area and the surface charge evaluated at the position Y of its center.

$$q = \int_A \sigma_s dydz \simeq A\sigma_s \tag{6}$$

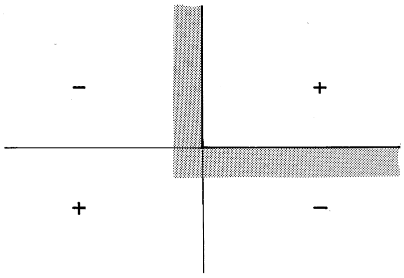


Fig. 4.7.3 Image charges arranged to satisfy equipotential conditions in two planes.

Thus, it follows from (4)–(6) that the induced voltage, $v_o = -R_s i_s$, is

$$v_o = -V_o \sin \omega t \frac{1}{1 + (Y/a)^2}; \qquad V_o \equiv \frac{R_s A C U_o \omega}{a\pi} \tag{7}$$

This distribution of the induced signal with probe position is shown in Fig. 4.7.2.

In the analysis, it is assumed that the plane $x = 0$, including the section of surface occupied by the probe, is constrained to zero potential. In first computing the current to the probe using this assumption and then finding the probe voltage, we are clearly making an approximation that is valid only if the voltage is "small." This can be insured by making the resistance R_s small.

The usual scope resistance is $1M\Omega$. It may come as a surprise that such a resistance is treated here as a short. However, the voltage given by (7) is proportional to the frequency, so the value of acceptable resistance depends on the frequency. As the frequency is raised to the point where the voltage of the probe does begin to influence the field distribution, some of the field lines that originally terminated on the electrode are diverted to the grounded part of the plane. Also, charges of opposite polarity are induced on the other side of the probe. The result is an output signal that no longer increases with frequency. A frequency response of the probe voltage that does not increase linearly with frequency is therefore telltale evidence that the resistance is too large or the frequency too high. In the demonstration, where "desk-top" dimensions are typical, the frequency response is linear to about 100 Hz with a scope resistance of $1M\Omega$.

As the frequency is raised, the system becomes one with two excitations contributing to the potential distribution. The multiple terminal-pair systems treated in Sec. 5.1 start to model the full frequency response of the probe.

Symmetry also motivates the use of image charges to satisfy boundary conditions on more than one planar surface. In Fig. 4.7.3, the objective is to find the field of the point charge in the first quadrant with the planes $x = 0$ and $y = 0$ at zero potential. One image charge gives rise to a field that satisfies one of the boundary conditions. The second is satisfied by introducing an image for the *pair* of charges.

Once an image or a system of images has been found for a point charge, the same principle of images can be used for a continuous charge distribution. The charge density distributions have density distributions of image charges, and the total field is again found using the superposition integral.

Even where symmetry is not involved, charges located outside the region of interest to produce fields that satisfy boundary conditions are often referred to

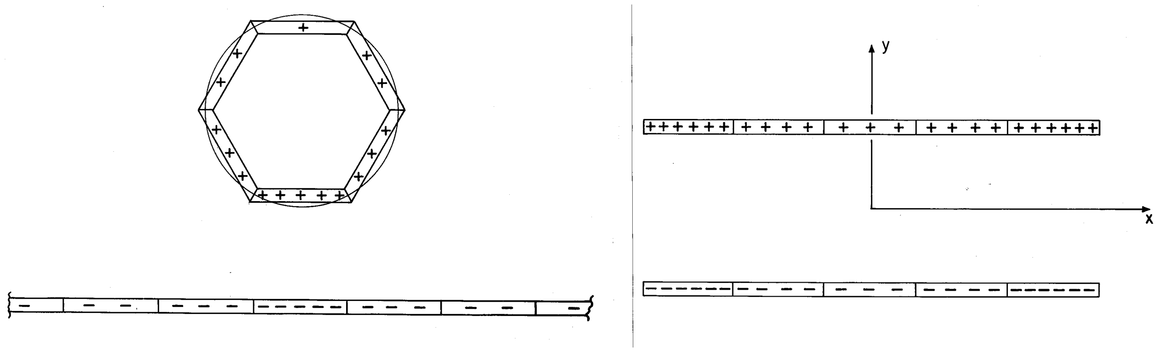


Fig. 4.8.1 (a) Surface of circular cylinder over a ground plane broken into planar segments, each having a uniform surface charge density. (b) Special case where boundaries are in planes $y =$ constant.

as image charges. Thus, the charge Q_1 located within the spherical electrode of Example 4.6.4 can be regarded as the image of q.

4.8 CHARGE SIMULATION APPROACH TO BOUNDARY VALUE PROBLEMS

In solving a boundary value problem, we are in essence finding that distribution of charges external to the region of interest that makes the total field meet the boundary conditions. Commonly, these external charges are actually on the surfaces of conductors bounding or embedded in the region of interest. By way of preparation for the boundary value point of view taken in the next chapter, we consider in this section a direct approach to adjusting surface charges so that the fields meet prescribed boundary conditions on the potential. Analytically, the technique is cumbersome. However, with a computer, it becomes one of a class of powerful numerical techniques[1] for solving boundary value problems.

Suppose that the fields are two dimensional, so that the region of interest can be "enclosed" by a surface that can be approximated by strip segments, as illustrated in Fig. 4.8.1a. This example becomes an approximation to the circular conductor over a ground plane (Example 4.7.1) if the magnitudes of the charges on the strips are adjusted to make the surfaces approximate appropriate equipotentials.

With the surface charge density on each of these strips taken as *uniform*, a "stair-step" approximation to the actual distribution of charge is obtained. By increasing the number of segments, the approximation is refined. For purposes of illustration, we confine ourselves here to boundaries lying in planes of constant y, as shown in Fig. 4.8.1b. Then the potential associated with a single uniformly charged strip is as found in Example 4.5.3.

Consider first the potential due to a strip of width (a) lying in the plane $y = 0$ with its center at $x = 0$, as shown in Fig. 4.8.2a. This is a special case of the configuration considered in Example 4.5.3. It follows from (4.5.24) with $x_1 = a/2$ and $x_2 = -a/2$ that the potential at the observer location (x, y) is

$$\Phi(x, y) = \sigma_o S(x, y) \tag{1}$$

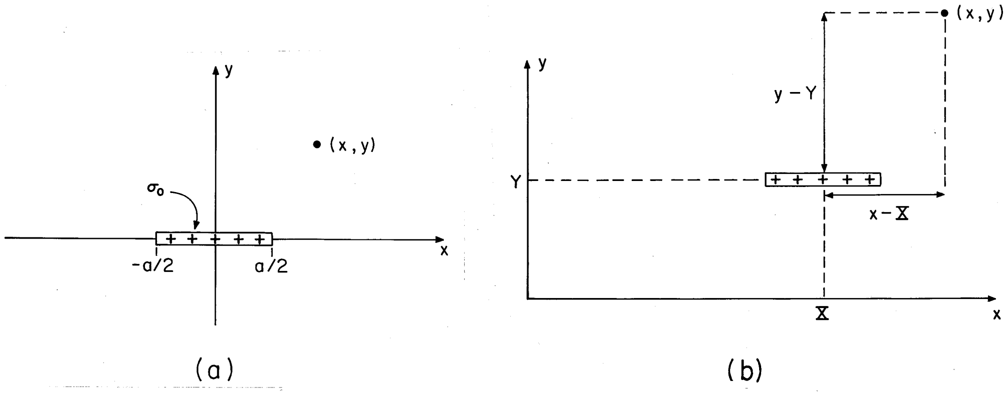


Fig. 4.8.2 (a) Charge strip of Fig. 4.5.8 centered at origin. (b) Charge strip translated so that its center is at (X, Y).

where

$$S(x,y) \equiv \Big[\big(x - \frac{a}{2}\big)\, ln \sqrt{\big(x - \frac{a}{2}\big)^2 + y^2} - \big(x + \frac{a}{2}\big) ln \sqrt{\big(x + \frac{a}{2}\big)^2 + y^2} + y \tan^{-1}\big(\frac{x - a/2}{y}\big) - y \tan^{-1}\big(\frac{x + a/2}{y}\big) + a \Big] / 2\pi\epsilon_o \qquad (2)$$

With the strip located at $(x, y) = (X, Y)$, as shown in Fig. 4.8.2b, this potential becomes

$$\Phi(x,y) = \sigma_o S(x - X, y - Y) \qquad (3)$$

In turn, by superposition we can write the potential due to N such strips, the one having the uniform surface charge density σ_i being located at $(x, y) = (X_i, Y_i)$.

$$\Phi(x,y) = \sum_{i=1}^{N} \sigma_i S_i; \qquad S_i \equiv S(x - X_i, y - Y_i) \qquad (4)$$

Given the surface charge densities, σ_i, the potential at any given location (x, y) can be evaluated using this expression. We assume that the net charge on the strips is zero, so that their collective potential goes to zero at infinity.

