Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.641 Electromagnetic Fields, Forces, and Motion

Final Exam SOLUTIONS
Spring Term, 2003

Problem 1

(A) From Ampere's Law, $B_z = \frac{\mu_0 N i_F}{D}$.

(B) $\lambda = NWTB_z \equiv Li \Rightarrow L = \frac{\mu_0 N^2 WT}{D}$.

(C) Apply Faraday's law to the armature circuit and assume perfectly conducting wires.

$$\oint_C E \cdot dl = -\frac{d}{dt}\int_S \underbrace{B}_{\text{ZERO}} \cdot ds = 0 \Rightarrow$$

$$\underbrace{\int_{(+)}^{(-)} E_y\, dy}_{\text{Fluid}} + \underbrace{\int_{(-)}^{(+)} -\nabla\phi \cdot dl}_{\text{Terminals}} = 0 \Rightarrow E_y W = v_A$$

Ohm's Law $\Rightarrow J = \sigma(E + v \times B) \Rightarrow E_y = \frac{J_y}{\sigma} + vB_z$

$\Rightarrow E_y = \frac{i_A}{\sigma DT} + vB_z \Rightarrow$

$$v_A = \underbrace{\left(\frac{W}{\sigma DT}\right)}_{R} i_A + \underbrace{\left(\frac{\mu_0 N W}{D}\right)}_{G} v\, i_F.$$

(D) Force density $= \bar{J} \times \bar{B} = J_y B_z \hat{x} = \frac{\mu_0 N i_F i_A}{TD}\hat{x}$

$$\text{Power} = J_y B_z v \cdot \underbrace{TDW}_{\text{Volume}} = \frac{\mu_0 N W}{D} i_F i_A v = G i_F i_A v$$

(E) $v_A = R i_A + G v\, i_F$ & $v_F = L \frac{di_F}{dt}$

& $v_F = v_A$ & $i_F = -i_A$

Together $\Rightarrow L\frac{di_F}{dt} = -R i_F + G v\, i_F$

Self excitation $\Rightarrow Gv > R \Rightarrow v > \frac{1}{\mu_0 \sigma N T}$

Problem 2

(A) In general, $\bar{E} = -\nabla\Phi$.

$$\bar{E}_{Fluid} = \hat{x}\,\mathrm{Real}\left\{ -k\left(\hat{\Phi}_A \frac{\cosh(kx)}{\sinh(k\Delta)} - \hat{\Phi}_B \frac{\cosh(k(x-\Delta))}{\sinh(k\Delta)}\right) e^{j(\omega t - kz)} \right\}$$

$$+ \hat{z}\,\mathrm{Real}\left\{ jk\left(\hat{\Phi}_A \frac{\sinh(kx)}{\sinh(k\Delta)} - \hat{\Phi}_B \frac{\sinh(k(x-\Delta))}{\sinh(k\Delta)}\right) e^{j(\omega t - kz)} \right\}$$

$$\bar{E}_{\substack{Free\\Space}} = \hat{x}\,\mathrm{Real}\left\{ k\hat{\Phi}_c e^{-k(x-\Delta)} e^{j(\omega t - kz)} \right\}$$

$$+ \hat{z}\,\mathrm{Real}\left\{ jk\hat{\Phi}_c e^{-k(x-\Delta)} e^{j(\omega t - kz)} \right\}$$

(B) At $x=0$: $\hat{\Phi}_B = \hat{V}$

At $x=\Delta$: $\hat{\Phi}_A = \hat{\Phi}_c$

$$\hat{\rho} = \epsilon_0 k \hat{\Phi}_c + \epsilon k \left(\hat{\Phi}_A \frac{\cosh(k\Delta)}{\sinh(k\Delta)} - \hat{\Phi}_B \frac{1}{\sinh(k\Delta)} \right)$$

(C) $\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)\rho = \sigma E_{Fluid_x}\Big|_{x=\Delta}$

$$j(\omega - kU)\hat{\rho} = -\sigma k \left(\hat{\Phi}_A \frac{\cosh(k\Delta)}{\sinh(k\Delta)} - \hat{\Phi}_B \frac{1}{\sinh(k\Delta)} \right)$$

