11

ENERGY, POWER FLOW, AND FORCES

11.0 INTRODUCTION

One way to decide whether a system is electroquasistatic or magnetoquasistatic is to consider the relative magnitudes of the electric and magnetic energy storages. The subject of this chapter therefore makes a natural transition from the quasistatic laws to the complete set of electrodynamic laws. In the order introduced in Chaps. 1 and 2, but now including polarization and magnetization,[1] these are Gauss' law [(6.2.1) and (6.2.3)]

$$\nabla \cdot (\epsilon_o \mathbf{E} + \mathbf{P}) = \rho_u \tag{1}$$

Ampère's law (6.2.11),

$$\nabla \times \mathbf{H} = \mathbf{J}_u + \frac{\partial}{\partial t}(\epsilon_o \mathbf{E} + \mathbf{P}) \tag{2}$$

Faraday's law (9.2.7),

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mu_o(\mathbf{H} + \mathbf{M}) \tag{3}$$

and the magnetic flux continuity law (9.2.2).

$$\nabla \cdot \mu_o(\mathbf{H} + \mathbf{M}) = 0 \tag{4}$$

Circuit theory describes the excitation of a two-terminal element in terms of the voltage v applied between the terminals and the current i into and out of the respective terminals. The power supplied through the terminal pair is vi. One objective in this chapter is to extend the concept of power flow in such a way that power is thought to flow throughout space, and is not associated only with

[1] For polarized and magnetized media at rest.

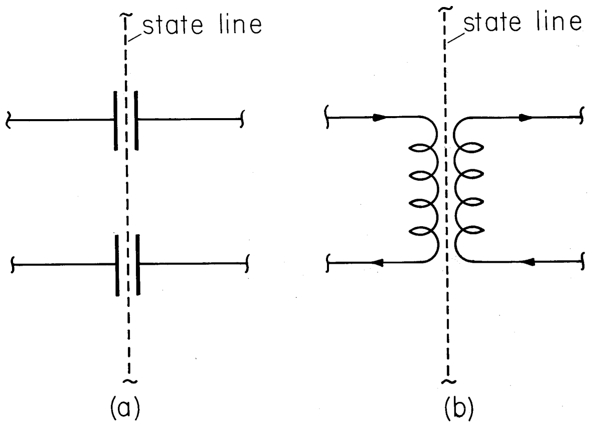

Fig. 11.0.1 If the border between two states passes between the plates of a capacitor or between the windings of a transformer, is there power flow that should be overseen by the federal government?

current flow into and out of terminals. The basis for this extension is the laws of electrodynamics, (1)–(4).

Even if a system can be represented by a circuit, the need for the generalization of the circuit-theoretical power flow concept is apparent if we try to understand how electrical energy is transferred within, rather than between, circuit elements. The limitations of the circuit viewpoint would be crucial to testimony of an expert witness in litigation concerning the authority of the Federal Power Commission[2] to regulate power flowing between states. If the view is taken that passage of current across a border is a prerequisite for power flow, either of the devices shown in Fig. 11.0.1 might be installed at the border to "launder" the power. In the first, the state line passes through the air gap between capacitor plates, while in the second, it separates the primary from the secondary in a transformer.[3] In each case, the current never leaves the state where it is generated. Yet in the examples shown, power generated in one state can surely be consumed in another, and a meaningful discussion of how this takes place must be based on a broadened view of power flow.

From the circuit-theoretical viewpoint, energy storage and rate of energy dissipation are assigned to circuit elements as a whole. Power flowing through a terminal pair is expressed as the product of a potential difference v between the terminals and the current i in one terminal and out of the other. Thus, the terminal voltage v and current i do provide a meaningful description of power flow into a surface S that encloses the circuit shown in Fig. 11.0.2. The surface S does not pass "inside" one of the elements.

Power Flow in a Circuit. For the circuit of Fig. 11.0.2, Kirchhoff's laws

[2] Now the Federal Energy Regulatory Commission.

[3] To be practical, the capacitor would be constructed with an enormous number of interspersed plates, so that in order to keep the state line in the air gap, a gerrymandered border would be required. Contemplation of the construction of a practical transformer, as described in Sec. 9.7, reveals that the state line would be even more difficult to explain in the MQS case.

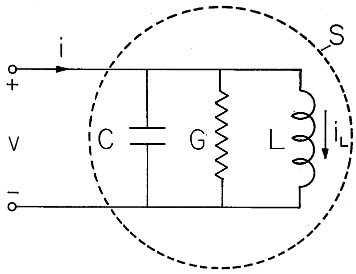

Fig. 11.0.2 Circuit used to review the derivation of energy conservation statement for circuits.

combine with the terminal relations for the capacitor, inductor, and resistor to give

$$i = C\frac{dv}{dt} + i_L + Gv \tag{5}$$

$$v = L\frac{di_L}{dt} \tag{6}$$

Motivated by the objective to obtain a statement involving vi, we multiply the first of these laws by the terminal voltage v. To eliminate the term vi_L on the right, we also multiply the second equation by i_L. Thus, with the addition of the two relations, we obtain

$$vi = vC\frac{dv}{dt} + i_L L\frac{di_L}{dt} + Gv^2 \tag{7}$$

Because L and C are assumed to be constant, we can use the relation $udu = d(\frac{1}{2}u^2)$ to rewrite this expression as

$$vi = \frac{dw}{dt} + Gv^2 \tag{8}$$

where

$$w = \frac{1}{2}Cv^2 + \frac{1}{2}Li_L^2$$

With its origins solely in the circuit laws, (8) can be regarded as giving no more information than inherent in the original laws. However, it gives insights into the circuit dynamics that are harbingers of what can be expected from the more general statement to be derived in Sec. 11.1. These come from considering some extremes.

- If the terminals are open ($i = 0$), and if the resistor is absent ($G = 0$), w is constant. Thus, the *energy* w is conserved in this limiting case. The solution to the circuit laws must lead to the conclusion that the sum of the electric energy $\frac{1}{2}Cv^2$ and the magnetic energy $\frac{1}{2}Li_L^2$ is constant.
- Again, with $G = 0$, but now with a current supplied to the terminals, (8) becomes

$$vi = \frac{dw}{dt} \tag{9}$$

Because the right-hand side is a perfect time derivative, the expression can be integrated to give

$$\int_0^t vidt = w(t) - w(0) \tag{10}$$

Regardless of the details of how the currents and voltage vary with time, the time integral of the *power* vi is solely a function of the initial and final total energies w. Thus, if w were zero to begin with and vi were positive, at some later time t, the total energy would be the positive value given by (10). To remove the total energy from the inductor and capacitor, vi must be reversed in sign until the integration has reduced w to zero. Because the process is reversible, we say that the energy w is *stored* in the capacitor and inductor.

- If the terminals are again open ($i = 0$) but the resistor is present, (8) shows that the stored energy w must decrease with time. Because Gv^2 is positive, this process is not reversible and we therefore say that the energy is *dissipated* in the resistor.

In circuit theory terms, (8) is an example of an energy conservation theorem. According to this theorem, *electrical energy* is *not* conserved. Rather, of the electrical energy supplied to the circuit at the rate vi, part is stored in the capacitor and inductor and indeed conserved, and part is dissipated in the resistor. The energy supplied to the resistor is *not conserved in electrical form.* This energy is dissipated in heat and becomes a new kind of energy, thermal energy.

Just as the circuit laws can be combined to describe the flow of power between the circuit elements, so Maxwell's equations are the basis for a field-theoretical view of power flow. The reasoning that casts the circuit laws into a power flow statement parallels that used in the next section to obtain the more general field-theoretical law, so it is worthwhile to review how the circuit laws are combined to obtain a statement describing power flow.

Overview. The energy conservation theorem derived in the next two sections will also not be a conservation theorem in the sense that electrical energy is conserved. Rather, in addition to accounting for the storage of energy, it will include conversion of energy into other forms as well. Indeed, one of the main reasons for our interest in power flow is the insight it gives into other subsystems of the physical world [e.g. the thermodynamic, chemical, or mechanical subsystems]. This will be evident from the topics of subsequent sections.

The conservation of energy statement assumes as many special forms as there are different constitutive laws. This is one reason for pausing with Sec. 11.1 to summarize the integral and differential forms of the conservation law, regardless of the particular application. We shall reference these expressions throughout the chapter. The derivation of Poynting's theorem, in the first part of Sec. 11.2, is motivated by the form of the general conservation theorem. As subsequent sections evolve, we shall also make continued reference to this law in its general form.

By specializing the materials to Ohmic conductors with linear polarization and magnetization constitutive laws, it is possible to make a clear identification of the origins of electrical energy storage and dissipation in media. Such systems are considered in Sec. 11.3, where the flow of power from source to "sinks" of thermal

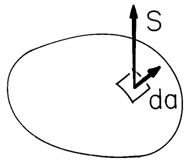

Fig. 11.1.1 Integral form of energy conservation theorem applies to system within arbitrary volume V enclosed by surface S.

dissipation is illustrated. Processes of energy storage and dissipation are developed in greater depth in Secs. 11.4 and 11.5.

Through Sec. 11.5, the assumption is that materials are at rest. In Secs. 11.6 and 11.7, the power input is studied in the presence of motion of materials. These sections illustrate how the energy conservation law is used to determine electric and magnetic forces on macroscopic media. The discussion in these sections is confined to a determination of total forces. Consistent with the field theory point of view is the concept of a distributed force per unit volume, a force density. Rigorous derivations of macroscopic force densities are based on energy arguments paralleling those of Secs. 11.6 and 11.7.

In Sec. 11.8, we shall look at microscopic models of force density distributions that provide a picture of the origin of these distributions. Finally, Sec. 11.9 is an introduction to the macroscopic force densities needed to put electromechanical coupling on a continuum basis.

11.1 INTEGRAL AND DIFFERENTIAL CONSERVATION STATEMENTS

The circuit with theoretical conservation theorem (11.0.8) equates the power flowing into the circuit to the rate of change of the energy stored and the rate of energy dissipation. In a field, theoretical generalization, the energy must be imagined distributed through space with an energy density W (joules/m^3), and the power is dissipated at a local rate of dissipation per unit volume P_d (watts/m^3). The power flows with a density $\mathbf{S}$ (watts/m^2), a vector, so that the power crossing a surface S_a is given by $\int_{S_a} \mathbf{S} \cdot d\mathbf{a}$. With these field-theoretical generalizations, the power flowing into a volume V, enclosed by the surface S must be given by

$$\boxed{-\oint_S \mathbf{S} \cdot d\mathbf{a} = \frac{d}{dt}\int_V W dv + \int_V P_d dv} \tag{1}$$

where the minus sign takes care of the fact that the term on the left is the power flowing *into* the volume.

According to the right-hand side of this equation, this input power is equal to the rate of increase of the total energy stored plus the power dissipation. The total energy is expressed as an integral over the volume of an *energy density*, W. Similarly, the total power dissipation is the integral over the volume of a *power dissipation density* P_d.

The volume is taken as being fixed, so the time derivative can be taken inside the volume integration on the right in (1). With the use of Gauss' theorem, the surface integral on the left is then converted to one over the volume and the term transferred to the right-hand side.

$$\int_V \Big(\nabla \cdot \mathbf{S} + \frac{\partial W}{\partial t} + P_d\Big) dv = 0 \tag{2}$$

Because V is arbitrary, the integrand must be zero and a *differential statement of energy conservation* follows.

$$\boxed{\nabla \cdot \mathbf{S} + \frac{\partial W}{\partial t} + P_d = 0} \tag{3}$$

With an appropriate definition of $\mathbf{S}$, W and P_d, (1) and (3) could describe the flow, storage, and dissipation not only of electromagnetic energy, but of thermal, elastic, or fluid mechanical energy as well. In the next section we will use Maxwell's equations to determine these variables for an electromagnetic system.

11.2 POYNTING'S THEOREM

The objective in this section is to derive a statement of energy conservation from Maxwell's equations in the form identified in Sec. 11.1. The conservation theorem includes the effects of both displacement current and of magnetic induction. The EQS and MQS limits, respectively, can be taken by neglecting those terms having their origins in the magnetic induction $\partial \mu_o(\mathbf{H} + \mathbf{M})/\partial t$ on the one hand, and in the displacement current density $\partial(\epsilon_o \mathbf{E} + \mathbf{P})/\partial t$ on the other.

Ampère's law, including the effects of polarization, is (11.0.2).

$$\nabla \times \mathbf{H} = \mathbf{J}_u + \frac{\partial \epsilon_o \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} \tag{1}$$

Faraday's law, including the effects of magnetization, is (11.0.3).

$$\nabla \times \mathbf{E} = -\frac{\partial \mu_o \mathbf{H}}{\partial t} - \frac{\partial \mu_o \mathbf{M}}{\partial t} \tag{2}$$

These field-theoretical laws play a role analogous to that of the circuit equations in the introductory section. What we do next is also analogous. For the circuit case, we form expressions that are quadratic in the dependent variables. Several considerations guide the following manipulations. One aim is to derive an expression involving power dissipation or conversion densities and time rates of change of energy storages. The power per unit volume imparted to the current density of unpaired charge follows directly from the Lorentz force law (at least in free space). The force on a particle of charge q is

$$\mathbf{f} = q(\mathbf{E} + \mathbf{v} \times \mu_o \mathbf{H}) \tag{3}$$

The rate of work on the particle is

$$\mathbf{f} \cdot \mathbf{v} = q\mathbf{v} \cdot \mathbf{E} \tag{4}$$

If the particle density is N and only one species of charged particles exists, then the rate of work per unit volume is

$$N\mathbf{f} \cdot \mathbf{v} = qN\mathbf{v} \cdot \mathbf{E} = \mathbf{J}_u \cdot \mathbf{E} \tag{5}$$

Thus, one must anticipate that an energy conservation law that applies to free space must contain the term $\mathbf{J}_u \cdot \mathbf{E}$. In order to obtain this term, one should dot multiply (1) by $\mathbf{E}$.

A second consideration that motivates the form of the energy conservation law is the aim to obtain a perfect divergence of density of power flow. Dot multiplication of (1) by $\mathbf{E}$ generates $(\nabla \times \mathbf{H}) \cdot \mathbf{E}$. This term is made into a perfect divergence if one adds to it $-(\nabla \times \mathbf{E}) \cdot \mathbf{H}$, i.e., if one subtracts (2) dot multiplied by $\mathbf{H}$.