With the strips representing surfaces that are constrained in potential (for example, perfectly conducting boundaries), the charge densities are adjusted to meet boundary conditions. Each strip represents part of an electrode surface. The potential V_j at the center of the j-th strip is set equal to the known voltage of the electrode to which it belongs. Evaluating (4) for the center of the j-th strip one obtains

$$\sum_{i=1}^{N} \sigma_i S_{ij} = V_j; \qquad S_{ij} \equiv S(x_j - X_i, y_j - Y_i), \qquad j = 1, \ldots N \qquad (5)$$

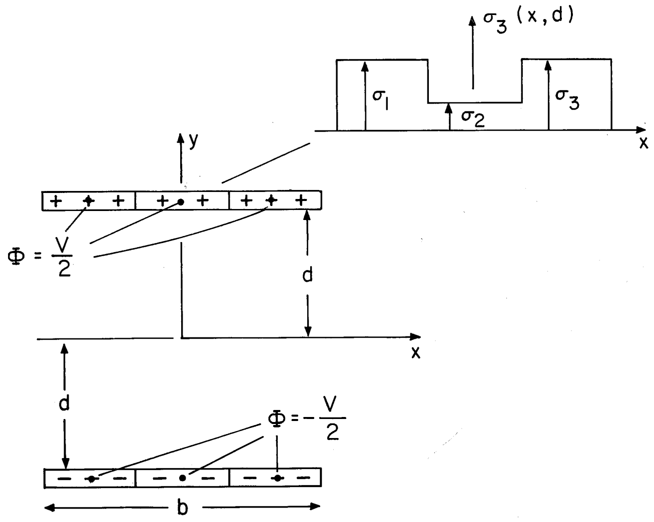


Fig. 4.8.3 Charge distribution on plane parallel electrodes approximated by six uniformly charged strips.

This statement can be made for each of the strips, so that it holds with $j = 1, \dots N$. These relations comprise N equations that are linear in the N unknowns $\sigma_1 \dots \sigma_N$.

$$\begin{pmatrix} C_{11} & C_{12} & \dots \\ C_{21} & & \\ \dots & & C_{NN} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_N \end{pmatrix} = \begin{pmatrix} V_1 \\ \vdots \\ V_N \end{pmatrix} \tag{6}$$

The potentials $V_1 \dots V_N$ on the right are known, so these expressions can be solved for the surface charge densities. Thus, the potential that meets the approximate boundary conditions, (4), has been determined. We have found an approximation to the surface charge density needed to meet the potential boundary condition.

Example 4.8.1. Fields of Finite Width Parallel Plate Capacitor

In Fig. 4.8.3, the parallel plates of a capacitor are divided into six segments. The potentials at the centers of those in the top row are required to be $V/2$, while those in the lower row are $-V/2$. In this simple case of six segments, symmetry gives

$$\sigma_1 = \sigma_3 = -\sigma_4 = -\sigma_6, \qquad \sigma_2 = -\sigma_5 \tag{7}$$

and the six equations in six unknowns, (6) with $N = 6$, reduces to two equations in two unknowns. Thus, it is straightforward to write analytical expressions for the surface charge densities (See Prob. 4.8.1).

The equipotentials and associated surface charge distributions are shown in Fig. 4.8.4 for increasing numbers of charge sheets. The first is a reminder of the distribution of potential for uniformly charged sheets. Shown next are the equipotentials that result from using the six-segment approximation just evaluated. In the last case, 20 segments have been used and the inversion of (6) carried out by means of a computer.

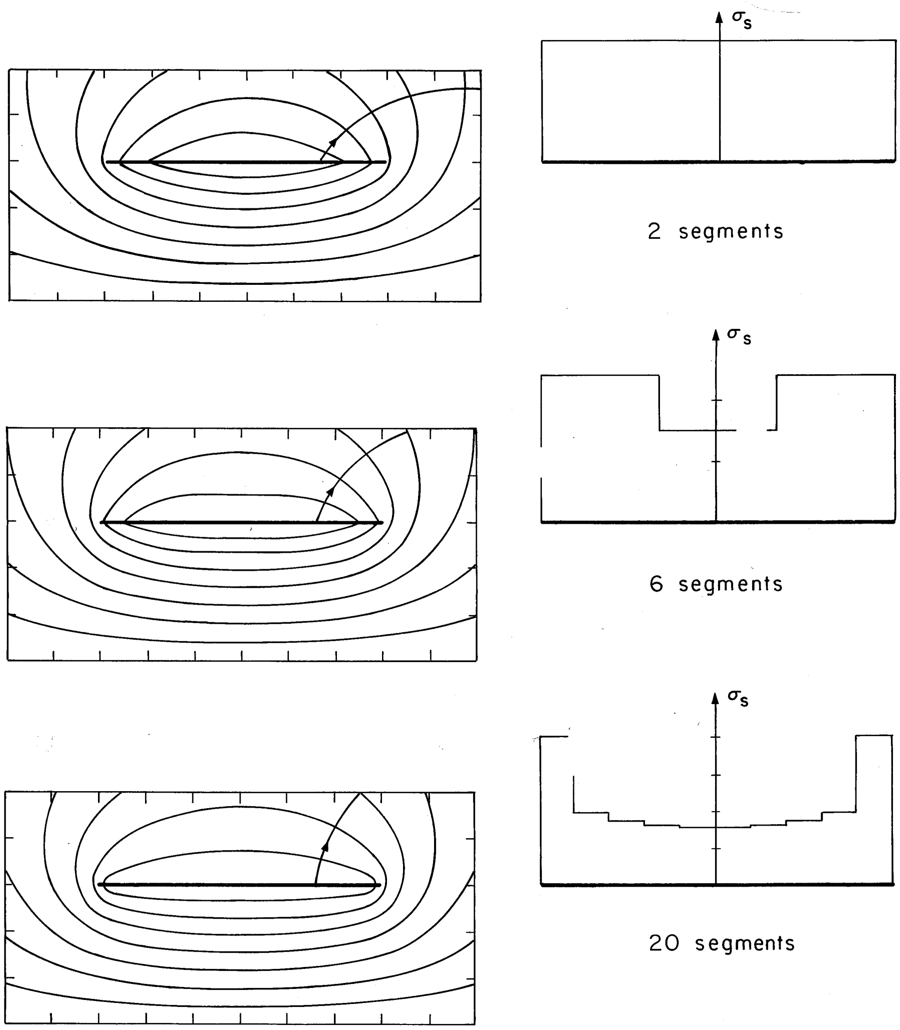


Fig. 4.8.4 Potential distributions using 2, 6, and 20 sheets to approximate the fields of a plane parallel capacitor. Only the fields in the upper half-plane are shown. The distributions of surface charge density on the upper plate are shown to the right.