(D) Substitute $\hat{\Phi}_B = \hat{V}$ and $\hat{\Phi}_c = \hat{\Phi}_A$.

$$\begin{bmatrix} 1 & -k\left(\epsilon_0 + \epsilon \frac{\cosh(k\Delta)}{\sinh(k\Delta)}\right) \\ j(\omega - kU) & \frac{\sigma k \cosh(k\Delta)}{\sinh(k\Delta)} \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{\phi}_A \end{bmatrix} = \begin{bmatrix} -\epsilon \\ \sigma \end{bmatrix} \frac{k\hat{V}}{\sinh(k\Delta)}$$

$$\begin{bmatrix} \hat{\rho} \\ \hat{\phi}_A \end{bmatrix} = \frac{\begin{bmatrix} \frac{\sigma k \cosh(k\Delta)}{\sinh(k\Delta)} & k\left(\epsilon_0 + \epsilon \frac{\cosh(k\Delta)}{\sinh(k\Delta)}\right) \\ -j(\omega - kU) & 1 \end{bmatrix} \begin{bmatrix} \frac{-\epsilon k \hat{V}}{\sinh(k\Delta)} \\ \frac{\sigma k \hat{V}}{\sinh(k\Delta)} \end{bmatrix}}{\frac{\sigma k \cosh(k\Delta)}{\sinh(k\Delta)} + j(\omega - kU)\, k\left(\epsilon_0 + \epsilon \frac{\cosh(k\Delta)}{\sinh(k\Delta)}\right)}$$

$$\hat{\phi}_A = \hat{\phi}_C = \frac{\left(1 + j(\omega - kU)\epsilon/\sigma\right)\hat{V}}{\cosh(k\Delta) + j(\omega - kU)\left(\epsilon_0 \sinh(k\Delta) + \epsilon\cosh(k\Delta)\right)/\sigma}$$

$$\hat{\phi}_B = \hat{V}$$

$$\hat{\rho} = \frac{+\epsilon_0 k \hat{V}}{\cosh(k\Delta) + j(\omega - kU)\left(\epsilon_0 \sinh(k\Delta) + \epsilon \cosh(k\Delta)\right)/\sigma}$$

(E) Use the stress tensor to compute the time and space average surface stress that pumps the fluid.

$$\tau = \epsilon_0 E_x E_z \Big|_{\substack{\text{Free Space} \\ @\, x=\Delta}} - \epsilon E_x E_z \Big|_{\substack{\text{Fluid} \\ @\, x=\Delta}} = \rho E_z \Big|_{x=\Delta}$$

$$\langle \tau \rangle_{z,t} = \frac{1}{2} \,\text{Real} \left\{ jk \hat{\phi}_A \hat{\rho}^* \right\}$$

$$= \frac{-\frac{1}{2}\epsilon_0 k^2 |\hat{V}|^2 (\omega - kU)\,\epsilon/\sigma}{\cosh^2(k\Delta) + (\omega-kU)^2\left(\epsilon_0 \sinh(k\Delta) + \epsilon \cosh(k\Delta)\right)^2/\sigma^2}$$

Problem 3

a) $E_1\,(d+\delta(d,t)) + E_2\,(s-d-\delta(d,t)) = 0$ (Short circuit)

$$\sigma_s = \epsilon_0 (E_2 - E_1) = \epsilon_0 E_2 \left(1 + \frac{(s-d-\delta(d,t))}{d+\delta(d,t)}\right) = \frac{\epsilon_0 E_2\, s}{d+\delta(d,t)}$$

$$E_2 = \frac{\sigma_s}{\epsilon_0 s}\,(d+\delta(d,t)) = C\,(D + d + \delta(d,t)) \Rightarrow D=0,\; C = \frac{\sigma_s}{\epsilon_0 s}$$

$$E_1 = -\frac{E_2\,(s-d-\delta(d,t))}{d+\delta(d,t)} = -\frac{\sigma_s}{\epsilon_0 s}\,(s-d-\delta(d,t)) = A\,(B + d + \delta(d,t))$$