Indeed,

$$(\nabla \times \mathbf{E}) \cdot \mathbf{H} - (\nabla \times \mathbf{H}) \cdot \mathbf{E} = \nabla \cdot (\mathbf{E} \times \mathbf{H}) \tag{6}$$

Thus, subtracting (2) dot multiplied by $\mathbf{H}$ from (1) dot multiplied by $\mathbf{E}$ one obtains

$$\boxed{\begin{aligned} -\nabla \cdot (\mathbf{E} \times \mathbf{H}) = & \frac{\partial}{\partial t}\Big(\frac{1}{2}\epsilon_o \mathbf{E} \cdot \mathbf{E}\Big) + \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} \\ & + \frac{\partial}{\partial t}\Big(\frac{1}{2}\mu_o \mathbf{H} \cdot \mathbf{H}\Big) \\ & + \mathbf{H} \cdot \frac{\partial \mu_o \mathbf{M}}{\partial t} + \mathbf{E} \cdot \mathbf{J}_u \end{aligned}} \tag{7}$$

In writing the first and third terms on the right, we have exploited the relation $u \cdot du = d(\frac{1}{2}u^2)$. These two terms now take the form of the energy storage term in the power theorem, (11.1.3). The desire to obtain expressions taking this form is a third consideration contributing to the choice of ways in which (1) and (2) were combined. We could have seen at the outset that dotting $\mathbf{E}$ with (1) and subtracting (2) after it had been dotted with $\mathbf{H}$ would result in terms on the right taking the desired form of "perfect" time derivatives.

In the electroquasistatic limit, the magnetic induction terms on the right in Faraday's law, (2), are neglected. It follows from the steps leading to (7) that in the EQS approximation, the third and fourth terms on the right of (7) are negligible. Similarly, in the magnetoquasistatic limit, the displacement current, the last two terms on the right in Ampère's law, (1), is neglected. This implies that for MQS systems, the first two terms on the right in (7) are negligible.

Systems Composed of Perfect Conductors and Free Space. Quasistatic examples in this category are the EQS systems of Chaps. 4 and 5 and the MQS systems of Chap. 8, where perfect conductors are surrounded by free space. Whether quasistatic or electrodynamic, in these configurations, $\mathbf{P} = 0,\ \mathbf{M} = 0$; and where there is a current density $\mathbf{J}_u$, the perfect conductivity insures that $\mathbf{E} = 0$. Thus, the second and last two terms on the right in (7) are zero. For perfect conductors surrounded by free space, the differential form of the power theorem becomes

$$\boxed{-\nabla \cdot \mathbf{S} = \frac{\partial W}{\partial t}} \tag{8}$$

with

$$\boxed{\mathbf{S} = \mathbf{E} \times \mathbf{H}} \tag{9}$$

and

$$\boxed{W = \frac{1}{2}\epsilon_o \mathbf{E} \cdot \mathbf{E} + \frac{1}{2}\mu_o \mathbf{H} \cdot \mathbf{H}} \tag{10}$$

where $\mathbf{S}$ is the *Poynting vector* and W is the sum of the *electric and magnetic energy densities.* The electric and magnetic fields are confined to the free space regions. Thus, power flow and energy storage pictured in terms of these variables occur entirely in the free space regions.

Limiting cases governed by the EQS and MQS laws, respectively, are distinguished by having predominantly electric and magnetic energy densities. The following simple examples illustrate the application of the power theorem to two simple quasistatic situations. Applications of the theorem to electrodynamic systems will be taken up in Chap. 12.

Example 11.2.1. Plane Parallel Capacitor

The plane parallel capacitor of Fig. 11.2.1 is familiar from Example 3.3.1. The circular electrodes are perfectly conducting, while the region between the electrodes is free space. The system is driven by a voltage source distributed around the edges of the electrodes. Between the electrodes, the electric field is simply the voltage divided by the plate spacing (3.3.6),

$$\mathbf{E} = \frac{v}{d}\mathbf{i_z} \tag{11}$$

while the magnetic field that follows from the integral form of Ampère's law is (3.3.10).

$$\mathbf{H} = \frac{r}{2}\epsilon_o \frac{d}{dt}\Big(\frac{v}{d}\Big)\mathbf{i}_\phi \tag{12}$$

Consider the application of the integral version of (8) to the surface S enclosing the region between the electrodes in Fig. 11.2.1. First we determine the power flowing into the volume through this surface by evaluating the left-hand side of (8). The density of power flow follows from (11) and (12).

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = -\frac{r}{2}\frac{\epsilon_o}{d^2}v\frac{dv}{dt}\mathbf{i_r} \tag{13}$$

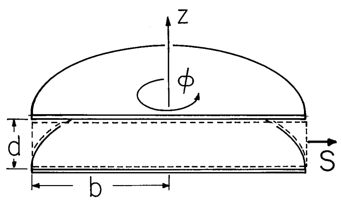

Fig. 11.2.1 Plane parallel circular electrodes are driven by a distributed voltage source. Poynting flux through surface denoted by dashed lines accounts for rate of change of electric energy stored in the enclosed volume.

The top and bottom surfaces have normals perpendicular to this vector, so the only contribution comes from the surface at $r = b$. Because $\mathbf{S}$ is constant on that surface, the integration amounts to a multiplication.

$$-\oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{a} = (2\pi bd)\Big(\frac{b}{2}\frac{\epsilon_o}{d^2}v\frac{dv}{dt}\Big) = \frac{d}{dt}\Big(\frac{1}{2}Cv^2\Big) \tag{14}$$

where

$$C \equiv \frac{\pi b^2\epsilon_o}{d}$$

Here the expression has been written as the rate of change of the energy stored in the capacitor. With $\mathbf{E}$ again given by (11), we double-check the expression for the time rate of change of energy storage.

$$\frac{d}{dt}\int_V \frac{1}{2}\epsilon_o\mathbf{E}\cdot\mathbf{E}dv = \frac{d}{dt}\left[\frac{1}{2}\epsilon_o(d\pi b^2)\Big(\frac{v}{d}\Big)^2\right] = \frac{d}{dt}\Big(\frac{1}{2}Cv^2\Big) \tag{15}$$

From the field viewpoint, power flows into the volume through the surface at $r = b$ and is stored in the form of electrical energy in the volume between the plates. In the quasistatic approximation used to evaluate the electric field, the magnetic energy storage is neglected at the outset because it is small compared to the electric energy storage. As a check on the implications of this approximation, consider the total magnetic energy storage. From (12),

$$\begin{aligned}\int_V \frac{1}{2}\mu_o\mathbf{H}\cdot\mathbf{H}dv &= \frac{1}{2}\mu_o\left[\frac{1}{2}\frac{\epsilon_o}{d}\Big(\frac{dv}{dt}\Big)\right]^2 d\int_0^b r^2 2\pi r dr \\ &= \frac{\mu_o\epsilon_o b^2}{16}C\Big(\frac{dv}{dt}\Big)^2\end{aligned} \tag{16}$$

Comparison of this expression with the electric energy storage found in (15) shows that the EQS approximation is valid provided that

$$\frac{\mu_o\epsilon_o b^2}{8}\Big|\frac{dv}{dt}\Big|^2 \ll v^2 \tag{17}$$

For a sinusoidal excitation of frequency ω, this gives

$$\Big(\frac{b\omega}{\sqrt{8}\,c}\Big)^2 \ll 1 \tag{18}$$

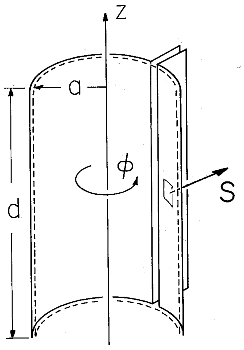

Fig. 11.2.2 One-turn solenoid surrounding volume enclosed by surface S denoted by dashed lines. Poynting flux through this surface accounts for the rate of change of magnetic energy stored in the enclosed volume.

where c is the free space velocity of light (3.1.16). The result is familiar from Example 3.3.1. *The requirement that the propagation time* b/c *of an electromagnetic wave be short compared to a period* $1/\omega$ *is equivalent to the requirement that the magnetic energy storage be negligible compared to the electric energy storage.*

A second example offers the opportunity to apply the integral version of (8) to a simple MQS system.

Example 11.2.2. Long Solenoidal Inductor

The perfectly conducting one-turn solenoid of Fig. 11.2.2 is familiar from Example 10.1.2. In terms of the terminal current $i = Kd$, the magnetic field intensity inside is (10.1.14),

$$\mathbf{H} = \frac{i}{d}\mathbf{i_z} \tag{19}$$

while the electric field is the sum of the particular and conservative homogeneous parts [(10.1.15) for the particular part and $\mathbf{E}_h$ for the conservative part].

$$\mathbf{E} = -\frac{\mu_o}{2}\frac{dH_z}{dt}r\mathbf{i}_\phi + \mathbf{E}_h \tag{20}$$

Consider how the power flow through the surface S of the volume enclosed by the coil is accounted for by the time rate of change of the energy stored. The Poynting flux implied by (19) and (20) is

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = \Big[-\frac{\mu_o a}{2d^2}\frac{d}{dt}\big(\frac{1}{2}i^2\big) + \frac{i}{d}E_{\phi h}\Big]\mathbf{i_r} \tag{21}$$

This Poynting vector has no component normal to the top and bottom surfaces of the volume. On the surface at $r = a$, the first term in brackets is constant, so the integration on S amounts to a multiplication by the area. Because $\mathbf{E_h}$ is irrotational, the integral of $\mathbf{E}_h \cdot d\mathbf{s} = E_{\phi h} r d\phi$ around a contour at $r = a$ must be zero. For this reason, there is no net contribution of $\mathbf{E}_h$ to the surface integral.

$$-\oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{a} = 2\pi a d\big(\frac{\mu_o a}{2d^2}\big)\frac{d}{dt}\big(\frac{1}{2}i^2\big) = \frac{d}{dt}\big(\frac{1}{2}Li^2\big); \tag{22}$$

where

$$L \equiv \frac{\mu_o \pi a^2}{d}$$

Here the result shows that the power flow is accounted for by the rate of change of the stored magnetic energy. Evaluation of the right hand side of (8), ignoring the electric energy storage, indeed gives the same result.

$$\frac{d}{dt}\int_V \frac{1}{2}\mu_o \mathbf{H}\cdot\mathbf{H}dv = \frac{d}{dt}\Big[\pi a^2 d\frac{1}{2}\mu_o\Big(\frac{i}{d}\Big)^2\Big] = \frac{d}{dt}\Big(\frac{1}{2}Li^2\Big) \tag{23}$$

The validity of the quasistatic approximation is examined by comparing the magnetic energy storage to the neglected electric energy storage. Because we are only interested in an order of magnitude comparison and we know that the homogeneous solution is proportional to the particular solution (10.1.21), the latter can be approximated by the first term in (20).

$$\begin{aligned}\int_V \frac{1}{2}\epsilon_o \mathbf{E}\cdot\mathbf{E}dv &\simeq \frac{1}{2}\epsilon_o\frac{\mu_o^2}{4d^2}\Big(\frac{di}{dt}\Big)^2\Big[d\int_0^a r^2 2\pi r dr\Big]\\ &= \frac{\mu_o\epsilon_o a^2}{16}L\Big(\frac{di}{dt}\Big)^2\end{aligned} \tag{24}$$

We conclude that *the MQS approximation is valid provided that the angular frequency ω is small compared to the time required for an electromagnetic wave to propagate the radius a of the solenoid and that this is equivalent to having an electric energy storage that is negligible compared to the magnetic energy storage.*

$$\frac{\mu_o\epsilon_o a^2}{8}\Big(\frac{di}{dt}\Big)^2 \ll i^2 \rightarrow \Big(\frac{\omega a}{\sqrt{8}c}\Big)^2 \ll 1 \tag{25}$$

A note of caution is in order. If the gap between the "sheet" terminals is made very small, the electric energy storage of the homogeneous part of the E field can become large. If it becomes comparable to the magnetic energy storage, the structure approaches the condition of resonance of the circuit consisting of the gap capacitance and solenoid inductance. In this limit, the MQS approximation breaks down. In practice, the electric energy stored in the gap would be dominated by that in the connecting plates, and the resonance could be described as the coupling of MQS and EQS systems as in Example 3.4.1.

In the following sections, we use (7) to study the storage and dissipation of energy in macroscopic media.

11.3 OHMIC CONDUCTORS WITH LINEAR POLARIZATION AND MAGNETIZATION

Consider a stationary material described by the constitutive laws

$$\begin{aligned}\mathbf{P} &= \epsilon_o\chi_e\mathbf{E}\\ \mu_o\mathbf{M} &= \mu_o\chi_m\mathbf{H}\end{aligned} \tag{1}$$

$$\mathbf{J}_u = \sigma \mathbf{E}$$

where the susceptibilities χ_e and χ_m, and hence the permittivity and permeability ϵ and μ, as well as the conductivity σ, are all independent of time. Expressed in terms of these constitutive laws for **P** and **M**, the polarization and magnetization terms in (11.2.7) become

$$\mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} = \frac{\partial}{\partial t}\Big(\frac{1}{2}\epsilon_o \chi_e \mathbf{E} \cdot \mathbf{E}\Big)$$

$$\mathbf{H} \cdot \frac{\partial \mu_o \mathbf{M}}{\partial t} = \frac{\partial}{\partial t}\Big(\frac{1}{2}\mu_o \chi_m \mathbf{H} \cdot \mathbf{H}\Big) \tag{2}$$

Because these terms now appear in (11.2.7) as perfect time derivatives, it is clear that in a material having "linear" constitutive laws, energy is *stored* in the polarization and magnetization processes.

With the substitution of these terms into (11.2.7) and Ohm's law for $\mathbf{J}_u$, a conservation law is obtained in the form discussed in Sec. 11.1. For an electrically and magnetically linear material that obeys Ohm's law, the integral and differential conservation laws are (11.1.1) and (11.1.3), respectively, with

$$\boxed{\mathbf{S} = \mathbf{E} \times \mathbf{H}} \tag{3a}$$

$$\boxed{W = \frac{1}{2}\epsilon \mathbf{E} \cdot \mathbf{E} + \frac{1}{2}\mu \mathbf{H} \cdot \mathbf{H}} \tag{3b}$$

$$\boxed{P_d = \sigma \mathbf{E} \cdot \mathbf{E}} \tag{3c}$$

The power flux density **S** and the energy density W appear as in the free space conservation theorem of Sec. 11.2. The energy storage in the polarization and magnetization is included by simply replacing the free space permittivity and permeability by ϵ and μ, respectively.