Note that the approximate capacitance per unit length is

$$C = \frac{1}{V} \sum_{i=1}^{N/2} \frac{b}{(N/2)} \sigma_i \tag{8}$$

This section shows how the superposition integral point of view can be the basis for a numerical approach to solving boundary value problems. But as we proceed to a more direct approach to boundary value problems, it is especially important to profit from the physical insight inherent in the method used in this section.

We have found a mathematical procedure for adjusting the distributions of surface charge so that boundaries are equipotentials. Conducting surfaces surrounded by insulating material tend to become equipotentials by similarly redistributing their surface charge. For example, consider how the surface charge redistributes itself on the parallel plates of Fig. 4.8.4. With the surface charge uniformly distributed, there is a strong electric field tangential to the surface of the plate. In the upper plate, the charges move radially outward in response to this tangential field. Thus, the charge redistributes itself as shown in the subsequent cases. The correct distribution of surface charge density is the one that makes this tangential electric field approach zero, which it is when the surfaces become equipotentials. Thus, the surface charge density is higher near the edges of the plates than it is in the middle. The additional surface charges near the edges result in just that inward-directed electric field which is needed to make the net field perpendicular to the surfaces of the electrodes.

We will find in Sec. 8.6 that the solution to a class of two-dimensional MQS boundary value problems is completely analogous to that for EQS systems of perfect conductors.

4.9 SUMMARY

The theme in this chapter is set by the two equations that determine $\mathbf{E}$, given the charge density ρ. The first of these, (4.0.1), requires that $\mathbf{E}$ be irrotational. Through the representation of $\mathbf{E}$ as the negative gradient of the electric potential, Φ, it is effectively integrated.

$$\mathbf{E} = -\nabla\Phi \tag{1}$$

This gradient operator, determined in Cartesian coordinates in Sec. 4.1 and found in cylindrical and spherical coordinates in the problems of that section, is summarized in Table I. The associated gradient integral theorem, (4.1.16), is added for reference to the integral theorems of Gauss and Stokes in Table II.

The substitution of (1) into Gauss' law, the second of the two laws forming the theme of this chapter, gives Poisson's equation.

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_o} \tag{2}$$

The Laplacian operator on the left, defined as the divergence of the gradient of Φ, is summarized in the three standard coordinate systems in Table I.

It follows from the linearity of (2) that the potential for the superposition of charge distributions is the superposition of potentials for the individual charge distributions. The potentials for dipoles and other singular charge distributions are therefore found by superimposing the potentials of point or line charges. The superposition integral formalizes the determination of the potential, given the distribution of charge. With the surface and line charges recognized as special (singular) volume charge densities, the second and third forms of the superposition integral summarized in Table 4.9.1 follow directly from the first. The fourth is convenient if the source and field are two dimensional.

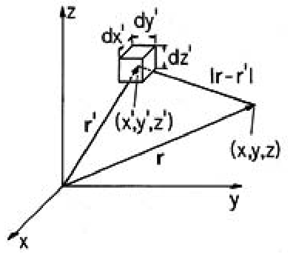
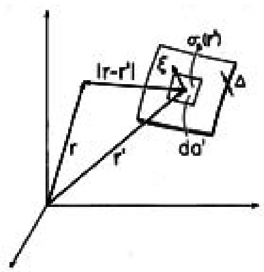
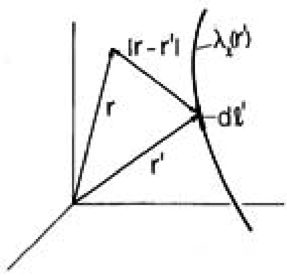
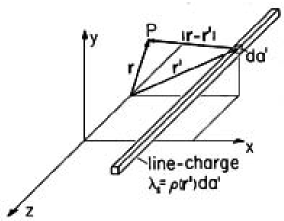
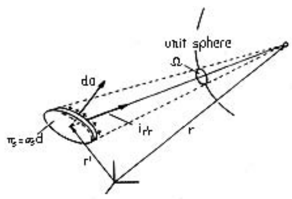
Through Sec. 4.5, the charge density is regarded as given throughout all space. From Sec. 4.6 onward, a shift is made toward finding the field in confined regions of space bounded by surfaces of constant potential. At first, the approach is opportunistic. Given a solution, what problems have been solved? However, the numerical convolution method of Sec. 4.8 is a direct and practical approach to solving boundary value problems with arbitrary geometry.

REFERENCES

[1] R. F. Harrington, **Field Computation by Moment Methods**, MacMillan, NY (1968).

TABLE 4.9.1

SUPERPOSITION INTEGRALS FOR ELECTRIC POTENTIAL

Volume Charge (4.5.3)	$\Phi = \int_{V'} \frac{\rho(\mathbf{r}')dv'}{4\pi\epsilon_o\lvert\mathbf{r}-\mathbf{r}'\rvert}$	
Surface Charge (4.5.5)	$\Phi = \oint_{A'} \frac{\sigma_s(\mathbf{r}')da'}{4\pi\epsilon_o\lvert\mathbf{r}-\mathbf{r}'\rvert}$	
Line Charge (4.5.12)	$\Phi = \int_{L'} \frac{\lambda_l(\mathbf{r}')dl'}{4\pi\epsilon_o\lvert\mathbf{r}-\mathbf{r}'\rvert}$	
Two-dimensional (4.5.20)	$\Phi = -\int_{S'} \frac{\rho(\mathbf{r}')ln\lvert\mathbf{r}-\mathbf{r}'\rvert da'}{2\pi\epsilon_o}$	
Double-layer (4.5.28)	$\Phi = \frac{\pi_s}{4\pi\epsilon_o}\Omega$ $\Omega \equiv \int_S \frac{\mathbf{i}_{r'r}\cdot d\mathbf{a}}{\lvert\mathbf{r}-\mathbf{r}'\rvert^2}$	

PROBLEMS

4.1 Irrotational Field Represented by Scalar Potential: The Gradient Operator and Gradient Integral Theorem

4.1.1 Surfaces of constant Φ that are spherical are given by

$$\Phi = \frac{V_o}{a^2}(x^2 + y^2 + z^2) \tag{a}$$

For example, the surface at radius a has the potential V_o.

(a) In Cartesian coordinates, what is grad(Φ)?
(b) By the definition of the gradient operator, the unit normal $\mathbf{n}$ to an equipotential surface is

$$\mathbf{n} = \frac{\nabla\Phi}{|\nabla\Phi|} \tag{b}$$

Evaluate $\mathbf{n}$ in Cartesian coordinates for the spherical equipotentials given by (a) and show that it is equal to $\mathbf{i_r}$, the unit vector in the radial direction in spherical coordinates.

4.1.2 For Example 4.1.1, carry out the integral of $\mathbf{E}\cdot d\mathbf{s}$ from the origin to $(x, y) = (a, a)$ along the line $y = x$ and show that it is indeed equal to $\Phi(0, 0) - \Phi(a, a)$.

4.1.3 In Cartesian coordinates, three two-dimensional potential functions are

$$\Phi = \frac{V_o x}{a} \tag{a}$$

$$\Phi = \frac{V_o y}{a} \tag{b}$$

$$\Phi = \frac{V_o}{a^2}(x^2 - y^2) \tag{c}$$

(a) Determine $\mathbf{E}$ for each potential.
(b) For each function, make a sketch of Φ and $\mathbf{E}$ using the conventions of Fig. 4.1.3.
(c) For each function, make a sketch using conventions of Fig. 4.1.4.

4.1.4* A cylinder of rectangular cross-section is shown in Fig. P4.1.4. The electric potential inside this cylinder is

$$\Phi = \frac{\rho_o(t)}{\epsilon_o\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]} \sin\frac{\pi}{a}x \sin\frac{\pi}{b}y \tag{a}$$

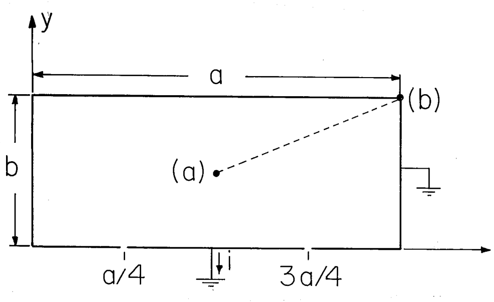


Fig. P4.1.4

where $\rho_o(t)$ is a given function of time.

