

6.641 Electromagnetic Fields, Forces and Motion
 Quiz 1 Solutions
 March 12, 2003

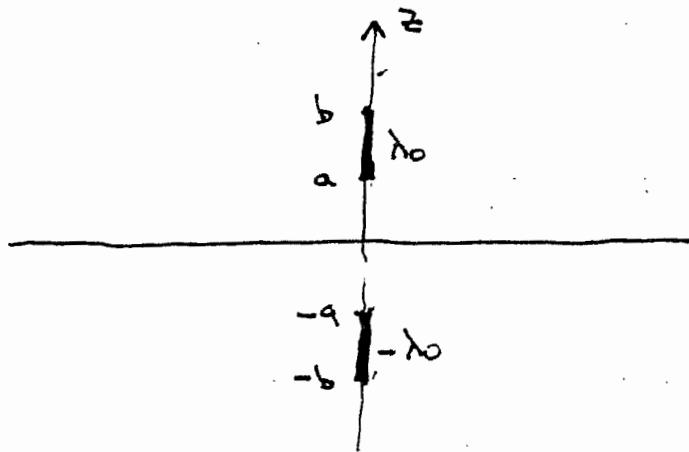


Image line charge $-\lambda_0$ extends from $-a < z < b$. Take charge element $+\lambda_0 dz'$ at $z=z'$. For $z > b$, $z > z'$.

$$a) \quad d\Phi(r=0, z) = \frac{\lambda_0}{4\pi\epsilon_0} \left[\frac{1}{z-z'} - \frac{1}{z+z'} \right] dz'$$

$$\Phi(r=0, z) = \frac{\lambda_0}{4\pi\epsilon_0} \int_a^b \left[\frac{1}{z-z'} - \frac{1}{z+z'} \right] dz'$$

$$= \frac{\lambda_0}{4\pi\epsilon_0} \left[-\ln(z'-z) - \ln(z'+z) \right] \Big|_{z'=a}^b$$

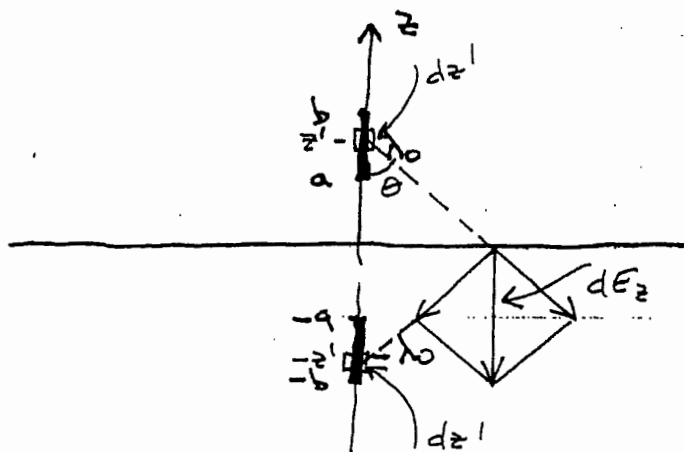
$$= \frac{\lambda_0}{4\pi\epsilon_0} \left[-\ln(b-z) - \ln(b+z) + \ln(a-z) + \ln(a+z) \right]$$

$$= \frac{\lambda_0}{4\pi\epsilon_0} \ln \left[\frac{(a-z)(a+z)}{(b-z)(b+z)} \right] = \frac{\lambda_0}{4\pi\epsilon_0} \ln \left[\frac{a^2 - z^2}{b^2 - z^2} \right]$$

$$b) \quad E_z(r=0, z) = -\frac{\partial \Phi(r=0, z)}{\partial z} = \frac{\lambda_0}{4\pi\epsilon_0} \left[-\frac{1}{b-z} + \frac{1}{b+z} + \frac{1}{a-z} - \frac{1}{a+z} \right]$$

$$= \frac{\lambda_0 z}{2\pi\epsilon_0} \frac{(b^2 - a^2)}{(a^2 - z^2)(b^2 - z^2)}$$

$$c) \quad \frac{z=0}{d} E_z = -\frac{\lambda_0 dz'}{4\pi\epsilon_0 [z'^2 + r^2]} \quad z \cos \theta \quad ; \quad \cos \theta = \frac{z'}{\sqrt{z'^2 + r^2}}$$