The term P_d is always positive and seems to represent a rate of power loss from the electromagnetic system. That P_d indeed represents power converted to thermal form is motivated by considering the origins of the Ohmic conduction law. In terms of the bipolar conduction model introduced in Sec. 7.1, positive and negative carriers, respectively, experience the forces $\mathbf{f}_+$ and $\mathbf{f}_-$. These forces are balanced by collisions with the surrounding particles, and hence the work done by the field in forcing the migration of the particles is converted into thermal energy. If the velocity of the families of particles are, respectively, $\mathbf{v}_+$ and $\mathbf{v}_-$, and the number densities N_+ and N_-, respectively, then the rate of work performed on the carriers (per unit volume) is

$$P_d = N_+ \mathbf{f}_+ \cdot \mathbf{v}_+ + N_- \mathbf{f}_- \cdot \mathbf{v}_- \tag{4}$$

In recognition of the balance between collision forces and electrical forces, the forces of (4) are replaced by $|q_+|\mathbf{E}$ and $-|q_-|\mathbf{E}$, respectively.

$$P_d = N_+|q_+|\mathbf{E}\cdot\mathbf{v}_+ - N_-|q_-|\mathbf{E}\cdot\mathbf{v}_- \tag{5}$$

If, in turn, the velocities are written as the products of the respective mobilities and the macroscopic electric field, (7.1.3), it follows that

$$P_d = (N_+|q_+|\mu_+ + N_-|q_-|\mu_-)\mathbf{E}\cdot\mathbf{E} = \sigma\mathbf{E}\cdot\mathbf{E} \tag{6}$$

where the definition of the conductivity σ (7.1.7) has been used.

The power dissipation density $P_d = \sigma\mathbf{E}\cdot\mathbf{E}$ (watts/m^3) represents a rate of energy loss from the electromagnetic system to the thermal system.

Example 11.3.1. The Poynting Vector of a Stationary Current Distribution

In Example 7.5.2, we studied the electric fields in and around a circular cylindrical conductor fed by a battery in parallel with a disk-shaped conductor. Here we determine the Poynting vector field and explore its spatial relationship to the dissipation density.

First, within the circular cylindrical conductor [region (b) in Fig. 11.3.1], the electric field was found to be uniform, (7.5.7),

$$\mathbf{E}^b = \frac{v}{L}\mathbf{i_z} \tag{7}$$

while in the surrounding free space region, it was [from (7.5.11)]

$$\mathbf{E}^a = -\frac{v}{L\,ln(a/b)}\Big[\frac{z}{r}\mathbf{i_r} + ln(r/a)\mathbf{i_z}\Big] \tag{8}$$

and in the disk-shaped conductor [from (7.5.9)]

$$\mathbf{E}^c = \frac{v}{ln(a/b)}\frac{1}{r}\mathbf{i_r} \tag{9}$$

By symmetry, the magnetic field intensity is ϕ directed. The ϕ component of $\mathbf{H}$ is most easily evaluated from the integral form of Ampère's law. The current density in the circular conductor follows from (7) as $J_o = \sigma v/L$. Then,

$$2\pi r H_\phi = J_o\pi r^2 \rightarrow H_\phi^b = \frac{J_o r}{2}; \qquad r < b \tag{10}$$

$$2\pi r H_\phi = J_o\pi b^2 \rightarrow H_\phi^a = \frac{J_o b^2}{2r}; \qquad b < r < a \tag{11}$$

The magnetic field distribution in the disk conductor is also deduced from Ampère's law. In this region, it is easiest to evaluate the r component of Ampère's differential law with the current density $\mathbf{J}^c = \sigma\mathbf{E}^c$, with $\mathbf{E}^c$ given by (9). Integration of this partial differential equation on z then gives a linear function of z plus

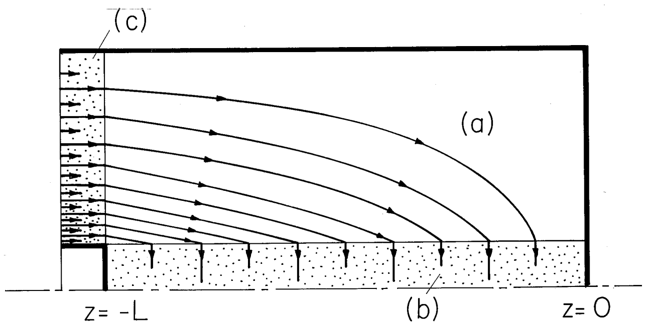

Fig. 11.3.1 Distribution of Poynting flux in coaxial resistors and associated free space. The configuration is the same as for Example 7.5.2. A source to the left supplies current to disk-shaped and circular cylindrical resistive materials. The outer and right-end conductors are perfectly conducting. Note that there is a Poynting flux in the free space interior region even when the currents are stationary.

an "integration constant" that is a function of r. The latter is determined by the requirement that H_ϕ be continuous at $z = -L$.

$$H_\phi^c = -\frac{\sigma}{ln(a/b)}\frac{v}{r}(L+z) + J_o\frac{b^2}{2r}; \qquad b < r < a \tag{12}$$

It follows from these last four equations that the Poynting vector inside the circular cylindrical conductor, in the surrounding space, and in the disk-shaped electrode is

$$\mathbf{S}^b = -\frac{v}{L}\frac{J_o}{2}r\mathbf{i_r} \tag{13}$$

$$\mathbf{S}^a = -\frac{vb^2 J_o}{ln(a/b)2rL}\Big(\frac{z}{r}\mathbf{i_z} - ln\frac{r}{a}\mathbf{i_r}\Big) \tag{14}$$

$$\mathbf{S}^c = \left[\frac{-\sigma}{ln^2(a/b)}\frac{v^2}{r^2}(L+z) + \frac{J_o v}{ln(a/b)}\frac{b^2}{2r^2}\right]\mathbf{i_z} \tag{15}$$

This distribution of **S** is sketched in Fig. 11.3.1. Wherever there is a dissipation density, there must be a negative divergence of **S**. Thus, in the conductors, the **S** lines terminate in the volume. In the free space region (a), **S** is solenoidal. Even with the fields perfectly stationary in time, the power is seen to flow through the *open space* to be absorbed in the volume where the dissipation takes place. The integral of the Poynting vector over the surface surrounding the inner conductor gives what we would expect either from the circuit point of view

$$-\oint \mathbf{E}\times\mathbf{H}\cdot d\mathbf{a} = (2\pi bL)\Big(\frac{v}{L}\Big)\Big(\frac{J_o b}{2}\Big) = v(\pi b^2 J_o) = vi \tag{16}$$

where i is the total current through the cylinder, or from an evaluation of the right-hand side of the integral conservation law.

$$\int_V \sigma\mathbf{E}\cdot\mathbf{E}dv = (\pi b^2 L)\sigma\Big(\frac{v}{L}\Big)^2 = v\Big(\pi b^2\sigma\frac{v}{L}\Big) = vi \tag{17}$$

An Alternative Conservation Theorem for Electroquasistatic Systems. In describing electroquasistatic systems, it is inconvenient to require that the magnetic field intensity be evaluated. We consider now an alternative conservation theorem that is specialized to EQS systems. We will find an alternative expression for $\mathbf{S}$ that does not involve $\mathbf{H}$. In the process of finding an alternative distribution of $\mathbf{S}$, we *illustrate the danger of ascribing meaning to* $\mathbf{S}$ *evaluated at a point, rather than integrated over a closed surface.*

In the EQS approximation, $\mathbf{E}$ is irrotational. Thus,

$$\mathbf{E} = -\nabla\Phi \tag{18}$$

and the power input term on the left in the integral conservation law, (11.1.1), can be expressed as

$$-\oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{a} = \oint_S \nabla\Phi\times\mathbf{H}\cdot d\mathbf{a} \tag{19}$$

Next, the vector identity

$$\nabla\times(\Phi\mathbf{H}) = \nabla\Phi\times\mathbf{H} + \Phi\nabla\times\mathbf{H} \tag{20}$$

is used to write the right-hand side of (19) as

$$-\oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{a} = \oint_S \nabla\times(\Phi\mathbf{H})\cdot d\mathbf{a} - \oint_S \Phi\nabla\times\mathbf{H}\cdot d\mathbf{a} \tag{21}$$

The first integral on the right is zero because the curl of a vector is divergence free and a field with no divergence has zero flux through a closed surface. Ampère's law can be used to eliminate *curl* $\mathbf{H}$ from the second.

$$-\oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{a} = -\oint_S \Phi\big(\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\big)\cdot d\mathbf{a} \tag{22}$$

In this way, we have determined an alternative expression for $\mathbf{S}$, *valid only in the electroquasistatic approximation.*

$$\boxed{\mathbf{S} = \Phi\big(\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\big)} \tag{23}$$

The density of power flow, expressed by (23) as the product of a potential and *total current density* consisting of the sum of the conduction and displacement current densities, has a form similar to that used in circuit theory.

The power flux density of (23) is convenient in describing EQS systems, where the effects of magnetic induction are not significant. To be consistent with the EQS approximation, the conservation law must be used with the magnetic energy density neglected.

Example 11.3.2. Alternative EQS Power Flux Density for Stationary Current Distribution

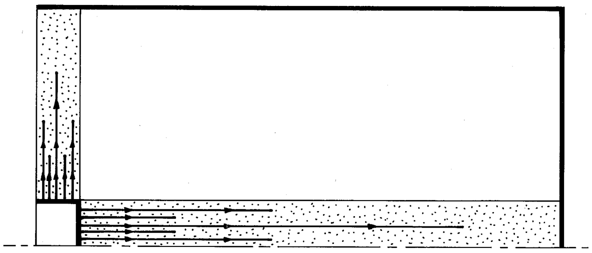

Fig. 11.3.2 Distribution of electroquasistatic flux density for the same system as shown in Fig. 11.3.1.

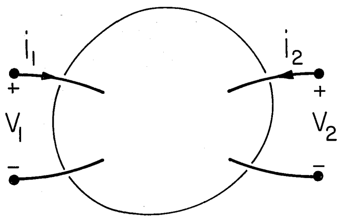

Fig. 11.3.3 Arbitrary EQS system accessed through terminal pairs.

To contrast the alternative EQS power flow density with the Poynting flux density, consider again the coaxial resistor configuration of Example 11.3.1. Because the fields are stationary, the EQS power flux density is

$$\mathbf{S} = \Phi\mathbf{J} \tag{24}$$

By contrast with the Poynting flux density, this vector field is zero in the free space region. In the circular cylindrical conductor, the potential and current density are [(7.5.6) and (7.5.7)]

$$\Phi^b = -\frac{v}{L}z; \qquad \mathbf{J}^b = \sigma\mathbf{E}^b = \frac{\sigma v}{L}\mathbf{i_z} \tag{25}$$

and it follows that the power flux density is simply

$$\mathbf{S} = -\frac{\sigma v^2}{L^2} z\mathbf{i_z} \tag{26}$$

There is a similar, radially directed flux density in the disk-shaped resistor.

The alternative distribution of **S**, shown in Fig. 11.3.2, is clearly very different from that shown in Fig. 11.3.1 for the Poynting flux density.

Poynting Power Density Related to Circuit Power Input. Suppose that the surface S described by the conservation theorem encloses a system that is accessed through terminal pairs, as shown in Fig. 11.3.3. Under what circumstances is the integral of $\mathbf{S} \cdot d\mathbf{a}$ over S equivalent to summing the voltage-current product of the terminals of the wires connected to the system?

Two attributes of the fields on the surface S enclosing the system are required. *First, the contribution of the magnetic induction to* $\mathbf{E}$ *must be negligible on* S. If this is so, then regardless of what is inside S (for example, both EQS and MQS systems), *on the surface* S, the electric field can be taken as irrotational. It follows that in taking the integral over a closed surface of the Poynting power density, we can just as well use (23).

$$-\oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{a} = -\oint_S \Phi\big(\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}\big)\cdot d\mathbf{a} \tag{27}$$

By contrast with the EQS systems treated in deriving this expression, it now holds only on the surface S, not necessarily on surfaces inside the volume enclosed by S.

Second, on the surface S, *the contribution of the displacement current must be negligible.* This is equivalent to requiring that S is chosen parallel to the displacement flux density. In this case, the total power into the system reduces to

$$-\oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{a} = -\oint_S \Phi\mathbf{J}\cdot d\mathbf{a} \tag{28}$$

The integrand has value only where the surface S intersects a wire. If taken as perfectly conducting (but nevertheless in a region where $\partial\mathbf{B}/\partial t$ is zero and hence $\mathbf{E}$ is irrotational), the wires have potentials that are uniform over their cross-sections. Thus, in (28), Φ is equal to the voltage of the terminal. In integrating the current density over the cross-section of the wire, note that $d\mathbf{a}$ is directed out of the surface, while a positive terminal current is directed into the surface. Thus,

$$-\oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{a} = \sum_{i=1}^{n} v_i i_i \tag{29}$$

and the input power expressed by (28) is equivalent to what would be expected from circuit theory.

Poynting Flux and Electromagnetic Radiation. Power cannot be supplied to or lost by a quasistatic system of finite extent through a surface at infinity. Such a power supply or loss requires radiation, and electromagnetic waves are neglected when either the magnetic induction or the displacement current density are neglected. To prove this statement, consider an EQS system of finite net charge. Its electric field intensity decays like $1/r^2$ at infinity, where r is the distance to a far-off point from some origin chosen within the system. At a great distance, the currents appear equivalent to current loop sources. Hence, the magnetic field intensity has the $1/r^3$ decay typical of a magnetic dipole. It follows that the Poynting vector decays at least as fast as $1/r^5$, so that the flux of $\mathbf{E}\times\mathbf{H}$ integrated over the "sphere" at infinity of area $4\pi r^2$ gives zero contribution. Because it is only that part of $\mathbf{E}\times\mathbf{H}$ resulting from electromagnetic radiation that contributes at infinity, Poynting's theorem is shown in Sec. 12.5 to be a powerful tool for dealing with antennae.

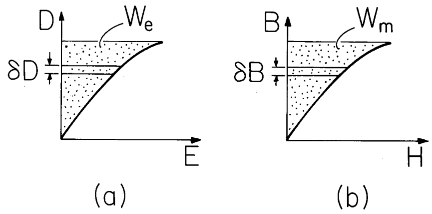

Fig. 11.4.1 Single-valued constitutive laws showing energy density associated with variables at the endpoints of the curves: (a) electric energy density; and (b) magnetic energy density.

11.4 ENERGY STORAGE

In the conservation theorem, (11.2.7), we have identified the terms $\mathbf{E} \cdot \partial \mathbf{P}/\partial t$ and $\mathbf{H} \cdot \partial \mu_o \mathbf{M}/\partial t$ as the rate of energy supplied per unit volume to the polarization and magnetization of the material. For a linear isotropic material, we found that these terms can be written as derivatives of energy density functions. In this section, we seek a more general description of energy storage. First, nonlinear materials are considered from the field viewpoint. Then, for those systems that can be described in terms of electrical terminal pairs, energy storage is formulated in terms of terminal variables. We will find the results of this section directly applicable to finding electric and magnetic forces in Secs. 11.6 and 11.7.