(a) Show that the electric field intensity is

$$\mathbf{E} = \frac{-\rho_o(t)}{\epsilon_o\big[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\big]}\Big[\frac{\pi}{a}\cos\frac{\pi}{a}x\sin\frac{\pi}{b}y\mathbf{i_x} + \frac{\pi}{b}\sin\frac{\pi}{a}x\cos\frac{\pi}{b}y\mathbf{i_y}\Big] \qquad (b)$$

(b) By direct evaluation, show that $\mathbf{E}$ is irrotational.
(c) Show that the charge density ρ is

$$\rho = \rho_o(t)\sin\frac{\pi}{a}x\sin\frac{\pi}{b}y \qquad (c)$$

(d) Show that the tangential $\mathbf{E}$ is zero on the boundaries.
(e) Sketch the distributions of Φ, ρ, and $\mathbf{E}$ using conventions of Figs. 2.7.3 and 4.1.3.
(f) Compute the line integral of $\mathbf{E}\cdot d\mathbf{s}$ between the center and corner of the rectangular cross-section (points shown in Fig. P4.1.4) and show that it is equal to $\Phi(a/2, b/2, t)$. Why would you expect the integration to give the same result for any path joining the point (a) to any point on the wall?
(g) Show that the net charge inside a length d of the cylinder in the z direction is

$$Q = d\rho_o 4\frac{ab}{\pi^2} \qquad (d)$$

first by integrating the charge density over the volume and then by using Gauss' integral law and integrating $\epsilon_o\mathbf{E}\cdot d\mathbf{a}$ over the surface enclosing the volume.
(h) Find the surface charge density on the electrode at $y = 0$ and use your result to show that the net charge on the electrode segment between $x = a/4$ and $x = 3a/4$ having depth d into the paper is

$$q = -\frac{\sqrt{2}\frac{a}{b}d\rho_o}{\big[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\big]} \qquad (e)$$

(i) Show that the current, $i(t)$, to this electrode segment is

$$i = \frac{\sqrt{2}\frac{ad}{b}\frac{d\rho_o}{dt}}{\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]} \tag{f}$$

4.1.5 Inside the cylinder of rectangular cross-section shown in Fig. P4.1.4, the potential is given as

$$\Phi = \frac{\rho_o(t)}{\epsilon_o\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]} \cos\frac{\pi}{a}x \cos\frac{\pi}{b}y \tag{a}$$

where $\rho_o(t)$ is a given function of time.

(a) Find $\mathbf{E}$.
(b) By evaluating the *curl*, show that $\mathbf{E}$ is indeed irrotational.
(c) Find ρ.
(d) Show that $\mathbf{E}$ is tangential to all of the boundaries.
(e) Using the conventions of Figs. 2.7.3 and 4.1.3, sketch Φ, ρ, and $\mathbf{E}$.
(f) Use $\mathbf{E}$ as found in part (a) to compute the integral of $\mathbf{E}\cdot d\mathbf{s}$ from (a) to (b) in Fig. P4.1.4. Check your answer by evaluating the potential difference between these points.
(g) Evaluate the net charge in the volume by first using Gauss' integral law and integrating $\epsilon_o\mathbf{E}\cdot d\mathbf{a}$ over the surface enclosing the volume and then by integrating ρ over the volume.

4.1.6 Given the potential

$$\Phi = A\sinh mx \sin k_y y \sin k_z z \sin\omega t \tag{a}$$

where A, m, and ω are given constants.

(a) Find $\mathbf{E}$.
(b) By direct evaluation, show that $\mathbf{E}$ is indeed irrotational.
(c) Determine the charge density ρ.
(d) Can you adjust m so that $\rho = 0$ throughout the volume?

4.1.7 The system, shown in cross-section in Fig. P4.1.7, extends to $\pm\infty$ in the z direction. It consists of a cylinder having a square cross-section with sides which are resistive sheets (essentially many resistors in series). Thus, the voltage sources $\pm V$ at the corners of the cylinder produce linear distributions of potential along the sides. For example, the potential between the corners at $(a, 0)$ and $(0, a)$ drops linearly from V to $-V$.

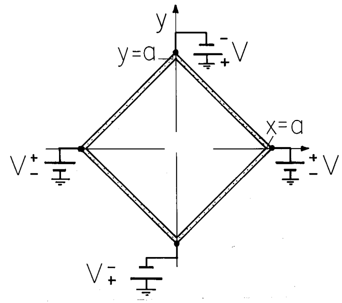


Fig. P4.1.7

(a) Show that the potential inside the cylinder can match that on the walls of the cylinder if it takes the form $A(x^2 - y^2)$. What is A?
(b) Determine $\mathbf{E}$ and show that there is no volume charge density ρ within the cylinder.
(c) Sketch the equipotential surfaces and lines of electric field intensity.

4.1.8 Figure P4.1.8 shows a cross-sectional view of a model for a "capacitance" probe designed to measure the depth h of penetration of a tool into a metallic groove. Both the "tool" and the groove can be considered constant potential surfaces having the potential difference $v(t)$ as shown. An insulating segment at the tip of the tool is used as a probe to measure h. This is done by measuring the charge on the surface of the segment. In the following, we start with a field distribution that can be made to fit the problem, determine the charge and complete some instructive manipulations along the way.

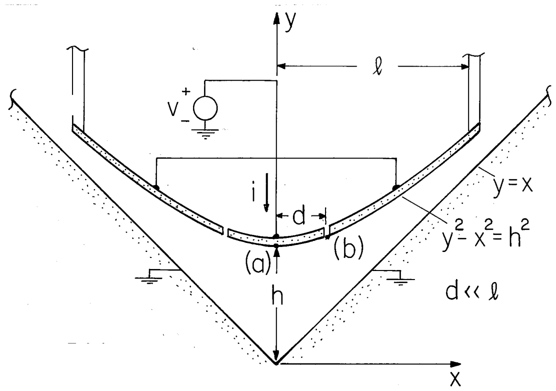


Fig. P4.1.8

(a) Given that the electric field intensity between the groove and tool takes the form

$$\mathbf{E} = C[x\mathbf{i_x} - y\mathbf{i_y}] \qquad (a)$$

show that $\mathbf{E}$ is irrotational and evaluate the coefficient C by computing the integral of $\mathbf{E} \cdot d\mathbf{s}$ between point (a) and the origin.

(b) Find the potential function consistent with (a) and evaluate C by inspection. Check with part (a).

(c) Using the conventions of Figs. 2.7.3 and 4.1.3, sketch lines of constant potential and electric field $\mathbf{E}$ for the region between the groove and the tool surfaces.

(d) Determine the total charge on the insulated segment, given $v(t)$. (Hint: Use the integral form of Gauss' law with a convenient surface S enclosing the electrode.)

4.1.9* In cylindrical coordinates, the incremental displacement vector, given in Cartesian coordinates by (9), is

$$\Delta\mathbf{r} = \Delta r\mathbf{i_r} + r\Delta\phi\mathbf{i}_\phi + \Delta z\mathbf{i_z} \tag{a}$$

Using arguments analogous to (7)–(12), show that the gradient operator in cylindrical coordinates is as given in Table I at the end of the text.

4.1.10* Using arguments analogous to those of (7)–(12), show that the gradient operator in spherical coordinates is as given in Table I at the end of the text.

4.2 Poisson's Equation

4.2.1* In Prob. 4.1.4, the potential Φ is given by (a). Use Poisson's equation to show that the associated charge density is as given by (c) of that problem.