$$B = -s,\; A = \frac{\sigma_s}{\epsilon_0 s}$$

b) $T_{xx}(x=d+\delta(d,t))_+ - T_{xx}(x=d+\delta(d,t))_- = \frac{\epsilon_0}{2}(E_2^2 - E_1^2)$

$$= \frac{\epsilon_0}{2}\left(\frac{\sigma_s}{\epsilon_0 s}\right)^2 \left[(d+\delta(d,t))^2 - (s-(d+\delta(d,t)))^2\right]$$

$$= \frac{\epsilon_0}{2}\left(\frac{\sigma_s}{\epsilon_0 s}\right)^2 \left[-s^2 + 2(d+\delta(d,t))s\right]$$

Another way: $\frac{F_e}{\text{Area}} = \frac{1}{2}\sigma_s(E_1+E_2)$

$$= \frac{1}{2}\sigma_s\left(\frac{\sigma_s}{\epsilon_0 s}\right)\left[d+\delta(d,t) - s + (d+\delta(d,t))\right]$$

$$= \frac{1}{2}\frac{\sigma_s^2}{\epsilon_0 s}\left[2d + 2\delta(d,t) - s\right]$$

$$= F + G\,\delta(d,t)$$

$$\frac{F_e}{\text{Area}} = \frac{1}{2}\frac{\sigma_s^2}{\epsilon_0 s}(2d - s),\quad G = \frac{\sigma_s^2}{\epsilon_0 s}$$

c) $E\frac{\partial^2 \delta_{ss}}{\partial x^2} = 0 \Rightarrow \delta_{ss} = ax + b$

$$\delta_{ss}(x=0) = b = 0$$

at $x = d + \delta_{ss}(d)$: $-E\left.\frac{\partial \delta_{ss}}{\partial x}\right|_{x=d} + \frac{F_e}{\text{Area}} = 0$

$$-Ea + \frac{1}{2}\left(\frac{\sigma_s^2}{\epsilon_0 s}\right)(2d - s + 2ad) = 0$$

$$a\left[\frac{\sigma_s^2 d}{\epsilon_0 s} - E\right] = \frac{1}{2}\left(\frac{\sigma_s^2}{\epsilon_0 s}\right)(s-2d)$$

$$a = \frac{\frac{1}{2}\left(\frac{\sigma_s^2}{\epsilon_0 s}\right)(s-2d)}{\frac{\sigma_s^2 d}{\epsilon_0 s} - E}$$

$$\delta_{ss}(x) = ax = \frac{\frac{1}{2}\left(\frac{\sigma_s^2}{\epsilon_0 s}\right)(s-2d)x}{\frac{\sigma_s^2 d}{\epsilon_0 s} - E}$$

d) $\delta(x,t) = \delta_{ss}(x) + \delta'(x,t)$

$$\rho \frac{\partial^2 \delta'}{\partial t^2} = E \frac{\partial^2 \delta'}{\partial x^2} \quad ; \quad \delta'(x,t) = \mathrm{Re}\left[\hat{\delta}(x) e^{j\omega t}\right]$$

$$\hat{\delta}(x) = A \sin kx + B \cos kx$$

$$\hat{\delta}(x=0) = 0 = B \quad \Rightarrow \hat{\delta}(x) = A \sin kx$$

e) $$-E \frac{\partial \hat{\delta}}{\partial x}\Big|_{x=d} + \frac{\sigma_s^2}{\epsilon_0 s} \hat{\delta}(d) = 0$$

$$-E k \cos kd + \frac{\sigma_s^2}{\epsilon_0 s} \sin kd = 0$$

$$\tan kd = \frac{E(kd)\epsilon_0 s}{\sigma_s^2 d} = H(kd)$$

$$H = \frac{E \epsilon_0 s}{\sigma_s^2 d}$$

f) For $H > 1$, solutions have k and ω real $\Rightarrow$ Stable

For $H < 1$, solutions have k and ω imaginary $\Rightarrow$ Unstable

$$k = j k_i$$

$$\tanh k_i d = H(k_i d)$$

Critical value of σ_s: $H > 1 \Rightarrow \sigma_s = \left[\frac{\epsilon_0 s E}{d}\right]^{1/2}$