Energy Densities. Consider a material in which $\mathbf{E}$ and $\mathbf{D} \equiv (\epsilon_o \mathbf{E} + \mathbf{P})$ are collinear. With E and D representing the magnitudes of these vectors, this material is presumed to be described by a constitutive law in which E is a single-valued function of D, such as that sketched in Fig. 11.4.1a. In the case of a linear constitutive law, the curve is a straight line with a slope equal to the permittivity ϵ.

Consider a material in which $\mathbf{E}$ and $\mathbf{P}$ are collinear (isotropic material). Then, of course, $\mathbf{E}$ and $\mathbf{D} \equiv \epsilon_o \mathbf{E} + \mathbf{P}$ are collinear as well. One may graph the magnitude of D versus E and obtain a complete characterization of the material. Now the power per unit volume imparted to the polarization is $\mathbf{E} \cdot \partial \mathbf{P}/\partial t$. If one adds to it the rate of energy supply to the field per unit volume (the free space part) $\mathbf{E} \cdot \partial \epsilon_o \mathbf{E}/\partial t$, one obtains for the power per unit volume

$$\boxed{\mathbf{E} \cdot \frac{\partial}{\partial t}(\epsilon_o \mathbf{E} + \mathbf{P}) = \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = E \frac{\partial D}{\partial t}} \tag{1}$$

The power supplied to the unit volume can now be written as the time derivative of a function of D, $W_e(D)$. Indeed, if we define the area above the graph in Fig. 11.4.1 as W_e, then

$$\boxed{\frac{\partial W_e}{\partial t} = \frac{\partial W_e}{\partial D} \frac{\partial D}{\partial t} = E \frac{\partial D}{\partial t}} \tag{2}$$

Thus, $E(\partial D/\partial t)$ is the derivative of the function $W_e(D)$. This function is the energy stored per unit volume, because the energy supplied per unit volume expressed by the integral

$$\int_{-\infty}^{t} dt E\frac{\partial D}{\partial t} = \int_{0}^{D} E\delta D = W_e(D) \tag{3}$$

is a function of the final value D of the displacement flux, and we assumed that the fields E and D were zero at $t = -\infty$. Here, δD represents the differential of D, usually denoted by dD. We will use δ rather than d to avoid confusion between differentials used in carrying out volume, surface and line integrals and the differential used here, which implies an integration in a "state space" having the "dimension" D.

Similar arguments show that if $\mathbf{B} \equiv \mu_o(\mathbf{H} + \mathbf{M})$ and $\mathbf{H}$ are collinear, and if H is a single-valued function of B, then

$$\boxed{\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial W_m}{\partial t}} \tag{4}$$

where

$$\boxed{W_m = W_m(B) = \int_0^B H\delta B} \tag{5}$$

With (1) and (4) replacing the first four terms on the right in the energy theorem of (11.2.7), it is clear that the energy density $W = W_e + W_m$. The electric and magnetic energy densities have the geometric interpretations as areas on the graphs representing the constitutive laws in Fig. 11.4.1.

Energy Storage in Terms of Terminal Variables. It was shown in Sec. 11.3 that the power input to a system could be represented by the sum of the vi products for each of the terminal pairs, (11.3.29), provided certain conditions were met in the neighborhoods of the terminals. The description of energy storage in a loss-free system in terms of terminal variables will be found useful in determining electric and magnetic forces. With the assumption that all of the power input to a system is accounted for by a time rate of change of the energy stored, the energy conservation statement for a system becomes

$$\sum_{i=1}^{n} v_i i_i = \frac{dw}{dt} \tag{6}$$

where

$$w = \int_V W dv$$

and the integral is carried over the volume of the system. If the system is *electroquasistatic*, conservation of charge requires that the terminal current be the time rate of change of the charge on the electrode to which the positive terminal is attached.

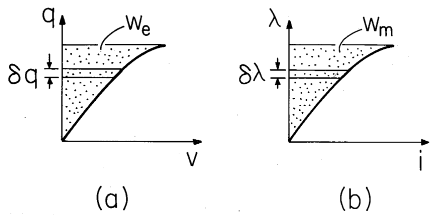

Fig. 11.4.2 Single-valued terminal relations showing total energy stored when variables are at the endpoints of the curves: (a) electric energy storage; and (b) magnetic energy storage.

$$\boxed{i_i = \frac{dq_i}{dt}} \tag{7}$$

Further, $w = w_e$, the stored electric energy. Thus, one concludes from (6) that

$$\sum_{i=1}^{n} v_i \frac{dq_i}{dt} = \frac{dw_e}{dt} \Rightarrow dw_e = \sum_{i=1}^{n} v_i dq_i \tag{8}$$

The second expression states that with the addition of an incremental amount of charge dq_i to an electrode having the voltage v_i goes an incremental change in the stored energy w_e. Integration on the charges then gives the total energy

$$\boxed{w_e = \sum_{i=1}^{n} \int v_i dq_i} \tag{9}$$

To complete this integral, each of the terminal voltages must be a known function of the associated charges.

$$\boxed{v_i = v_i(q_1, \dots\ q_n)} \tag{10}$$

Integration is then carried out along any path in the state space $(q_1 \dots q_n)$ that begins at the origin and ends with the desired charges on the electrodes (and hence the desired terminal voltages). For a single terminal pair, the energy can be pictured as the area shown in Fig. 11.4.2a.

If the system is magnetoquasistatic, the conservation law for a lossless system that can be described by terminal relations again takes the form of (6). However, rather than expressing the currents as derivatives of electrode charges, the voltages are derivatives of the fluxes linked by the respective terminal pairs.

$$\boxed{v_i = \frac{d\lambda_i}{dt}} \tag{11}$$

Then, (6) leads to

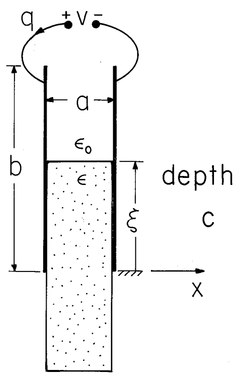

Fig. 11.4.3 Capacitor partially filled by free space and by dielectric having permittivity ϵ.

$$\boxed{w_m = \sum_{i=1}^{n} \int i_i d\lambda_i} \tag{12}$$

To complete this integral, we require the terminal currents as functions of the terminal flux linkages.

$$\boxed{i_i = i_i(\lambda_i \ldots \lambda_n)} \tag{13}$$

For a single terminal pair system, w_m is portrayed in Fig. 11.4.2b.

The most general way to compute the total energy stored in a system is to integrate the energy densities given by (3) and (5) over the volumes of the respective systems. If systems can be described in terms of terminal relations and are loss free, (9) and (12) must lead to the same answers. Note that (D, E) and (q, v) are the field and circuit variables in the EQS systems, while (B, H) and (λ, i) have corresponding roles in MQS systems.

Example 11.4.1. An Electrically Linear System

A dielectric slab of permittivity ϵ partially fills the region between plane parallel perfectly conducting electrodes, as shown in Fig. 11.4.3. With the fringing field ignored, we find the total energy stored by two methods. First, the energy density is integrated over the volume. Then, the terminal relation is used to evaluate the total energy.

An exact solution for the electric field well between the electrodes is simply $\mathbf{E} = \mathbf{i_x}(v/a)$. Note that this field satisfies the boundary conditions at the interface between the dielectric slab and the free space region above and at the electrodes. We assume that $a \ll b$ and therefore neglect the fringing fields.

The energy density in the linear dielectric, where $\mathbf{D} = \epsilon\mathbf{E}$, follows from evaluation of (3).

$$W_e = \int_0^D E\delta D = \frac{1}{2}\frac{D^2}{\epsilon} = \frac{1}{2}\epsilon E^2; \tag{14}$$

$$E = \frac{v}{a}$$

In the free space region, the same result applies with $\epsilon \to \epsilon_o$.

Integration of these energy densities over the regions in which they apply amounts to a multiplication by the respective volumes. Thus, the total energy is

$$w_e = \int_V W_e dV = \frac{1}{2}\epsilon\Big(\frac{v}{a}\Big)^2(\xi ca) + \frac{1}{2}\epsilon_o\Big(\frac{v}{a}\Big)^2[(b-\xi)ca] \quad (15)$$

Note that this expression takes the form

$$w_e = \frac{1}{2}Cv^2 \quad (16)$$

where

$$C \equiv \frac{c}{a}[\epsilon_o b + \xi(\epsilon - \epsilon_o)]$$

In terms of the terminal variables, where $q = Cv$, the total energy follows from an evaluation of (9).

$$w_e = \int v dq = \int \frac{q}{C} dq = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}Cv^2 \quad (17)$$

Once the integration has been carried out, the last expression is written by again using the relation $q = Cv$. Note that the volume integration of the energy density and the integration in terms of the terminal variables give the same result.

The next example considers an MQS system with two terminal pairs and thus illustrates the integration called for in evaluating the energy from the terminal relations. Also, the energy stored in coupled inductors is often of practical interest.

Example 11.4.2. Coupled Coils; Transformers

An example of a two terminal pair lossless MQS system is a pair of coupled coils having the terminal relations

$$\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \begin{bmatrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{bmatrix}\begin{bmatrix}i_1\\ i_2\end{bmatrix} \quad (18)$$

In this case, (12) becomes

$$dw_m = i_1 d\lambda_1 + i_2 d\lambda_2 \quad (19)$$

To evaluate this expression, we need to substitute for the currents written in terms of the flux linkages. This requires the inversion of (18). For linear systems, this is easily done, but not for nonlinear systems. To avoid inversion, we rewrite the right-hand side of (19), which becomes

$$dw_m = d(i_1\lambda_1 + i_2\lambda_2) - \lambda_1 di_1 - \lambda_2 di_2 \quad (20)$$

and regroup terms

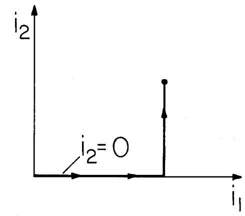

Fig. 11.4.4 Integration path in state space consisting of terminal currents.

$$\boxed{dw'_m = \lambda_1 di_1 + \lambda_2 di_2} \tag{21}$$

where the *coenergy* is defined as

$$\boxed{w'_m = (i_1\lambda_1 + i_2\lambda_2) - w_m} \tag{22}$$

Equation (21) can be integrated when the flux linkages are expressed in terms of the currents, and that is the form in which the terminal relations are given by (18). Once the coenergy w'_m has been found, w_m follows from (22).

The integration of (19) is a line integral in a state space (i_1, i_2). If energy is conserved, we must be able to carry out this integration along any path that begins with the currents turned off and ends with the currents at the desired values. In the path represented by Fig. 11.4.4, the current i_1 is turned up first while holding the current i_2 to zero. Then, with i_1 held fixed at its final value, the current i_2 is raised from zero to its final value. For this path, the integration of (22) becomes

$$w'_m = \int_0^{i_1} \lambda_1(i'_1, 0)di'_1 + \int_0^{i_2} \lambda_2(i_1, i'_2)di'_2 \tag{23}$$

Substitution for the flux linkages from (18) and evaluation of the integrals then gives

$$w'_m = \frac{1}{2}L_{11}i_1^2 + L_{12}i_1i_2 + \frac{1}{2}L_{22}i_2^2 \tag{24}$$

If the integration is carried out along a path where the roles of i_1 and i_2 are reversed, the expression obtained is (24) with $L_{12} \rightarrow L_{21}$. To make the energy stored independent of path, the mutual inductances must be equal.

$$L_{12} = L_{21} \tag{25}$$

This relation, which we found to hold for the transformer of Example 9.7.4, is required if energy is to be conserved. The energy is now evaluated by substituting this expression and the flux linkages expressed using (18) into (22) solved for w_m. It follows that

$$w'_m = w_m \tag{26}$$

Evaluation of the energy stored in a unity-coupled transformer, where the inductances take the form of (9.7.20), gives

$$w_m = \frac{A\mu}{l}\Big(\frac{1}{2}N_1^2i_1^2 + N_1N_2i_1i_2 + \frac{1}{2}N_2^2i_2^2\Big) \tag{27}$$

Operating under "ideal" conditions [in the sense that $i_2/i_1 = -N_1/N_2$, (9.7.13)], the transformer does not store energy, $w_m = 0$. Thus, according to the power theorem in the form of (6), under ideal operating conditions, the power input at one terminal pair instantaneously appears as a power output at the second terminal pair.

Examples have so far involved linear polarization and magnetization constitutive laws. In the following, the EQS energy storage in a material having a nonlinear polarization constitutive law is determined.

Example 11.4.3. Energy Storage in Electrically Nonlinear Material

To represent the tendency of the polarization to saturate as the electric field is raised, a constitutive law might take the form

$$D = \left(\frac{\alpha_1}{\sqrt{1+\alpha_2 E^2}} + \epsilon_o\right)E \tag{28}$$

Here, α_1 and α_2 are parameters descriptive of the specific material, and $\mathbf{D}$ is collinear with $\mathbf{E}$. This constitutive law is portrayed graphically in Fig. 11.4.5.

Because D is given as a function of E that is not easily solved for E as a function of D, the computation of the electric energy density using (3) is inconvenient. However, we can observe that

$$\delta W_e = E\delta D = \delta(ED) - D\delta E \tag{29}$$

and then regroup terms so that the expression becomes

$$\boxed{\delta W_e' = D\delta E} \tag{30}$$

where

$$\boxed{W_e' = ED - W_e} \tag{31}$$

Integration now leads to the *coenergy density* W_e', but the energy density W_e can then be found using (31) and the constitutive law.

Specifically, evaluation of (30) using (28) gives the coenergy density

$$W_e' = \int D\delta E = \frac{\alpha_1}{\alpha_2}\left(\sqrt{1+\alpha_2 E^2} - 1\right) + \frac{1}{2}\epsilon_o E^2 \tag{32}$$

It follows from (31) that the energy density is

$$W_e = ED - W_e' = \frac{\alpha_1}{\alpha_2}\left(\frac{\sqrt{1+\alpha_2 E^2} - 1}{\sqrt{1+\alpha_2 E^2}}\right) + \frac{1}{2}\epsilon_o E^2 \tag{33}$$

A graphical representation of the energy and coenergy functions is given in Fig. 11.4.5. The area "under the curve" with D as the integration variable is W_e, (3), and the area under the curve with E as the integration variable is W_e', (31).