4.2.2 In Prob. 4.1.5, Φ is given by (a). Use Poisson's equation to find the charge density.

4.2.3 Use the expressions for the divergence and gradient in cylindrical coordinates from Table I at the end of the text to show that the Laplacian operator is as summarized in that table.

4.2.4 Use the expressions from Table I at the end of the text for the divergence and gradient in spherical coordinates to show that the Laplacian operator is as summarized in that table.

4.3 Superposition Principle

4.3.1 A current source $I(t)$ is connected in parallel with a capacitor C and a resistor R. Write the ordinary differential equation that can be solved for the voltage $v(t)$ across the three parallel elements. Follow steps analogous to those used in this section to show that if $I_a(t) \Rightarrow v_a(t)$ and $I_b(t) \Rightarrow v_b(t)$, then $I_a(t) + I_b(t) \Rightarrow v_a(t) + v_b(t)$.

4.4 Fields Associated with Charge Singularities

4.4.1* A two-dimensional field results from parallel uniform distributions of line charge, $+\lambda_l$ at $x = d/2,\ y = 0$ and $-\lambda_l$ at $x = -d/2,\ y = 0$, as shown in Fig. P4.4.1. Thus, the potential distribution is independent of z.

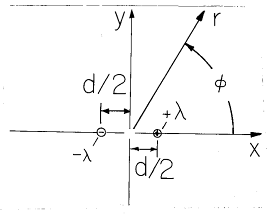


Fig. P4.4.1

(a) Start with the electric field of a line charge, (1.3.13), and determine Φ.
(b) Define the two-dimensional dipole moment as $p_\lambda = d\lambda_l$ and show that in the limit where $d \to 0$ (while this moment remains constant), the electric potential is

$$\Phi = \frac{p_\lambda}{2\pi\epsilon_o}\frac{\cos\phi}{r} \qquad (a)$$

4.4.2* For the configuration of Prob. 4.4.1, consider the limit in which the line charge spacing d goes to infinity. Show that, in polar coordinates, the potential distribution is of the form

$$\Phi \to Ar\cos\phi \qquad (a)$$

Express this in Cartesian coordinates and show that the associated $\mathbf{E}$ is uniform.

4.4.3 A two-dimensional charge distribution is formed by pairs of positive and negative line charges running parallel to the z axis. Shown in cross-section in Fig. P4.4.3, each line is at a distance $d/2$ from the origin. Show that in the limit where $d \ll r$, this potential takes the form $A\cos 2\phi/r^n$. What are the constants A and n?

4.4.4 The charge distribution described in Prob. 4.4.3 is now at infinity ($d \gg r$).

(a) Show that the potential in the neighborhood of the origin takes the form $A(x^2 - y^2)$.
(b) How would you position the line charges so that in the limit where they moved to infinity, the potential would take the form of (4.1.18)?

4.5 Solution of Poisson's Equation for Specified Charge Distributions

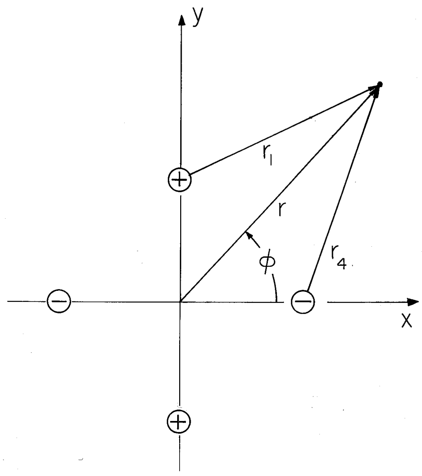


Fig. P4.4.3

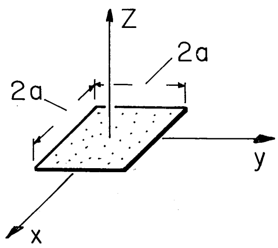


Fig. P4.5.1

4.5.1 The only charge is restricted to a square patch centered at the origin and lying in the $x - y$ plane, as shown in Fig. P4.5.1.

(a) Assume that the patch is very thin in the z direction compared to other dimensions of interest. Over its surface there is a given surface charge density $\sigma_s(x, y)$. Express the potential Φ along the z axis for $z > 0$ in terms of a two-dimensional integral.

(b) For the particular surface charge distribution $\sigma_s = \sigma_o |xy|/a^2$ where σ_o and a are constants, determine Φ along the positive z axis.

(c) What is Φ at the origin?

(d) Show that Φ has a z dependence for $z \gg a$ that is the same as for a point charge at the origin. In this limit, what is the equivalent point charge for the patch?

(e) What is $\mathbf{E}$ along the positive z axis?

4.5.2* The highly insulating spherical shell of Fig. P4.5.2 has radius R and is "coated" with a surface charge density $\sigma_s = \sigma_o \cos\theta$, where σ_o is a given constant.

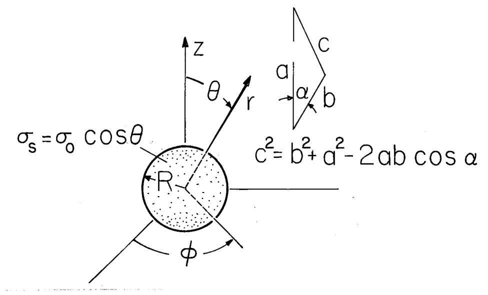


Fig. P4.5.2

(a) Show that the distribution of potential along the z axis in the range $z > R$ is

$$\Phi = \frac{\sigma_o R^3}{3\epsilon_o z^2} \tag{a}$$

[Hint: Remember that for the triangle shown in the figure, the law of cosines gives $c = (b^2 + a^2 - 2ab\cos\alpha)^{1/2}$.]

(b) Show that the potential distribution for the range $z < R$ along the z axis inside the shell is

$$\Phi = \frac{\sigma_o z}{3\epsilon_o} \tag{b}$$

(c) Show that along the z axis, $\mathbf{E}$ is

$$\mathbf{E} = \mathbf{i_z} \begin{cases} \frac{2\sigma_o R^3}{3\epsilon_o z^3} & R < z \\ -\frac{\sigma_o}{3\epsilon_o} & R > z \end{cases} \tag{c}$$

(d) By comparing the z dependence of the potential to that of a dipole polarized in the z direction, show that the equivalent dipole moment is $qd = (4\pi/3)\sigma_o R^3$.

4.5.3 All of the charge is on the surface of a cylindrical shell having radius R and length $2l$, as shown in Fig. P4.5.3. Over the top half of this cylinder at $r = R$ the surface charge density is σ_o (coulomb/m^2), where σ_o is a positive constant, while over the lower half it is $-\sigma_o$.

(a) Find the potential distribution along the z axis.
(b) Determine $\mathbf{E}$ along the z axis.
(c) In the limit where $z \gg l$, show that Φ becomes that of a dipole at the origin. What is the equivalent dipole moment?

4.5.4* A uniform line charge of density λ_l and length d is distributed parallel to the y axis and centered at the point $(x, y, z) = (a, 0, 0)$, as shown in Fig. P4.5.4. Use the superposition integral to show that the potential $\Phi(x, y, z)$ is

$$\Phi = \frac{\lambda_l}{4\pi\epsilon_o} ln \left[\frac{\frac{d}{2} - y + \sqrt{(x-a)^2 + \left(\frac{d}{2} - y\right)^2 + z^2}}{-\frac{d}{2} - y + \sqrt{(x-a)^2 + \left(\frac{d}{2} + y\right)^2 + z^2}} \right] \tag{a}$$

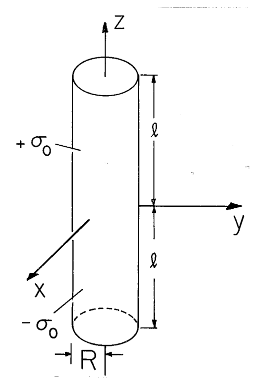


Fig. P4.5.3

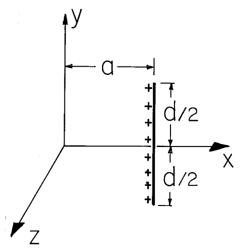


Fig. P4.5.4

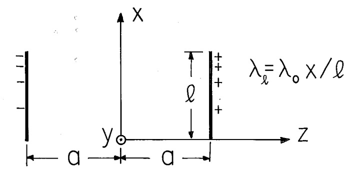


Fig. P4.5.5

4.5.5 Charge is distributed with density $\lambda_l = \pm\lambda_o x/l$ coulomb/m along the lines $z = \pm a$, $y = 0$, respectively, between the points $x = 0$ and $x = l$, as shown in Fig. P4.5.5. Take λ_o as a given charge per unit length and note that λ_l varies from zero to λ_o over the lengths of the line charge distributions. Determine the distribution of Φ along the z axis in the range $0 < z < a$.