$z=0$

$$dE_z = -\frac{\lambda_0 z' dz'}{2\pi\epsilon_0 [z'^2 + r^2]^{3/2}}$$

$$E_z = -\frac{\lambda_0}{2\pi\epsilon_0} \int_a^b \frac{z' dz'}{[z'^2 + r^2]^{3/2}}$$

$$= \frac{\lambda_0}{2\pi\epsilon_0} \left. \frac{1}{\sqrt{z'^2 + r^2}} \right|_{z'=a}^b$$

$$= \frac{\lambda_0}{2\pi\epsilon_0} \left[\frac{1}{\sqrt{b^2 + r^2}} - \frac{1}{\sqrt{a^2 + r^2}} \right]$$

$$\mathcal{V}_S(z=0) = \epsilon_0 E_z(z=0_+) = \frac{\lambda_0}{2\pi} \left[\frac{1}{\sqrt{b^2 + r^2}} - \frac{1}{\sqrt{a^2 + r^2}} \right]$$

a) $p_f(t) = p_0 e^{-t/\tau}$, $\tau = \epsilon / \sigma$

b) $\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} = \frac{p_f(t)}{\epsilon} \Rightarrow E_x = \frac{p_f(t)}{\epsilon} x + C(t)$

$$\int_0^s E_x dx = \frac{p_f(t) s^2}{2\epsilon} + C(t) s = 0$$

$$C(t) = -\frac{p_f(t) s}{2\epsilon}$$

$$E_x = \frac{p_f(t)}{\epsilon} \left(x - \frac{s}{2} \right)$$

c) $\nabla_s(x=0) = \epsilon E_x(x=0) = -\frac{p_f(t) s}{2}$

$$\nabla_s(x=s) = -\epsilon E_x(x=s) = -\frac{p_f(t) s}{2}$$

d)
$$\begin{aligned} \frac{\dot{i}(t)}{A} &= \nabla E_x(x=s) + \epsilon \frac{\partial E_x}{\partial t} \Big|_{x=s} \\ &= \frac{\nabla p_f(t) s}{2\epsilon} + \frac{s}{2\epsilon} \frac{\partial p_f}{\partial t} \\ &= \frac{\sigma s}{2\epsilon} p_0 e^{-t/\tau} + \frac{s}{2} \left(-\frac{1}{\tau} \right) p_0 e^{-t/\tau} \\ &= \frac{\sigma s}{2\epsilon} p_0 e^{-t/\tau} - \frac{\sigma s}{2\epsilon} p_0 e^{-t/\tau} \\ &= 0 \end{aligned}$$

Problem 3

(A) \vec{J} is only in the \hat{y} direction so therefore \vec{A} is only in the \hat{y} direction. The system is symmetric in y , so only x and z dependencies are expected. Therefore, $\vec{A} = A_y(x, z)\hat{y}$. Following $\vec{B} = \mu_0 \vec{H} = \nabla \times \vec{A}$,
 $\vec{B} = \hat{z} \frac{\partial A_y}{\partial x} - \hat{x} \frac{\partial A_y}{\partial z}$. In summary,
 $\vec{B} = B_x(x, z)\hat{x} + B_z(x, z)\hat{z}$.

(B)
$$\vec{H}(\vec{r}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\vec{K}(\vec{r}') \times (\vec{r} - \vec{r}')}{4\pi |\vec{r} - \vec{r}'|^3} dx' dy'$$

$$\vec{K}(\vec{r}') = \begin{cases} K_0 \hat{y} & \text{for } x' > 0 \\ -K_0 \hat{y} & \text{for } x' < 0 \end{cases}$$

$\vec{r} = x\hat{x} + z\hat{z}$ (y dependence is not necessary)

$\vec{r}' = x'\hat{x} + y'\hat{y}$ (integration only in the x - y plane)

$$\begin{aligned} \vec{H}(\vec{r}) &= \int_0^{\infty} \int_{-\infty}^{\infty} \frac{K_0 \hat{y} \times ((x-x')\hat{x} - y'\hat{y} + z\hat{z})}{4\pi [(x-x')^2 + (y')^2 + (z)^2]^{3/2}} dy' dx' \\ &- \int_{-\infty}^0 \int_{-\infty}^{\infty} \frac{K_0 \hat{y} \times ((x-x')\hat{x} - y'\hat{y} + z\hat{z})}{4\pi [(x-x')^2 + (y')^2 + (z)^2]^{3/2}} dy' dx' \end{aligned}$$

$$\vec{H}(x, z) = \int_0^{\infty} \int_{-\infty}^{\infty} \frac{K_0 z \hat{x} - K_0 (x-x') \hat{z}}{2\pi [(x-x')^2 + (y')^2 + (z)^2]^{3/2}} dy' dx'$$

(c) Let K^* be the distribution of the surface

(8) ~~(7)~~

current in the \hat{y} direction at $z=0$ in Parts (A) and (B). Then, the distribution of the surface current in the \hat{y} direction at $z=D$ in this part is given by $K = \frac{1}{2}(K^*(x-L, y) + K^*(x+L, y))$. Here, superposition is used to create the new surface current distribution. The resulting magnetic fields are similarly superimposed, after translation by D in the \hat{z} direction. A second set of superpositions is used to match the boundary conditions at the perfect conductor following the method of images. Hence,

$$\bar{H} = \frac{1}{2} \left[\underbrace{\bar{H}^*(x-L, z-D) + \bar{H}^*(x+L, z-D)}_{\text{Translated and superimposed original solutions}} - \underbrace{\bar{H}^*(x-L, z+D) - \bar{H}^*(x+L, z+D)}_{\text{Image solutions}} \right]$$

where \bar{H}^* is the magnetic field found in Part B.