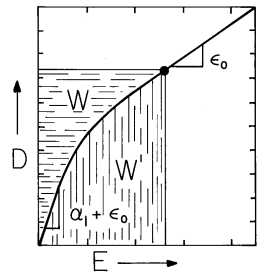

Fig. 11.4.5 Single-valued nonlinear constitutive law. Areas representing energy density W and coenergy density W' are not equal in this case.

11.5 ELECTROMAGNETIC DISSIPATION

The heat generated by electromagnetic fields is often the controlling feature of an engineering design. Semiconductors inevitably produce heat, and the distribution and magnitude of the heat source is an important consideration whether the application is to computers or power conversion. Often, the generation of heat poses a fundamental limitation on the performance of equipment. Examples where the generation of heat is desirable include the heating coil of an electric stove and the microwave irradiation of food in a microwave oven.

Ohmic conduction is the primary cause of heat generation in metals, but it also operates in semiconductors, electrolytes, and (at low frequencies) in semi-insulating liquids and solids. The mechanism responsible for this type of heating was discussed in Sec. 11.3. The dissipation density associated with Ohmic conduction is $\sigma\mathbf{E}\cdot\mathbf{E}$.

An Ohmic current can be imposed by making electrical contact with the material, as for the heating element in a stove. If the material is a good conductor, such currents can also be induced by magnetic induction (without electrical contact). The currents induced by time-varying magnetic fields in Chap. 10 are an example. *Induction heating* is an MQS process and often used in processing metals. Currents induced in transformer cores by the time-varying magnetic flux are an example of undesirable heating. In this context, the associated losses (which are minimized by laminating the core) are said to be due to *eddy currents*.

Ohmic heating can also be induced by "capacitive" coupling. In the EQS examples of Sec. 7.9, *dielectric heating* is caused by the currents associated with the accumulation of unpaired charges.

Whether due to magnetic induction or capacitive coupling, the generation of heat is described by the dissipation density $P_d = \sigma\mathbf{E}\cdot\mathbf{E}$ identified in Sec. 11.3. However, the polarization and magnetization terms in the conservation theorem, (11.2.7), can also be responsible for energy dissipation. This occurs when the (electric or magnetic) dipoles do not align instantaneously with the fields. The polarization and magnetization constitutive laws differ from the laws postulated in Sec. 11.3.

As an example suggesting how the polarization term in (11.2.7) can represent dissipation, picture the artificial dielectric of Demonstration 6.6.1 (the ping-pong ball dielectric) but with spheres that are highly resistive rather than perfectly conducting. The accumulation of charge on the poles of the spheres in response to the application of an electric field is described by a *rate*, rather than a *magnitude*, that is proportional to the field. Thus, we would expect $\partial\mathbf{P}/\partial t$ rather than $\mathbf{P}$ to be proportional to $\mathbf{E}$. With γ a coefficient representing the properties and geometry of the spheres, the polarization constitutive law would then take the form

$$\frac{\partial\mathbf{P}}{\partial t} = \gamma\mathbf{E} \tag{1}$$

If this law is used to express the polarization term in the conservation law, the second term on the right in (11.2.7), a positive definite quantity results.

$$\mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial t} = \gamma\mathbf{E}\cdot\mathbf{E} \tag{2}$$

As might be expected from the physical origins of the constitutive law, the polarization term now represents dissipation rather than energy storage.

When materials are placed in electric fields having frequencies so high that conduction effects are negligible, losses due to the polarization of dipoles become the dominant heating mechanism. The artificial diamagnetic material considered in Demonstration 9.5.1 suggests how analogous losses are associated with the dynamic magnetization of a material. If the spherical particles comprising the artificial diamagnetic material have a finite conductivity, the induced dipole moments are not in phase with an applied sinusoidal field. What amounts to Ohmic dissipation on the particle scale is accounted for on the macroscopic scale by a modified constitutive law of magnetization.

The most common losses due to magnetization are encountered in ferromagnetic materials. *Hysteresis losses* occur because of the coercion required to obtain alignment of ferromagnetic domains. We will end this section with the relationship between the hysteresis curve of Fig. 9.4.6 and the dissipation density.

Energy Conservation for Temporally Periodic Systems. Many practical situations involve fields that vary with time in a periodic fashion. The sinusoidal steady state is the most common example. If the energy conservation law (11.0.8) is integrated over one period T, the energy storage term makes no contribution.

$$\int_0^T \frac{dw}{dt}dt = w(T) - w(0) = 0 \tag{3}$$

As a result, the time average of the conservation law states that the time average of the input power goes into the time average of the dissipation. The time average of the integral form of the conservation law, (11.1.1), becomes

$$-\langle\oint_S \mathbf{S}d\mathbf{a}\rangle = \langle\int_V P_d dv\rangle \tag{4}$$

This expression, which assumes that the dynamics are periodic but not necessarily sinusoidal, gives us two ways to compute the total energy dissipation. Either we can use the right-hand side and integrate the power dissipation density over the volume, or we can use the left-hand side and integrate the time average of $\mathbf{S} \cdot d\mathbf{a}$ over the surface enclosing the volume.

Consider the sinusoidal steady state as a particular case. If $\mathbf{P}$ and $\mathbf{M}$ are related to $\mathbf{E}$ and $\mathbf{H}$ by linear differential equations, an approach can be taken that is familiar from circuit theory. The phase and amplitude of each field at a given location are represented by a complex amplitude. For example, the electric and magnetic field intensities are written as

$$\mathbf{E} = Re\hat{\mathbf{E}}(\mathbf{r})e^{j\omega t}; \qquad \mathbf{H} = Re\hat{\mathbf{H}}(\mathbf{r})e^{j\omega t} \tag{5}$$

A complex vector $\hat{\mathbf{E}}(\mathbf{r})$ has three complex scalar components $\hat{E}_x(\mathbf{r})$, $\hat{E}_y(\mathbf{r})$, and $\hat{E}_z(\mathbf{r})$. The meaning of each is the same as the meaning of a complex voltage in circuit theory: e.g., the magnitude of $\hat{E}_x(\mathbf{r})$, $|\hat{E}_x(\mathbf{r})|$, gives the peak amplitude of the x component of the electric field varying cosinusoidally with time, and the phase of $\hat{E}_x(\mathbf{r})$ gives the phase advance of the cosine time function.

In determining the time averages of products of quantities that are in the sinusoidal steady state, it is helpful to make use of the *time average theorem.* With $*$ designating the complex conjugate,

$$\boxed{\langle Re\hat{A}e^{j\omega t} Re\hat{B}e^{j\omega t}\rangle = \frac{1}{2}Re\hat{A}\hat{B}^*} \tag{6}$$

This can be shown by using the identity

$$Re\hat{C}e^{j\omega t} = \frac{1}{2}(\hat{C}e^{j\omega t} + \hat{C}^*e^{-j\omega t}) \tag{7}$$

Induction Heating. In this case, the heating is represented by Ohmic conduction and P_d given by (11.3.3c). The examples from Chaps. 7 and 10 involving conductors of finite conductivity offer the opportunity to apply this relation to the evaluation of the right-hand side of (4). If the same total time average power is calculated using the left-hand side of this expression, it may seem that Ohm's law is not required. However, remember that this law is also reflected in the field quantities used to calculate $\mathbf{S}$.

Example 11.5.1. Induction Heating of the Thin Shell

The thin conducting shell of Fig. 11.5.1, in a field $H_o(t)$ applied collinear with its axis, was described in Example 10.3.1. Here the applied field is in the sinusoidal steady state

$$H_o = Re\hat{H}_oe^{j\omega t} \tag{8}$$

According to (10.3.9), the complex amplitude of the response, the magnetic field inside the shell, is

$$\hat{H}_i = \frac{\hat{H}_o}{1 + j\omega\tau_m} \tag{9}$$

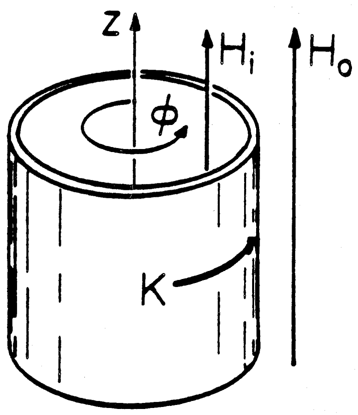

Fig. 11.5.1 Circular cylindrical conducting shell in imposed axial magnetic field intensity $H_o(t)$.

where $\tau_m = \frac{1}{2}\mu_o\sigma\Delta a$.

The complex amplitude of the surface current density circulating in the shell follows from (10.3.8).

$$\hat{K} = -\hat{H}_o + \frac{\hat{H}_o}{1 + j\omega\tau_m} = -\frac{j\omega\tau_m\hat{H}_o}{1 + j\omega\tau_m} \tag{10}$$

Because the current density is uniform over the radial cross-section of the shell, the dissipation density can be written in terms of the surface current density $K = \Delta\sigma E$.

$$P_d = \sigma\mathbf{E}\cdot\mathbf{E} = \frac{K^2}{\Delta^2\sigma} \tag{11}$$

It follows from the application of the time average theorem, (6), that the total time average dissipation is

$$\langle\int_V P_d dv\rangle = \langle\int_V \frac{1}{2} Re\frac{\hat{K}\hat{K}^*}{\Delta^2\sigma}dv\rangle = \frac{2\pi a\Delta l}{2\Delta^2\sigma}Re\hat{K}\hat{K}^* \tag{12}$$

where l is the shell length. To complete the derivation based on an integration of the density over the volume of the conductor, this expression can be evaluated using (10).

$$\langle p_d\rangle \equiv \langle\int_V P_d dv\rangle = p_o\frac{(\omega\tau_m)^2}{1 + (\omega\tau_m)^2}; \qquad p_o \equiv \frac{\pi a l}{\sigma\Delta}|\hat{H}_o|^2 \tag{13}$$

The same result is found by evaluating the time average of the Poynting flux density integrated over a surface that is just outside the shell at $r = a$. To see this, we again use the time average theorem, (6), and recognize that the surface integral amounts to a multiplication by the surface area of the shell.

$$-\langle\oint_S(\mathbf{E}\times\mathbf{H})\cdot d\mathbf{a}\rangle = \oint_S \frac{1}{2}Re\hat{E}_\phi\hat{H}_o^* da = \frac{2\pi a l}{2}Re\hat{E}_\phi\hat{H}_o^* \tag{14}$$

To evaluate this expression, (10) is used to determine E_ϕ

$$\hat{E}_\phi = \frac{\hat{K}}{\Delta\sigma} = -\frac{j\omega\tau_m\hat{H}_o}{\Delta\sigma(1 + j\omega\tau_m)} = -\frac{j\omega\tau_m(1 - j\omega\tau_m)\hat{H}_o}{\Delta\sigma[1 + (\omega\tau_m)^2]} \tag{15}$$

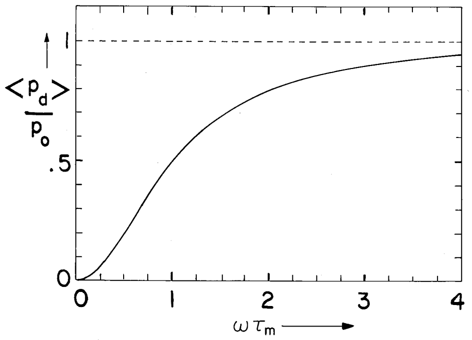

Fig. 11.5.2 Time average power dissipation density normalized to p_o as defined with (13) as a function of the frequency normalized to the magnetic diffusion time defined with (9).

Evaluation of (14) then gives

$$-\langle \oint \mathbf{E} \times \mathbf{H} \cdot d\mathbf{a} \rangle = p_o \frac{(\omega\tau_m)^2}{1 + (\omega\tau_m)}; \qquad p_o \equiv \frac{\pi a l}{\sigma\Delta} |\hat{H}_o|^2 \tag{16}$$

which is the same result as found by integrating the dissipation density over the volume, (13).

The dependence of the time average power dissipation on the normalized frequency is shown in Fig. 11.5.2. At very low frequencies, the induced current is not large enough to have an appreciable effect on the imposed field. Thus, the electric field is proportional to the time rate of change of the *applied field*, and because the dissipation is proportional to the square of $\mathbf{E}$, the power dissipation increases as the square of ω. At high frequencies, the induced current can be no more than that required to shield the imposed field from the region inside the shell. As a result, the dissipation reaches an asymptotic limit.

Which of the two approaches is best for finding the total power dissipation? The answer depends on what field information is available. Certainly, the notion that the total heat generated can be found by integrating over a surface that is completely outside the heated material is a fundamental consequence of Poynting's theorem.

Dielectric Heating. In the sinusoidal steady state, we can identify the power dissipation density associated with polarization by finding the time average

$$\langle P_d \rangle = \langle \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \rangle \tag{17}$$

In view of the time average theorem, (6), this becomes

$$\langle P_d \rangle = \frac{1}{2} Re j\omega \hat{\mathbf{D}} \cdot \hat{\mathbf{E}}^* \tag{18}$$

If the polarization $\mathbf{P}$ does not follow the electric field $\mathbf{E}$ instantaneously, yet the material is still linear and isotropic, the complex vector $\hat{\mathbf{P}}$ can be related to $\hat{\mathbf{E}}$ by

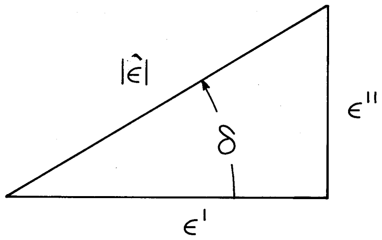

Fig. 11.5.3 Definition of angle δ defining the loss tangent $\tan(\delta)$ in terms of the real and the negative of the imaginary parts of the complex permittivity.

a complex susceptibility. Or, instead, the complex displacement flux density vector $\hat{\mathbf{D}}$ is related to $\hat{\mathbf{E}}$ by a complex dielectric constant.

$$\hat{\mathbf{D}} = \hat{\epsilon}\hat{\mathbf{E}} = (\epsilon' - j\epsilon'')\hat{\mathbf{E}} \tag{19}$$

Here $\hat{\epsilon}$ is the *complex permittivity* with real and imaginary parts ϵ' and $-\epsilon''$, respectively.

Evaluation of (18) using this constitutive law gives

$$\langle P_d \rangle = \frac{\omega}{2}|\hat{\mathbf{E}}|^2 Re j\hat{\epsilon} = \frac{\omega}{2}|\hat{\mathbf{E}}|^2\epsilon'' \tag{20}$$

Thus, ϵ'' represents the electrical dissipation associated with the polarization process.