4.5.6 Charge is distributed along the z axis such that the charge per unit length $\lambda_l(z)$ is given by

$$\lambda_l = \begin{cases} \frac{\lambda_o z}{a} & -a < z < a \\ 0 & z < -a;\ a < z \end{cases} \qquad (a)$$

Determine Φ and $\mathbf{E}$ at a position $z > a$ on the z axis.

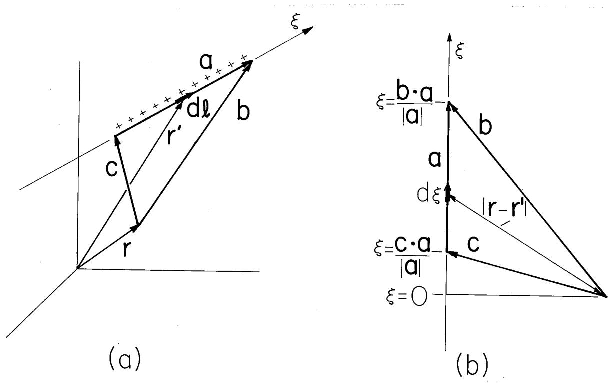


Fig. P4.5.9

4.5.7* A strip of charge lying in the $x-z$ plane between $x = -b$ and $x = b$ extends to $\pm\infty$ in the z direction. On this strip the surface charge density is

$$\sigma_s = \sigma_o \frac{(d-b)}{(d-x)} \qquad (a)$$

where $d > b$. Show that at the location $(x, y) = (d, 0)$, the potential is

$$\Phi(d, 0) = \frac{\sigma_o}{4\pi\epsilon_o}(d-b)\{[ln(d-b)]^2 - [ln(d+b)]^2\} \qquad (b)$$

4.5.8 A pair of charge strips lying in the $x-z$ plane and running from $z = +\infty$ to $z = -\infty$ are each of width $2d$ with their left and right edges, respectively, located on the z axis. The one between the z axis and $(x, y) = (2d, 0)$ has a uniform surface charge density σ_o, while the one between $(x, y) = (-2d, 0)$ and the z axis has $\sigma_s = -\sigma_o$. (Note that the symmetry makes the plane $x = 0$ one of zero potential.) What must be the value of σ_o if the potential at the center of the right strip, where $(x, y) = (d, 0)$, is to be V?

4.5.9* Distributions of line charge can be approximated by piecing together uniformly charged segments. Especially if a computer is to be used to carry out the integration by summing over the fields due to the linear elements of line charge, this provides a convenient basis for calculating the electric potential for a given line distribution of charge. In the following, you determine the potential at an arbitrary observer coordinate **r** due to a line charge that is uniformly distributed between the points **r** + **b** and **r** + **c**, as shown in Fig. P4.5.9a. The segment over which this charge (of line charge density λ_l) is distributed is denoted by the vector **a**, as shown in the figure.

Viewed in the plane in which the position vectors **a**, **b**, and **c** lie, a coordinate ξ denoting the position along the line charge is as shown in Fig. P4.5.9b. The origin of this coordinate is at the position on the line segment collinear with **a** that is nearest to the observer position **r**.

(a) Argue that in terms of ξ, the base and tip of the $\mathbf{a}$ vector are as designated in Fig. P4.5.9b along the ξ axis.

(b) Show that the superposition integral for the potential due to the segment of line charge at $\mathbf{r}'$ is

$$\Phi = \int_{\mathbf{c}\cdot\mathbf{a}/|\mathbf{a}|}^{\mathbf{b}\cdot\mathbf{a}/|\mathbf{a}|} \frac{\lambda_l d\xi}{4\pi\epsilon_o |\mathbf{r}-\mathbf{r}'|} \qquad (a)$$

where

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{\xi^2 + \frac{|\mathbf{b}\times\mathbf{a}|^2}{|\mathbf{a}|^2}} \qquad (b)$$

(c) Finally, show that the potential is

$$\Phi = \frac{\lambda}{4\pi\epsilon_o} ln \frac{\left|\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|} + \sqrt{\left(\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|}\right)^2 + \frac{|\mathbf{b}\times\mathbf{a}|^2}{|\mathbf{a}|^2}}\right|}{\left|\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|} + \sqrt{\left(\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|}\right)^2 + \frac{|\mathbf{b}\times\mathbf{a}|^2}{|\mathbf{a}|^2}}\right|} \qquad (c)$$

(d) A straight segment of line charge has the uniform density λ_o between the points $(x,y,z) = (0,0,d)$ and $(x,y,z) = (d,d,d)$. Using (c), show that the potential $\phi(x,y,z)$ is

$$\Phi = \frac{\lambda_o}{4\pi\epsilon_o} ln \left| \frac{2d - x - y + \sqrt{2[(d-x)^2 + (d-y)^2 + (d-z)^2]}}{-x - y + \sqrt{2[x^2 + y^2 + (d-z)^2]}} \right| \qquad (d)$$

4.5.10* Given the charge distribution, $\rho(\mathbf{r})$, the potential Φ follows from (3). This expression has the disadvantage that to find $\mathbf{E}$, derivatives of Φ must be taken. Thus, it is not enough to know Φ at one location if $\mathbf{E}$ is to be determined. Start with (3) and show that a superposition integral for the electric field intensity is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_o} \int_{V'} \frac{\rho(\mathbf{r}')\mathbf{i}_{r'r} dv'}{|\mathbf{r}-\mathbf{r}'|^2} \qquad (a)$$

where $\mathbf{i}_{r'r}$ is a unit vector directed from the source coordinate $\mathbf{r}'$ to the observer coordinate $\mathbf{r}$. (Hint: Remember that when the gradient of Φ is taken to obtain $\mathbf{E}$, the derivatives are with respect to the observer coordinates with the source coordinates held fixed.) A similar derivation is given in Sec. 8.2, where an expression for the magnetic field intensity $\mathbf{H}$ is obtained from a superposition integral for the vector potential $\mathbf{A}$.

4.5.11 For a better understanding of the concepts underlying the derivation of the superposition integral for Poisson's equation, consider a hypothetical situation where a somewhat different equation is to be solved. The charge density is assumed in part to be a predetermined density $s(x,y,z)$, and in part to be induced at a given point (x,y,z) in proportion to the potential itself at that same point. That is,

$$\rho = s - \epsilon_o \kappa^2 \Phi \tag{a}$$

(a) Show that the expression to be satisfied by Φ is then not Poisson's equation but rather

$$\nabla^2 \Phi - \kappa^2 \Phi = -\frac{s}{\epsilon_o} \tag{b}$$

where $s(x,y,z)$ now plays the role of ρ.

(b) The first step in the derivation of the superposition integral is to find the response to a point source at the origin, defined such that

$$\lim_{R \to 0} \int_0^R s 4\pi r^2 dr = Q \tag{c}$$

Because the situation is then spherically symmetric, the desired response to this point source must be a function of r only. Thus, for this response, (b) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2 \frac{\partial \Phi}{\partial r}\Big) - \kappa^2 \Phi = -\frac{s}{\epsilon_o} \tag{d}$$

Show that for $r \neq 0$, a solution is

$$\Phi = A\frac{e^{-\kappa r}}{r} \tag{e}$$

and use (c) to show that $A = Q/4\pi\epsilon_o$.