Problem 4

a) $i = \frac{V_0}{R_s + 2Z_0}$, $v = \frac{2Z_0 V_0}{R_s + 2Z_0}$

b) $t=0$: $V_+ + V_- = \frac{2Z_0 V_0}{R_s + 2Z_0}$ $\qquad$ $t=\infty$: $V_+ + V_- = V_0 \frac{Z_0}{Z_0 + R_s}$

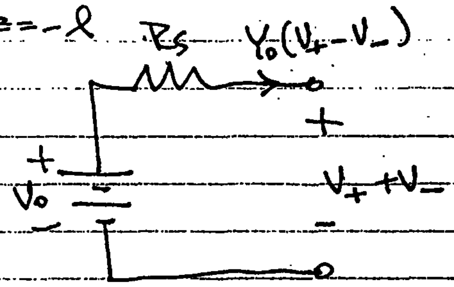
$V_+ - V_- = \frac{V_0 Z_0}{R_s + 2Z_0}$ $\qquad$ $V_+ - V_- = \frac{V_0 Z_0}{R_s + Z_0}$

$V_+ = \frac{1}{2} \frac{V_0 (3Z_0)}{R_s + 2Z_0}$ $\qquad$ $V_+ = \frac{V_0 Z_0}{R_s + Z_0}$

$V_- = \frac{1}{2} \frac{V_0 Z_0}{R_s + 2Z_0}$ $\qquad$ $V_- = 0$

c) $\left.\frac{V_-}{V_+}\right|_{z=0} = 0$ matched end

d) At $z = -l$



$$-V_0 + Y_0 R_s (V_+ - V_-) + V_+ + V_- = 0$$

$$V_+(Y_0 R_s + 1) + V_-(1 - Y_0 R_s) = V_0$$

$$V_+ = \frac{V_0}{1 + Y_0 R_s} + V_- \frac{(Y_0 R_s - 1)}{Y_0 R_s + 1}$$

$$= \frac{V_0}{1 + \frac{R_s}{Z_0}} + V_- \frac{(R_s - Z_0)}{R_s + Z_0}$$

$$= A + B V_-$$

$$A = \frac{V_0}{1 + \frac{R_s}{Z_0}}, \quad B = \frac{R_s - Z_0}{R_s + Z_0} = \Gamma_s$$

source reflection coefficient

e) $\Rightarrow \left. V_+ \right|_{t=0} = \frac{3}{2} \frac{V_0}{2} = \frac{3}{4} V_0$

$\left. V_- \right|_{t=0} = \frac{1}{2} \frac{V_0}{2} = \frac{V_0}{4}$

At $x=0$, $V_- = 0$

At $x=-l$, $V_+ = V_0 - V_-$

Region 1: $V_+ = \frac{3}{4} V_0$, $V_- = \frac{V_0}{4}$

Region 2: $V_+ = \frac{3}{4} V_0$, $V_- = 0$

Region 3: $V_+ = V_0 - \frac{V_0}{4} = \frac{3}{4} V_0$, $V_- = \frac{V_0}{4}$

Region 4: $V_+ = \frac{3}{4} V_0$, $V_- = 0$

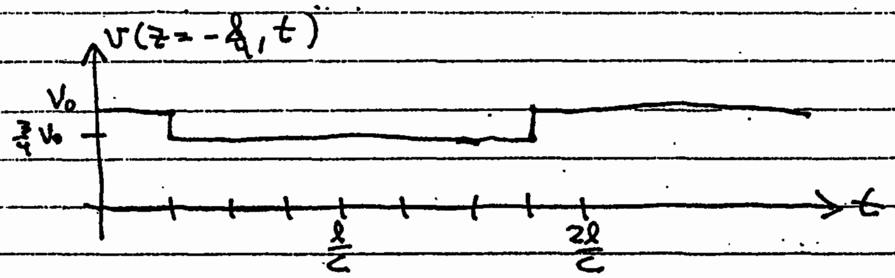
Region 5: $V_+ = \frac{3}{4} V_0$, $V_- = 0$