In the literature, the *loss tangent* $\tan\delta$ is often used to represent dissipation. It is the tangent of the phase angle δ of the complex dielectric constant defined in terms ϵ' and ϵ'' in Fig. 11.5.3. Thus,

$$\tan\delta = \frac{\epsilon''}{\epsilon'}; \qquad \cos\delta = \frac{\epsilon'}{|\hat{\epsilon}|}; \qquad \sin\delta = \frac{\epsilon''}{|\hat{\epsilon}|} \tag{21}$$

From this definition, it follows from Eulers formula that

$$\epsilon' - j\epsilon'' = |\hat{\epsilon}|(\cos\delta - j\sin\delta) = |\hat{\epsilon}|e^{-j\delta} \tag{22}$$

Given the complex amplitude of the electric field, $\mathbf{D}$ is

$$\mathbf{D} = Re\{|\hat{\epsilon}|\hat{\mathbf{E}}e^{j(\omega t - \delta)}\} \tag{23}$$

If the electric field is $\mathbf{E}_o\cos(\omega t)$, then $\mathbf{D}$ is $|\hat{\epsilon}|\mathbf{E}_o\cos(\omega t - \delta)$. The electric displacement lags the electric field by the phase angle δ.

In terms of the loss tangent defined by (21), the time average electrical dissipation density of (20) becomes

$$\langle P_d \rangle = \frac{\omega\epsilon'}{2}|\hat{E}|^2\tan\delta \tag{24}$$

Usually the loss tangent and ϵ' are measured. In the following example, we compute the complex permittivity from a model of the polarizable medium and find the electrical dissipation on a macroscopic basis. In this special case we have the option of finding the time average loss by considering each of the dipoles on a microscopic basis. This is not generally possible, because the interactions among dipoles that are neglected in this example are usually too complicated for an analytic treatment.

Example 11.5.2. An Artificial Lossy Dielectric

By putting together examples considered in Chaps. 6 and 7, we can illustrate the origins of the complex permittivity. The artificial dielectric of Example 6.6.1 and Demonstration 6.6.1 had "molecules" consisting of perfectly conducting spheres. As a result, the polarization was pictured as instantaneously in step with the applied field. We consider now the result of having spheres that have finite conductivity.

The response of a single sphere having a finite conductivity σ and permittivity ϵ surrounded by free space is a special case of Example 7.9.3. The response to a sinusoidal drive is summarized by (7.9.36), where we set $\sigma_a = 0$, $\epsilon_a = \epsilon_o$, $\sigma_b = \sigma$, and $\epsilon_b = \epsilon$. All that is required from this solution for the potential is the moment of a dipole that would give rise to the same exterior field as does the sphere. Comparison of the potential of a dipole, (4.4.10), to that given by (7.9.36a) shows that the complex amplitude of the moment is

$$\hat{p} = 4\pi\epsilon_o R^3 \frac{\left[1 + j\omega\tau_e \frac{(\epsilon-\epsilon_o)}{2\epsilon_o+\epsilon}\right]}{1 + j\omega\tau_e}\hat{E} \tag{25}$$

where $\tau_e \equiv (2\epsilon_o + \epsilon)/\sigma$. If mutual interactions between dipoles are ignored, the polarization density $\mathbf{P}$ is this moment of a single dipole multiplied by the number of dipoles per unit volume, N. For a cubic array with a distance s between the dipoles (the centers of the spheres), $N = 1/s^3$. Thus, the complex amplitude of the electric displacement is

$$\hat{D} = \epsilon_o\hat{E} + \hat{P} = \epsilon_o\hat{E} + \frac{\hat{p}}{s^3} \tag{26}$$

Combining this result with the moment given by (25) yields the desired constitutive law in the form $\hat{\mathbf{D}} = \hat{\epsilon}\hat{\mathbf{E}}$, where the complex permittivity is

$$\hat{\epsilon} = \epsilon_o\left\{1 + 4\pi\left(\frac{R}{s}\right)^3 \frac{\left[1 + j\omega\tau_e \frac{(\epsilon-\epsilon_o)}{2\epsilon_o+\epsilon}\right]}{1 + j\omega\tau_e}\right\} \tag{27}$$

The time average power dissipation density follows from this expression and (20).

$$\langle P_d\rangle = \frac{2\pi\epsilon_o}{\tau_e}\left(\frac{R}{s}\right)^3\left[\frac{3\epsilon_o}{2\epsilon_o+\epsilon}\right]\frac{(\omega\tau_e)^2}{1+(\omega\tau_e)^2}|\hat{E}|^2 \tag{28}$$

The dependence of the power dissipation on frequency has the same form as for the induction heating example, Fig. 11.5.2. At low frequencies, the surface charges induced at the north and south poles of each sphere are completely determined by the external field. Thus, the current density within the sphere that makes possible the accumulation of these surface charges is proportional to the time rate of change of the applied field. At low frequencies, the dissipation is proportional to the square of the volume current and hence to the square of the time rate of change of the applied field. As a result, at low frequencies, the dissipation density increases with the square of the frequency.

As the frequency is raised, less surface charge is induced on the spheres. Although the amount of charge induced is inversely proportional to the frequency, there is a compensating effect because the volume currents are responsible for the dissipation, and these are proportional to the time rate of change of the charge. Thus, the dissipation density reaches a saturation value as the frequency becomes very high.

One tool used to form a picture of atomic, molecular, and domain physics is dielectric spectroscopy. Using this approach, the frequency dependence of the complex permittivity is used to gain insight into the microscopic structure.

Magnetization, like polarization, can also be the source of dissipation. The time average dissipation density due to magnetization follows by taking the time averge of the third and fourth terms on the right in the basic power theorem, (11.2.7). Combined, these terms give

$$\langle P_d \rangle = \langle \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \rangle \tag{29}$$

For small-signal applications, this source of dissipation is dealt with by introducing a complex permeability $\hat{\mu}$ such that $\hat{\mathbf{B}} = \hat{\mu}\hat{\mathbf{H}}$. The role of the complex permeability is similar to that of the complex permittivity. The artificial diamagnetic material of Example 9.5.2 and Demonstration 9.5.1 can be used to exemplify the concept. Instead of perfectly conducting spheres that give rise to a magnetic moment instantaneously induced antiparallel to the applied field, spherical shells of finite conductivity would be used. The dipole moment induced in the individual spherical shells would be deduced following the same approach as in Sec. 10.4. The resulting dipole moment would not be in phase with an applied sinusoidally varying magnetic field. The derivation of an equivalent complex permeability would follow from the same line of reasoning as used in the previous example.

Hysteresis Losses. Under periodic conditions in magnetizable solids, B and H are related by the hysteresis curve described in Sec. 9.4 and illustrated again in Fig. 11.5.4. What time average power dissipation is implied by the hysteresis?

As before, B and H are collinear. However, neither is now a single-valued function of the other. Evaluation of (29) is accomplished by breaking the cycle into two parts, each involving a single-valued relationship between B and H. The first is the upswing "trajectory" from $A \rightarrow C$ in Fig. 11.5.4. Over this half-cycle, which takes B from B_A to B_C, the trajectory is $H_+(B)$. With B taken as B_A when $t = 0$, it follows from (11.4.4) and (11.4.5) that

$$\int_0^{T/2} \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} dt = \int_{B_A}^{B_C} \frac{dW_m}{dt} dt = \int_{B_A}^{B_C} H_+ \delta B \tag{30}$$

This is the area under the curve of H versus B between A and C in Fig. 11.5.4, traversed on the "upswing." A similar evaluation for the "downswing," where the

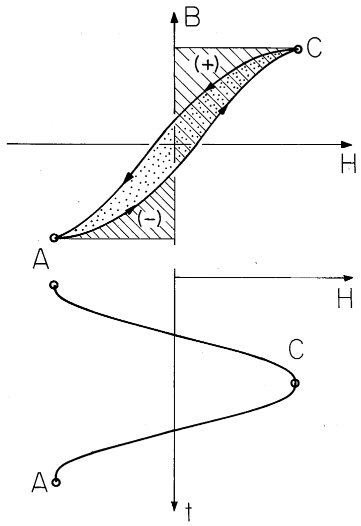

Fig. 11.5.4 With the application of a sinusoidal magnetic field intensity, a steady state is reached in which the hysteresis loop shown in the $B - H$ plane is traced out in the direction shown. The dashed area represents the energy density associated with upward traversal from A to C. The dotted area inside the loop represents the energy density dissipated per traversal of the loop.

trajectory is $H_-(B)$, gives

$$\int_{T/2}^{T} \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} dt = \int_{B_C}^{B_A} H_- \delta B = - \int_{B_A}^{B_C} H_- \delta B \tag{31}$$

The time average power dissipation, (29), then is the sum of these two contributions divided by T.

$$\langle P_d \rangle = \frac{1}{T} \Big[\int_{B_A}^{B_C} H_+ \delta B - \int_{B_A}^{B_C} H_- \delta B \Big] \tag{32}$$

Thus, the area within the hysteresis loop is the energy dissipated in one cycle.

11.6 ELECTRICAL FORCES ON MACROSCOPIC MEDIA

Electrical forces on macroscopic materials have their origins in the forces exerted on the microscopic particles of which the materials are composed. Macroscopic fields have been used to describe conduction, polarization, and magnetization. In Chaps. 6, 7 and 9, polarization, current, and magnetization densities, respectively, were related to the macroscopic field variables through constitutive laws. Typically, the parameters in these laws are determined from measurements. Thus, the experimentally determined relations make it unnecessary to take detailed account of how the microscopic fields are averaged.

Because the definition of the average is already implicit in our macroscopic formulation of Maxwell's equations, we must now take care that our use of macroscopic field quantities for representing electromagnetic forces is self-consistent. The

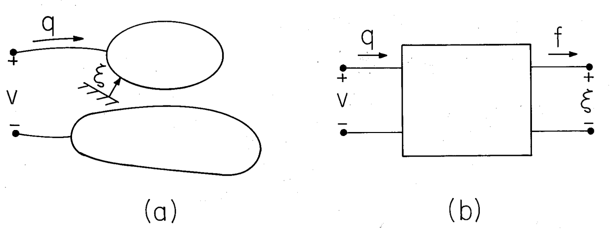

Fig. 11.6.1 (a) Electroquasistatic system having one electrical terminal pair and one mechanical degree of freedom. (b) Schematic representation of EQS subsystem with coupling to external mechanical system represented by a mechanical terminal pair.

force on a macroscopic volume element ΔV of a material is the sum of the forces on the charged particles and magnetic dipoles constituting the material. Consider the simple case in which no magnetic dipoles are present. Then

$$\mathbf{f} = \sum_i q_i\{\mathbf{E}(\mathbf{r}_i) + \mathbf{v}_i \times \mu_o \mathbf{H}(\mathbf{r}_i)\} \tag{1}$$

where the summation is over all the charges within ΔV at their respective positions. Now, the fields $\mathbf{E}(\mathbf{r}_i)$ and $\mathbf{H}(\mathbf{r}_i)$ are the *microscopic* fields that vary greatly from point $\mathbf{r}_i$ to point $\mathbf{r}_j$ in the material. The macroscopic fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ are averaged (smoothed) versions of these fields, whose sources are the averaged charge densities

$$\rho \equiv \sum_i \frac{q_i}{\Delta V} \tag{2}$$

and

$$\mathbf{J} \equiv \sum_i \frac{q_i \mathbf{v}_i}{\Delta V} \tag{3}$$

where the velocity $\mathbf{v}_i$ of the microscopic particles should be distinguished from that of the macroscopic material in which they are embedded or through which they move. The average of a product is not equal to the product of the averages. Thus, one could not find the force density $\mathbf{F} = \mathbf{f}/\Delta V$ from the expression $\rho\mathbf{E} + \mathbf{J} \times \mu_o \mathbf{H}$, as the product of the averaged charge density and averaged electric field plus averaged current density times averaged magnetic flux density. Other methods have to be used to determine the force. One of the most useful is the energy method. Given the constitutive law for the material, which represents the interrelationship between macroscopic field variables, conservation of energy provides a way of deducing the self-consistent force acting on the material.

In this and the next section, we illustrate how total forces can be determined using conservation of energy as a *premise.* In this section, the EQS systems considered have only one mechanical degree of freedom and only one electrical terminal pair. In the next section, MQS systems are considered and the approach is broadened to a somewhat more general class of systems. A parallel approach determines the force density rather than the total force. After expanding on microscopic forces in Sec. 11.8, we shall review macroscopic force densities in Sec. 11.9.

Typical of the electroquasistatic problems considered in this section is the pair of metallic electrodes shown in Fig. 11.6.1. With the application of a voltage, unpaired charges of opposite polarity are induced on the electrode surfaces. The electrical state of the system is specified by giving the geometry and the potential difference v between the electrodes. Here we picture one electrode as movable, with its position denoted by ξ. The two terminal pair system of Fig. 11.6.1b is useful to include mechanical effects via an additional terminal pair. If we think of the net unpaired charge q on the electrode as an electrical terminal variable complementing v, then the force of electrical origin f complements the mechanical displacement ξ.

Given the electrical terminal relation $v = v(q, \xi)$, we now use an energy conservation principle to determine the force $f = f(q, \xi)$ that acts to increase the displacement ξ. The electrical terminal relation can either be regarded as a measured function or be predicted using the macroscopic field laws and constitutive laws for the materials within the "box."

It is now *assumed* that there is no conversion of electrical energy to thermal form within the box of Fig. 11.6.1b. Mechanisms for conversion of energy to heat are modeled by elements outside the box. For example, the finite conductivity of any dielectric is taken into account by a resistance external to the system. Thus, the electrical power input to what is defined as the "box," the electroquasistatic subsystem, must either result in a change in the electrical energy stored or mechanical power expended as the force f acts on the mechanical system. The integral form of the power conservation theorem, (11.1.1), is generalized to include the rate of work by the force f

$$-\oint_S \mathbf{S} \cdot d\mathbf{a} = \frac{d}{dt} \int_V W_e dv + f \frac{d\xi}{dt} \tag{4}$$

In Sec. 11.4, we represented the quasistatic net electrical power input on the left in this expression in circuit theory terms. With the total energy w_e defined as the integral of the energy density over the entire volume of the system, (4) becomes

$$v \frac{dq}{dt} = \frac{dw_e}{dt} + f \frac{d\xi}{dt} \tag{5}$$

where the electrical power input is the product $vi = vdq/dt$. Multiplication of (4) by dt converts a statement of power flow to one of energy conservation.

$$\boxed{vdq = dw_e + f d\xi} \tag{6}$$

If an increment of charge dq is placed on an electrode at potential v, an increment of energy vdq is added to the system that produces a change in the total stored energy dw_e, an increment of work $fd\xi$ done on an external mechanical system, or some combination of both. Here, $f(q, \xi)$ is the as yet unknown force. Solved for dw_e, this energy conservation statement is

$$dw_e = vdq - fd\xi \tag{7}$$

This expression describes what might be termed a quasistatic electrical and mechanical *subsystem*. The *state* of this subsystem is specified by prescribing the geometry (ξ) and the charge on the electrode, for then the voltage of the electrode follows

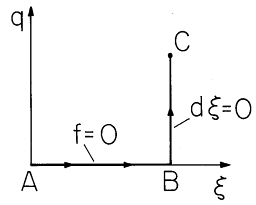

Fig. 11.6.2 Path of line integration in state space (q, ξ) used to find energy at location C.

from the terminal relation $v(q, \xi)$. The state of the subsystem is fully determined by the variables (q, ξ), which are therefore regarded as *independent variables.* In terms of the two terminal pairs shown in Fig. 11.6.1b, one of each pair of terminal variables has been chosen as an independent variable.