(c) What is the superposition integral for Φ?

4.5.12* Because there is a jump in potential across a dipole layer, given by (31), there is an infinite electric field within the layer.

(a) With **n** defined as the unit normal to the interface, argue that this internal electric field is

$$\mathbf{E}_{int} = -\epsilon_o \sigma_s \mathbf{n} \tag{a}$$

(b) In deriving the continuity condition on **E**, (1.6.12), using (4.1.1), it was assumed that **E** was finite everywhere, even within the interface. With a dipole layer, this assumption cannot be made. For example, suppose that a nonuniform dipole layer $\pi_s(x)$ is in the plane $y = 0$. Show that there is a jump in tangential electric field, E_x, given by

$$E_x^a - E_x^b = -\epsilon_o \frac{\partial \pi_s}{\partial x} \tag{b}$$

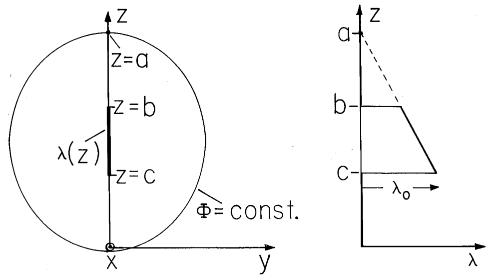


Fig. P4.6.1

4.6 Electroquasistatic Fields in the Presence of Perfect Conductors

4.6.1* A charge distribution is represented by a line charge between $z = c$ and $z = b$ along the z axis, as shown in Fig. P4.6.1a. Between these points, the line charge density is given by

$$\lambda_l = \lambda_o \frac{(a-z)}{(a-c)} \tag{a}$$

and so it has the distribution shown in Fig. P4.6.1b. It varies linearly from the value λ_o where $z = c$ to $\lambda_o(a-b)/(a-c)$ where $z = b$. The only other charges in the system are at infinity, where the potential is defined as being zero.

An equipotential surface for this charge distribution passes through the point $z = a$ on the z axis. [This is the same "a" as appears in (a).] If this equipotential surface is replaced by a perfectly conducting electrode, show that the capacitance of the electrode relative to infinity is

$$C = 2\pi\epsilon_o(2a - c - b) \tag{b}$$

4.6.2 Charges at "infinity" are used to impose a uniform field $\mathbf{E} = E_o\mathbf{i_z}$ on a region of free space. In addition to the charges that produce this field, there are positive and negative charges, of magnitude q, at $z = +d/2$ and $z = -d/2$, respectively, as shown in Fig. P4.6.2. Spherical coordinates (r, θ, ϕ) are defined in the figure.

(a) The potential, radial coordinate and charge are normalized such that

$$\underline{\Phi} = \frac{\Phi}{E_o d}; \qquad \underline{r} = \frac{r}{d}; \qquad \underline{q} = \frac{q}{4\pi\epsilon_o E_o d^2} \tag{a}$$

Show that the normalized electric potential $\underline{\Phi}$ can be written as

$$\underline{\Phi} = -\underline{r}\cos\theta + \underline{q}\{\left[\underline{r}^2 + \frac{1}{4} - \underline{r}\,\cos\theta\right]^{-1/2} - \left[\underline{r}^2 + \frac{1}{4} + \underline{r}\cos\theta\right]^{-1/2}\} \tag{b}$$

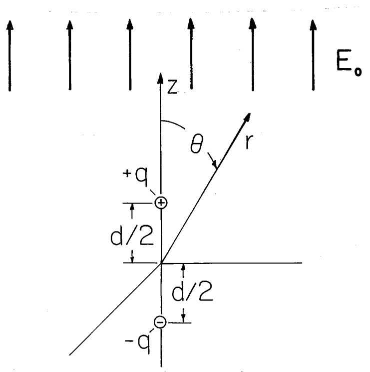


Fig. P4.6.2

(b) There is an equipotential surface $\Phi = 0$ that encloses these two charges. Thus, if a "perfectly conducting" object having a surface taking the shape of this $\Phi = 0$ surface is placed in the initially uniform electric field, the result of part (a) is a solution to the boundary value problem representing the potential, and hence electric field, around the object. The following establishes the shape of the object. Use (b) to find an implicit expression for the radius r at which the surface intersects the z axis. Use a graphical solution to show that there will always be such an intersection with $r > d/2$. For $\underline{q} = 2$, find this radius to two-place accuracy.

(c) Make a plot of the surface $\underline{\Phi} = 0$ in a $\phi =$ constant plane. One way to do this is to use a programmable calculator to evaluate $\underline{\Phi}$ given $\underline{r}$ and θ. It is then straightforward to pick a θ and iterate on $\underline{r}$ to find the location of the surface of zero potential. Make $\underline{q} = 2$.

(d) We expect $\mathbf{E}$ to be largest at the poles of the object. Thus, it is in these regions that we expect electrical breakdown to first occur. In terms of $\underline{E}_o$ and with $\underline{q} = 2$, what is the electric field at the north pole of the object?

(e) In terms of E_o and d, what is the total charge on the northern half of the object. [Hint: A numerical calculation is *not* required.]

4.6.3* For the disk of charge shown in Fig. 4.5.3, there is an equipotential surface that passes through the point $z = d$ on the z axis and encloses the disk. Show that if this surface is replaced by a perfectly conducting electrode, the capacitance of this electrode relative to infinity is

$$C = \frac{2\pi R^2 \epsilon_o}{(\sqrt{R^2 + d^2} - d)} \tag{a}$$

4.6.4 The purpose of this problem is to get an estimate of the capacitance of, and the fields surrounding, the two conducting spheres of radius R shown in Fig. P4.6.4, with the centers separated by a distance h. We construct

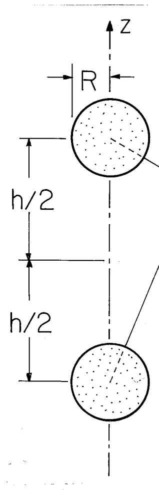


Fig. P4.6.4

an approximate field solution for the field produced by charges $\pm Q$ on the two spheres, as follows:

(a) First we place the charges at the centers of the spheres. If $R \ll h$, the two equipotentials surrounding the charges at $r_1 \approx R$ and $r_2 \approx R$ are almost spherical. If we assume that they *are* spherical, what is the potential difference between the two spherical conductors? Where does the maximum field occur and how big is it?

(b) We can obtain a better solution by noting that a spherical equipotential coincident with the top sphere is produced by a set of three charges. These are the charge $-Q$ at $z = -h/2$ and the two charges inside the top sphere properly positioned according to (33) of appropriate magnitude and total charge $+Q$. Next, we replace the charge $-Q$ by two charges, just like we did for the charge $+Q$. The net field is now due to four charges. Find the potential difference and capacitance for the new field configuration and compare with the previous result. Do you notice that you have obtained higher-order terms in R/h? You are in the process of obtaining a rapidly convergent series in powers of R/h.

4.6.5 This is a continuation of Prob. 4.5.4. The line distribution of charge given there is the only charge in the region $0 \leq x$. However, the $y - z$ plane is now a perfectly conducting surface, so that the electric field is normal to the plane $x = 0$.

(a) Determine the potential in the half-space $0 \leq x$.

(b) For the potential found in part (a), what is the equation for the equipotential surface passing through the point $(x, y, z) = (a/2, 0, 0)$?

(c) For the remainder of this problem, assume that $d = 4a$. Make a sketch of this equipotential surface as it intersects the plane $z = 0$. In doing this, it is convenient to normalize x and y to a by defining $\xi = x/a$ and

$\eta = y/a$. A good way to make the plot is then to compute the potential using a programmable calculator. By iteration, you can quickly zero in on points of the desired potential. It is sufficient to show that in addition to the point of part (a), your curve passes through three well-defined points that suggest its being a closed surface.