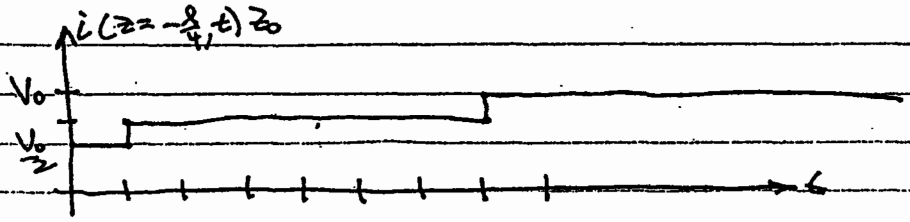
Region 6: $V_+ = V_0$, $V_- = 0$

Region 7: $V_+ = V_0$, $V_- = 0$

Region 8: $V_+ = V_0$, $V_- = 0$

Region 9: $V_+ = V_0$, $V_- = 0$





Problem 5

a) Using method of images, $f = \lambda E = \dfrac{\lambda^2}{2\pi\epsilon_0 (2)(s - \xi(x,t))}$

$$= \frac{\lambda^2}{4\pi\epsilon_0 s\left(1 - \frac{\xi(x,t)}{s}\right)}$$

$$\approx \frac{\lambda^2}{4\pi\epsilon_0 s}\left(1 + \frac{\xi(x,t)}{s}\right)$$

b) $mg = \dfrac{\lambda^2}{4\pi\epsilon_0 s}$ when $\xi(x,t) = 0$

$$\lambda = \sqrt{4\pi\epsilon_0 s m g}$$

c) $m\dfrac{\partial^2 \xi}{\partial t^2} = T\dfrac{\partial^2 \xi}{\partial x^2} - mg + \dfrac{\lambda^2}{4\pi\epsilon_0 s}\left(1 + \dfrac{\xi(x,t)}{s}\right)$

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{T}{m}\frac{\partial^2 \xi}{\partial x^2} + \frac{\lambda^2}{4\pi\epsilon_0 s^2 m}\xi(x,t)$$

$$\xi(x,t) = \mathrm{Re}\left[\hat{\xi} e^{j(\omega t - kx)}\right]$$

$$-\omega^2 = -k^2 v_p^2 + \omega_c^2 \quad ; \quad v_p = \sqrt{\frac{T}{m}} \, , \quad \omega_c^2 = \frac{\lambda^2}{4\pi\epsilon_0 s^2 m}$$

$$\omega^2 = k^2 v_p^2 - \omega_c^2$$

d) $\xi(x,t) = \mathrm{Re}\left[\hat{\xi}(x) e^{j\omega t}\right]$

$\hat{\xi}(x) = A\sin kx + B\cos kx \, , \quad k = \dfrac{\omega^2 + \omega_c^2}{v_p^2}$

$\xi(x=0) = 0 = B$

$$T\frac{\partial \xi}{\partial x}\bigg|_{x=-l} - K\xi(x=-l,t) = 0$$

$$T\cancel{A}k\cos kl + K\cancel{A}\sin kl = 0$$

$$\tan kl = -\frac{T}{K}k = C(kl) \Rightarrow C = -\frac{T}{Kl}$$

e) $K = 0 \Rightarrow \tan k\ell = \infty \Rightarrow k\ell = (2n+1)\frac{\pi}{2}, \quad n = 0, 1, 2, \ldots$

f) $(k\ell)_{min} = \frac{\pi}{2}$

For stability $\omega_c^2 < (k v_p)^2$

$$\frac{\lambda^2}{4\pi\epsilon_0 s^2 m} < \left(\frac{\pi}{2\ell}\right)^2 v_p^2 \Rightarrow \frac{\lambda^2}{4\pi\epsilon_0 s^2 \cancel{m}} < \left(\frac{\pi}{2\ell}\right)^2 \frac{T}{\cancel{m}}$$

From part (b), $\frac{\lambda^2}{4\pi\epsilon_0 s} = mg \Rightarrow$

$$\frac{mg}{s} < \left(\frac{\pi}{2\ell}\right)^2 T$$

$$m < \left(\frac{\pi}{2\ell}\right)^2 \frac{sT}{g}$$