The incremental change in $w_e(q, \xi)$ associated with incremental changes of dq and $d\xi$ in the independent variables is

$$dw_e = \frac{\partial w_e}{\partial q} dq + \frac{\partial w_e}{\partial \xi} d\xi \tag{8}$$

Because q and ξ can be independently specified, (7) and (8) must hold for any combination of dq and dξ. For example, they must hold if the position of the electrode is held fixed so that $d\xi = 0$ and the charge is changed by the incremental amount dq. They must also describe the change in energy resulting from making an incremental displacement $d\xi$ of the electrode under open circuit conditions, where $dq = 0$. Indeed, (7) and (8) hold if q and ξ are changed by *arbitrary* incremental amounts, and so it follows that the coefficients of dq in (7) and (8) must be equal to each other, as must the coefficients of dξ.

$$\boxed{v = \frac{\partial}{\partial q} w_e(q, \xi); \qquad f = -\frac{\partial}{\partial \xi} w_e(q, \xi)} \tag{9}$$

Given the total energy, written in terms of the independent variables (q, ξ), the second of these relations provides the desired force. Integration of the energy density over the volume of the system is one way to determine w_e. Another is to integrate (7) along a line in the state space (q, ξ) designed so that the integral can be carried out without having to know f.

$$w_e(q, \xi) = \int (v dq - f d\xi) \tag{10}$$

Such a path[4] is shown in Fig. 11.6.2, where it is assumed that the force of electrical origin f is zero if the charge q is zero. Thus, in integrating along the contour $q = 0$ from $A \rightarrow B$, $dq = 0$ and $f = 0$, so there is no contribution. The remainder of the integral, from $B \rightarrow C$, is carried out with ξ fixed, so $d\xi = 0$, and (10) reduces to

[4] Note the analogy with the line integral $\int (E_x dx + E_y dy)$ of a two-dimensional conservation field that results in the potential $\phi(x, y)$.

$$\boxed{w_e = -\int_0^\xi f(0,\xi')d\xi' + \int_0^q v(q',\xi)dq' = \int_0^q v(q',\xi)dq'} \tag{11}$$

We have accounted for the energy required to place the subsystem in the state (q, ξ). In physical terms, the mathematical steps represent first assembling the subsystem mechanically with no electrical excitation. Because there is no force acting on the electrode as it is put in place, no work is involved. Then, with its location fixed, the electrode is charged by means of an electrical source.

Suppose that the subsystem is electrically linear, so that either as a result of mathematical modeling or of measurements on the actual system, the electrical terminal relation takes the form

$$v = \frac{q}{C(\xi)} \tag{12}$$

Then, with this relation used to evaluate (11), it follows that the energy is

$$w_e = \int_0^q \frac{q'}{C}dq' = \frac{1}{2}\frac{q^2}{C} \tag{13}$$

Finally, the desired force of electrical origin follows from substituting this expression into (9b).

$$f = -\frac{1}{2}q^2\frac{dC^{-1}}{d\xi} = \frac{1}{2}\Big(\frac{q}{C}\Big)^2\frac{dC}{d\xi} \tag{14}$$

Note that with a similar substitution into (9a), the terminal relation of (12) is obtained.

Once the partial derivative with respect to ξ has been taken while holding the proper independent variable (q) fixed, the force can be written in terms of variables other than the independent ones. Thus, with the use of the terminal relation, (12), the force is written in terms of the terminal voltage v as

$$\boxed{f = \frac{1}{2}v^2\frac{dC}{d\xi}} \tag{15}$$

The following example gives the opportunity to apply this result to a specific configuration.

Example 11.6.1. Force on a Capacitor Plate

The region between the plane parallel electrodes shown in Fig. 11.6.3 is filled by a layer of dielectric having permittivity ϵ and thickness b and an air gap ξ. The total distance between electrodes, $b + \xi$, is small compared to the linear dimensions of the plates, so fringing fields will be ignored. Thus, the electric fields E_a and E_b in the air gap and in the dielectric, respectively, are uniform. What force on the upper electrode results from applying the voltage v between the electrodes?

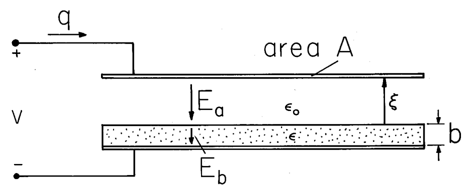

Fig. 11.6.3 Specific example of EQS systems having one electrical and one mechanical terminal pair.

First we determine the charge q on the upper electrode. To this end, the integral of $\mathbf{E}$ from the upper electrode to the lower one must be equal to the applied voltage, so

$$v = \xi E_a + bE_b \tag{16}$$

Further, there is presumably no unpaired surface charge at the interface between the dielectric layer and the air gap. Thus, Gauss' continuity condition requires that

$$\epsilon_o E_a = \epsilon E_b \tag{17}$$

Elimination of E_b between these equations gives

$$E_a = \frac{v}{\xi + b\frac{\epsilon_o}{\epsilon}} \tag{18}$$

In terms of this electric field at the surface of the upper electrode, Gauss' continuity condition shows that the total charge on the upper electrode is

$$q = D_a A = \epsilon_o E_a A \tag{19}$$

and so it follows from (18) that the electrical terminal relation can be written in terms of a capacitance C.

$$q = Cv; \qquad C \equiv \frac{\epsilon_o A}{\xi + b\frac{\epsilon_o}{\epsilon}} \tag{20}$$

Because the dielectric is described by a linear constitutive law, we have obtained an electrical terminal relation where v is a linear function of q.

The force acting on the upper electrode follows from a substitution of (20) into (15).

$$f = -\frac{1}{2}v^2 \frac{\epsilon_o A}{\left(\xi + b\frac{\epsilon_o}{\epsilon}\right)^2} \tag{21}$$

By definition, if f is positive, it acts in the direction of ξ. Here we find that regardless of the polarity of the applied voltage, f is negative. This is to be expected, because charges of one polarity on the upper electrode are attracted toward those of opposite polarity on the lower electrode.

In describing energy conversion, a minus sign can be extremely important. For example, vdq is the incremental energy *into* the electroquasistatic subsystem, while $fd\xi$ is the energy *leaving* that subsystem as the force of electrical origin acts *on* the external mechanical system. Thus, if f is positive, it acts on the mechanical system in such a direction as to increase the associated displacement.

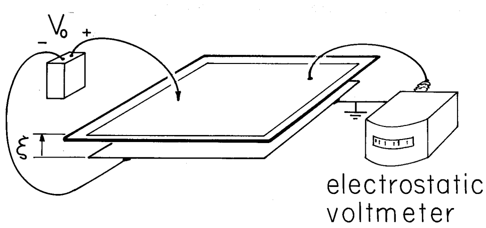

Fig. 11.6.4 Apparatus used to demonstrate amplification of voltage as the upper electrode is raised. (The electrodes are initially charged and then the voltage source is removed so q = constant.) The electrodes, consisting of foil mounted on insulating sheets, are about 1 m × 1 m, with the upper one insulated from the frame, which is used to control its position. The voltage is measured by the electrostatic voltmeter, which "loads" the system with a capacitance that is small compared to that of the electrodes and (at least on a dry day) a negligible resistance.

Rotating motors and generators are examples where the conversion of energy between electrical and mechanical form is a cyclic process. In these cases, the subsystem returns to its original state once each cycle. The energy converted per cycle is determined by integrating the energy conservation law, (6), around the closed path in the state space representing this process.

$$\oint v dq = \oint dw_e + \oint f d\xi \tag{22}$$

Because the energy stored in the system must return to its original value, there is no net contribution of the energy storage term in (22). For a cyclic process, the net electrical energy input per cycle must be equal to the net mechanical power output per cycle.

$$\oint v dq = \oint f d\xi \tag{23}$$

The following demonstration is primarily intended to give further insight into the implications of the conservation of energy principle for a cyclic process.

Demonstration 11.6.1. An Energy Conversion Cycle

The experiment shown in Fig. 11.6.4 is based on the plane parallel capacitor configuration analyzed in Example 11.6.1. The lower electrode, aluminum foil mounted on a table top, is covered by a thin sheet of plastic. The upper one is also foil, but taped to an insulating sheet which is attached to a frame. This electrode can then be manually raised and lowered to effectively control the displacement ξ.

With the letters A through D used to designate states of the system, we consider the following energy conversion cycle.

- $A \rightarrow B$. With $v = 0$, the upper electrode rests on the plastic sheet. A voltage V_o is applied.
- $B \rightarrow C$. With the voltage source removed so that the upper electrode is electrically isolated, it is raised to the position $\xi = L$.
- $C \rightarrow D$. The upper electrode is shorted, so that its voltage returns to zero.
- $D \rightarrow A$. The upper electrode is returned to its original position at $\xi = 0$.

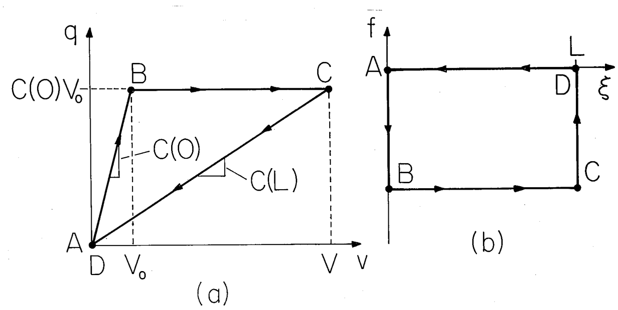

Fig. 11.6.5 Closed paths followed in cyclic conversion of energy from mechanical to electrical form: (a) in (q, v) plane; and (b) in (f, ξ) plane.

Is electrical energy converted to mechanical form, or vice versa?

The process in carrying out the closed integrals on the left-hand and right-hand sides of (23) as the cycle is carried out can be pictured in the (q, v) and (f, ξ) planes, respectively, as shown in Fig. 11.6.5. From $A \rightarrow B$, $q = C(0)v$, where $C(\xi)$ is given by (20). Thus, the trajectory in the (q, v) plane is a straight line ending at the voltage $v = V_o$ of the source. Because the upper electrode has remained at its original position, the trajectory in the (f, ξ) plane is along the $\xi = 0$ axis. The force on the electrode caused by raising its voltage to V_o follows from (21).

$$f = -\frac{1}{2}\frac{v^2C^2}{\epsilon_o A} \Rightarrow f_B = -\frac{1}{2}\frac{[V_oC(0)]^2}{\epsilon_o A} \tag{24}$$

Now, from $B \rightarrow C$, the voltage source is removed so that as the upper electrode is raised to $\xi = L$, its charge is conserved. This means that the trajectory in the (q, v) plane is one where $q = \text{constant} = V_oC(0)$. The voltage reached by the upper electrode can be found by requiring that q be conserved.

$$V_oC(0) = VC(L) \Rightarrow V = V_o\frac{C(0)}{C(L)} \tag{25}$$

In the experiment, the thickness b of the dielectric sheet is a fraction of a millimeter, while the final elevation $\xi = L$ might be 20 cm. (If the displacement is larger than this, the fringing field comes into play and the expression for the capacitance is no longer valid.) Thus, as the sheet is raised, an original voltage of 500 V is easily amplified to 10 - 20 kV. This is readily observed by means of an electrostatic voltmeter attached to the upper electrode, as shown in Fig. 11.6.4.

To determine the trajectory $B \rightarrow C$ in the (f, ξ) plane, observe from (14) or (24) that as a function of q, the force is independent of ξ.

$$f = -\frac{1}{2}\frac{q^2}{\epsilon_o A} \tag{26}$$

The trajectory $B \rightarrow C$ in the (f, ξ) plane is therefore one of constant f_B given by (24). In general, $f(q, \xi)$ is not independent of ξ, but in plane parallel geometry, it is.

The system is now returned to its original state in two steps. First, from $C \rightarrow D$, the upper electrode remains at $\xi = L$ and is shorted to ground. In the (q, v) plane, the state returns to the origin along the straight line given by $q = C(L)v$, (20). In the (f, ξ) plane, the force drops to zero with $\xi = L$. Second, from $D \rightarrow A$, the upper electrode is returned to its original position. The values of (q, v) remain zero, while the trajectory in the (f, ξ) plane is $f = 0$.

The experiment is simple enough so that we can use physical reasoning to decide the direction of energy conversion. Although the force of gravity is likely to exceed the electrical force of attraction between the electrodes, as far as the electrical subsystem is concerned, the upper electrode is raised against a downward electrical force. Because the charge is removed before it is lowered, there is no electrical force on the electrode as it is lowered. Thus, net work is done on the EQS subsystem. The right-hand side of (23), the net work done by the subsystem on the external mechanical system, is thus negative.

Evaluation of one or the other of the two sides of the energy conversion law, (23), provides two other ways to determine the direction of energy conversion. Consider first the electrical input energy. The integral has contributions from $A \rightarrow B$ (where the source is used to charge the upper electrode) and from $C \rightarrow D$ (where the electrode is discharged). The areas under the respective triangles representing the integral of vdq are

$$\oint vdq = \int_A^B vdq + \int_C^D vdq = \frac{1}{2}C(0)V_o^2 - \frac{1}{2}C(L)V^2 \tag{27}$$

In view of the expression for $C(\xi)$, (20), this expression can be written as

$$\oint vdq = \frac{1}{2}C(0)V_o^2\left[1 - \frac{C(0)}{C(L)}\right] = -\frac{1}{2}C(0)V_o^2\frac{L}{b}\frac{\epsilon}{\epsilon_o} \tag{28}$$

This expression is clearly negative, indicating that the net electrical energy flow is out of the electrical terminal pair. This is consistent with having a net mechanical energy input to the system.