(d) Suppose that this closed surface having potential V is actually a metallic (perfect) conductor. Sketch the lines of electric field intensity in the region between the electrode and the ground plane.

(e) The capacitance of the electrode relative to the ground plane is defined as $C = q/V$, where q is the total charge on the surface of the electrode having potential V. For the electrode of part (c), what is C?

4.7 Method of Images

4.7.1* A point charge Q is located on the z axis a distance d above a perfect conductor in the plane $z = 0$.

(a) Show that Φ above the plane is

$$\Phi = \frac{Q}{4\pi\epsilon_o}\left\{\frac{1}{[x^2+y^2+(z-d)^2]^{1/2}} - \frac{1}{[x^2+y^2+(z+d)^2]^{1/2}}\right\} \tag{a}$$

(b) Show that the equation for the equipotential surface $\Phi = V$ passing through the point $z = a < d$ is

$$[x^2+y^2+(z-d)^2]^{-1/2} - [x^2+y^2+(z+d)^2]^{-1/2} = \frac{2a}{d^2-a^2} \tag{b}$$

(c) Use intuitive arguments to show that this surface encloses the point charge. In terms of $a, d,$ and ϵ_o, show that the capacitance relative to the ground plane of an electrode having the shape of this surface is

$$C = \frac{2\pi\epsilon_o(d^2-a^2)}{a} \tag{c}$$

4.7.2 A positive uniform line charge is along the z axis at the center of a perfectly conducting cylinder of square cross-section in the $x-y$ plane.

(a) Give the location and sign of the image line charges.

(b) Sketch the equipotentials and **E** lines in the $x-y$ plane.

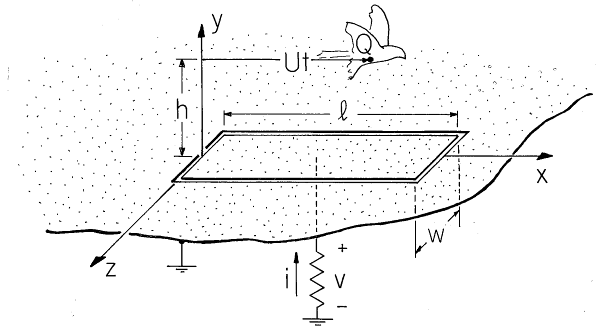


Fig. P4.7.3

4.7.3 When a bird perches on a dc high-voltage power line and then flies away, it does so carrying a net charge.

(a) Why?
(b) For the purpose of measuring this net charge Q carried by the bird, we have the apparatus pictured in Fig. P4.7.3. Flush with the ground, a strip electrode having width w and length l is mounted so that it is insulated from ground. The resistance, R, connecting the electrode to ground is small enough so that the potential of the electrode (like that of the surrounding ground) can be approximated as zero. The bird flies in the x direction at a height h above the ground with a velocity U. Thus, its position is taken as $y = h$ and $x = Ut$.
(c) Given that the bird has flown at an altitude sufficient to make it appear as a point charge, what is the potential distribution?
(d) Determine the surface charge density on the ground plane at $y = 0$.
(e) At a given instant, what is the net charge, q, on the electrode? (Assume that the width w is small compared to h so that in an integration over the electrode surface, the integration in the z direction is simply a multiplication by w.)
(f) Sketch the time dependence of the electrode charge.
(g) The current through the resistor is dq/dt. Find an expression for the voltage, v, that would be measured across the resistance, R, and sketch its time dependence.

4.7.4* Uniform line charge densities $+\lambda_l$ and $-\lambda_l$ run parallel to the z axis at $x = a, y = 0$ and $x = b, y = 0$, respectively. There are no other charges in the half-space $0 < x$. The $y - z$ plane where $x = 0$ is composed of finely segmented electrodes. By connecting a voltage source to each segment, the potential in the $x = 0$ plane can be made whatever we want. Show that the potential distribution you would impose on these electrodes to insure that there is no normal component of $\mathbf{E}$ in the $x = 0$ plane, $E_x(0, y, z)$, is

$$\Phi(0, y, z) = -\frac{\lambda_l}{2\pi\epsilon_o} ln \frac{(a^2 + y^2)}{(b^2 + y^2)} \qquad (a)$$

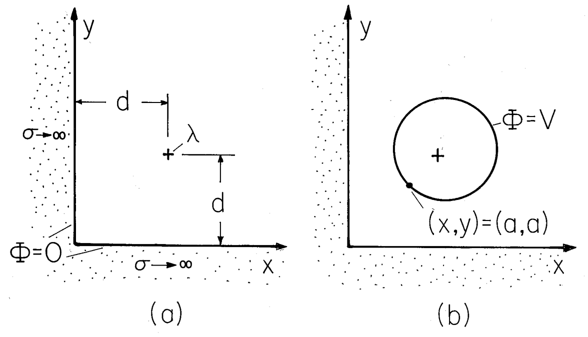


Fig. P4.7.5

4.7.5 The two-dimensional system shown in cross-section in Fig. P4.7.5 consists of a uniform line charge at $x = d, y = d$ that extends to infinity in the $\pm z$ directions. The charge per unit length in the z direction is the constant λ. Metal electrodes extend to infinity in the $x = 0$ and $y = 0$ planes. These electrodes are grounded so that the potential in these planes is zero.

(a) Determine the electric potential in the region $x > 0, y > 0$.

(b) An equipotential surface passes through the line $x = a, y = a(a < d)$. This surface is replaced by a metal electrode having the same shape. In terms of the given constants a, d, and ϵ_o, what is the capacitance per unit length in the z direction of this electrode relative to the ground planes?

4.7.6* The disk of charge shown in Fig. 4.5.3 is located at $z = s$ rather than $z = 0$. The plane $z = 0$ consists of a perfectly conducting ground plane.

(a) Show that for $0 < z$, the electric potential along the z axis is given by

$$\Phi = \frac{\sigma_o}{2\epsilon_o}\Big[\big(\sqrt{R^2 + (z-s)^2} - |z-s|\big) - \big(\sqrt{R^2 + (z+s)^2} - |z+s|\big)\Big] \qquad (a)$$

(b) Show that the capacitance relative to the ground plane of an electrode having the shape of the equipotential surface passing through the point $z = d < s$ on the z axis and enclosing the disk of charge is

$$C = \frac{2\pi R^2 \epsilon_o}{\big[\sqrt{R^2 + (d-s)^2} - \sqrt{R^2 + (d+s)^2} + 2d\big]} \qquad (b)$$

4.7.7 The disk of charge shown in Fig. P4.7.7 has radius R and height h above a perfectly conducting plane. It has a surface charge density $\sigma_s = \sigma_o r/R$. A perfectly conducting electrode has the shape of an equipotential surface

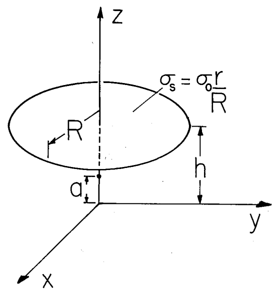


Fig. P4.7.7

that passes through the point $z = a < h$ on the z axis and encloses the disk. What is the capacitance of this electrode relative to the plane $z = 0$?

4.7.8 A straight segment of line charge has the uniform density λ_o between the points $(x, y, z) = (0, 0, d)$ and $(x, y, z) = (d, d, d)$. There is a perfectly conducting material in the plane $z = 0$. Determine the potential for $z \geq 0$. [See part (d) of Prob. 4.5.9.]

4.8 Charge Simulation Approach to Boundary Value Problems

4.8.1 For the six-segment approximation to the fields of the parallel plate capacitor in Example 4.8.1, determine the respective strip charge densities in terms of the voltage V and dimensions of the system. What is the approximate capacitance?