The net mechanical output energy per cycle expressed by the right-hand side of (23) should be equal to (28). To see that this is so, we recognize that the integral consists of two possible contributions, from $B \rightarrow C$ and $D \rightarrow A$. During the latter, $f = 0$, so the magnitude of the integral is simply the area of the rectangle in Fig. 11.6.5b.

$$\oint fd\xi = -\frac{1}{2}V_o^2\frac{C^2(0)}{\epsilon_o A}L = -\frac{1}{2}C(0)V_o^2\frac{L}{b}\frac{\epsilon}{\epsilon_o} \tag{29}$$

As required by the conservation law, this mechanical energy output is negative and is equal to the net electrical input energy given by (28).

With the sequence of electrical and mechanical terminal constraints described above, there is a net conversion of energy from mechanical to electrical form. The system acts as an electrical generator with energy provided by whoever raises and lowers the upper electrode. With the voltage applied when the electrode is at its largest displacement, and the electrode grounded before it is raised, the energy flow is from the electrical voltage source to the mechanical system. In this case, the system acts as a motor. We will have the opportunity to exemplify a practical motor in the next section. Most motors are MQS rather than EQS. However, a practical EQS device that is designed to convert energy from mechanical to electrical form is the capacitor microphone, a version of which was described in Example 6.3.3.

Electrical forces have their origins in forces on unpaired charges and on dipoles. The force on the upper capacitor plate of Example 11.6.1 is due to the unpaired charges. The equal and opposite force on the combination of electrode and dielectric

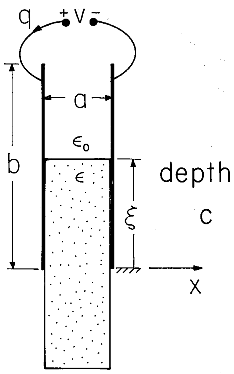

Fig. 11.6.6 Slab of dielectric partially extending between capacitor plates. The spacing, a, is much less than either b or the depth c of the system into the paper. Further, the upper surface at ξ is many spacings a away from the upper and lower edges of the capacitor plates, as is the lower surface as well.

is in part due to unpaired charges and in part to dipoles induced in the dielectric. The following exemplifies a total force that is entirely due to polarization.

Example 11.6.2. Force on a Dielectric Material

We return to the configuration of Example 11.4.1, where a dielectric slab extends a distance ξ into the region between plane parallel electrodes, as shown in Fig. 11.6.6. The capacitance was found in Example 11.4.1 to be (11.4.16).

$$C = \frac{c}{a}[\epsilon_o b + \xi(\epsilon - \epsilon_o)] \tag{30}$$

It follows from (15) that there is a force tending to draw the dielectric into the region between the electrodes.

$$f = \frac{1}{2}v^2\frac{c}{a}(\epsilon - \epsilon_o) \tag{31}$$

This force results because dipoles induced by the fringing field experience forces in the ξ direction that are passed on to the material in which they are embedded. Even though the force is due to the fringing fields, the net force does not depend on the details of that field. This is evident from our energy arguments because, at least as long as the upper edge of the slab is well within the region between the plates and the lower edge never reaches the vicinity of the electrodes, the energy storage in *the fringing fields does not change when the slab is moved.*

Further discussion of the force density responsible for the force on the dielectric will be given in Sec. 11.9. Its physical reality is demonstrated next.

Demonstration 11.6.2. Force on a Liquid Dielectric

In the experiment shown in Fig. 11.6.7, capacitor plates are dipped into a dish full of dielectric liquid. Thus, with the application of the voltage, it rises against gravity. To demonstrate the relationship between the voltage and the force, the spacing between the electrodes is a slowly varying function of radial position. With r denoting the radial distance from an axis where an extension of the electrodes would join, and α

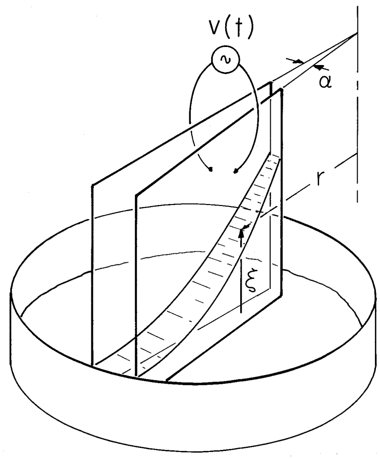

Fig. 11.6.7 In a demonstration of the polarization force, a pair of conducting transparent electrodes are dipped into a liquid (corn oil dyed with food coloring). They are closer together at the upper right than at the lower left, so when a voltage is applied, the electric field intensity decreases with increasing distance, r, from the apex. As a result, the liquid is seen to rise to a height that varies as $1/r^2$. The electrodes are about 10 cm × 10 cm, with an electric field exceeding the nominal breakdown strength of air at atmospheric pressure, 3×10^6 V/m. The experiment is therefore carried out under pressurized nitrogen.

the angle between the electrodes, the spacing at a distance r is αr. Thus, in (31), the spacing between electrodes $a \to \alpha r$ and the force per unit radial distance tending to push the liquid upward is a function of the radial position.

$$\frac{f}{c} = \frac{1}{2}\frac{v^2}{\alpha r}(\epsilon - \epsilon_o) \tag{32}$$

This force must raise a column of liquid having a height ξ and width αr. With the mass density of the liquid defined as ρ, the total mass per radial distance raised by this force is therefore $\alpha r \xi \rho$. Force equilibrium is therefore represented by setting the force per unit radial length equal to this mass multiplied by the gravitational acceleration g.

$$\frac{1}{2}\frac{v^2}{\alpha r}(\epsilon - \epsilon_o) = \alpha r \xi \rho g \tag{33}$$

This expression can be solved for $\xi(r)$.

$$\xi = \frac{1}{2}\frac{v^2(\epsilon - \epsilon_o)}{\alpha^2 \rho g}\frac{1}{r^2} \tag{34}$$

In the experiment,[5] the electrodes are constructed from tin-oxide coated glass. They are then both conducting and transparent. As a result, the height to which the liquid rises can be seen to obey (34), both as to its magnitude and its radial dependence on r.

[5] See film *Electric Fields and Moving Media* from series by National Committee for Electrical Engineering Films, Education Development Center, 39 Chapel St., Newton, Mass. 02160.

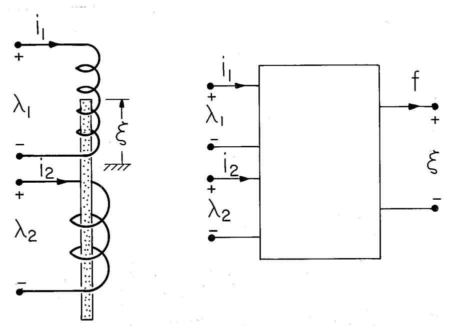

Fig. 11.7.1 (a) Magnetoquasistatic system with two electrical terminal pairs and one mechanical degree of freedom. (b) MQS subsystem representing (a).

To obtain an appreciable rise of the liquid without exceeding the field strength for electrical breakdown between the electrodes, the atmosphere over the liquid must be pressurized. Also, to avoid effects of unpaired charges injected at high field strengths by the electrodes, the applied voltage is alternating and, because the force is proportional to the square of the applied field, the height of rise is proportional to the rms value of the voltage.

11.7 MACROSCOPIC MAGNETIC FORCES

In this section, an energy principle is applied to the determination of net forces in MQS systems. With Sec. 11.6 as background, it is appropriate to include the case of multiple terminal pairs. As in Sec. 11.4, the coenergy is again found to be a convenient alternative to the energy.

The MQS system is shown schematically in Fig. 11.7.1. It has two electrical terminal pairs and one mechanical degree of freedom. The magnetoquasistatic subsystem now described by an energy principle excludes electrical dissipation and all aspects of the mechanical system, mechanical energy storage and dissipation. The energy principle then states that the input of electrical power through the electrical terminal pairs either goes into a rate of change of the stored magnetic energy or into a rate of change of the work done on the external mechanical world.

$$v_1 i_1 + v_2 i_2 = \frac{dw_m}{dt} + f\frac{d\xi}{dt} \tag{1}$$

As in Sec. 11.6, our starting point in finding the force is a postulated principle of energy conservation. Because the system is presumably MQS, in accordance with (11.3.29), the left-hand side represents the net flux of power into the system. With the addition of the last term and the inherent assumption that there is no electrical dissipation in the subsystem being described, (1) is more than the recasting of Poynting's theorem.

In an MQS system, the voltages are the time rates of change of the flux linkages. With these derivatives substituted into (1) and the expression multiplied by dt, it becomes

$$\boxed{i_1 d\lambda_1 + i_2 d\lambda_2 = dw_m + f d\xi} \tag{2}$$

This energy principle states that the increments of electrical energy put into the MQS subsystem (as increments of flux $d\lambda_1$ and $d\lambda_2$ through the terminals multiplied by their currents i_1 and i_2, respectively) either go into the total energy, which is increased by the amount dw_m, or into work on the external mechanical system, subject to the force f and experiencing a displacement $d\xi$.

With the energy principle written as in (2), the flux linkages are the independent variables. We saw in Example 11.4.2 that it is inconvenient to specify the flux linkages as functions of the currents. With the objective of casting the currents as the independent variables, we now recognize that

$$i_1 d\lambda_1 = d(i_1\lambda_1) - \lambda_1 di_1; \qquad i_2 d\lambda_2 = d(i_2\lambda_2) - \lambda_2 di_2 \tag{3}$$

and substitute into (2) to obtain

$$dw'_m = \lambda_1 di_1 + \lambda_2 di_2 + f d\xi \tag{4}$$

where the coenergy function, seen before in Sec. 11.4, is defined as

$$w'_m = i_1\lambda_1 + i_2\lambda_2 - w_m \tag{5}$$

We picture the MQS subsystem as having flux linkages λ_1 and λ_2, a force f and a total energy w_m that are specified once the currents i_1 and i_2 and the displacement ξ are stipulated. According to (4), the coenergy is a function of the independent variables i_1, i_2, and ξ, $w'_m = w'_m(i_1, i_2, \xi)$, and the change in w'_m can also be written as

$$dw'_m = \frac{\partial w'_m}{\partial i_1} di_1 + \frac{\partial w'_m}{\partial i_2} di_2 + \frac{\partial w'_m}{\partial \xi} d\xi \tag{6}$$

Because the currents and displacement are independent variables, (4) and (6) can hold only if the coefficients of like terms on the right are equal. Thus,

$$\boxed{\lambda_1 = \frac{\partial w'_m}{\partial i_1}; \qquad \lambda_2 = \frac{\partial w'_m}{\partial i_2}; \qquad f = \frac{\partial w'_m}{\partial \xi}} \tag{7}$$

The last of these three expressions is the key to finding the force f.

Reciprocity Condition. Before we find the coenergy and hence f, consider the implication of the first two expressions in (7) for the electrical terminal relations. Taking the derivative of λ_1 with respect to i_2, and of λ_2 with respect to i_1, shows that

$$\boxed{\frac{\partial \lambda_1}{\partial i_2} = \frac{\partial^2 w'_m}{\partial i_1 \partial i_2} = \frac{\partial^2 w'_m}{\partial i_2 \partial i_1} = \frac{\partial \lambda_2}{\partial i_1}} \tag{8}$$

Although this *reciprocity condition* must reflect conservation of energy for any lossless system, magnetically linear or not, consider its implications for a system described by the linear terminal relations.

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{9}$$

Application of (8) shows that energy conservation requires the equality of the mutual inductances.

$$L_{12} = L_{21} \tag{10}$$

This relation has been derived in Example 11.4.2 from a related but different point of view.

Finding the Coenergy. To find w'_m, we integrate (4) along a path in the state space (i_1, i_2, ξ) arranged so that the integral can be carried out without having to know f.

$$\boxed{w'_m = \int (\lambda_1 di_1 + \lambda_2 di_2 + f d\xi)} \tag{11}$$

Thus, the first leg of the line integral is carried out on ξ with the currents equal to zero. Provided that $f = 0$ in the absence of these currents, this means that the integral of $f d\xi$ makes no contribution.

The payoff from our formulation in terms of the coenergy rather than the energy comes in being able to carry out the remaining integration using terminal relations in which the flux linkages are expressed in terms of the currents. For the linear terminal relations of (9), this line integration was illustrated in Example 11.4.2, where it was found that

$$w'_m = \frac{1}{2} L_{11} i_1^2 + L_{12} i_1 i_2 + \frac{1}{2} L_{22} i_2^2 \tag{12}$$

Evaluation of the Force. In general, the inductances in this expression are functions of ξ. Thus, the force f follows from substituting this expression into (7c).

$$\boxed{f = \frac{1}{2} i_1^2 \frac{dL_{11}}{d\xi} + i_1 i_2 \frac{dL_{12}}{d\xi} + \frac{1}{2} i_2^2 \frac{dL_{22}}{d\xi}} \tag{13}$$

Of course, this expression applies to systems having a single electrical terminal pair as a special case where $i_2 = 0$.

This generalization of the energy method to multiple electrical terminal pair systems suggests how systems with two or more mechanical degrees of freedom are treated.

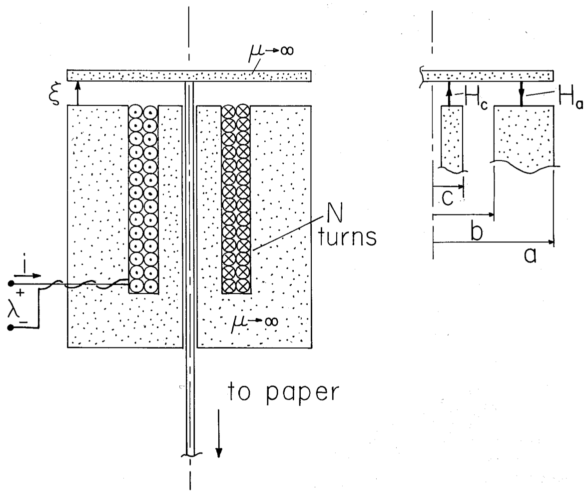

Fig. 11.7.2 Cross-section of axially symmetric transducer similar to the ones used to drive dot matrix printers.

Example 11.7.1. Driver for a Matrix Printer

A transducer that is similar to those used to drive an impact printer is shown in Fig. 11.7.2. The device, which is symmetric about the axis, might be one of seven used to drive wires in a high-speed matrix printer. The objective is to transduce a current i that drives the N-turn coil into a longitudinal displacement ξ of the permeable disk at the top. This disk is attached to one end of a wire, the other end of which is used to impact the ribbon against paper, imprinting a dot.

The objective here is to determine the force f acting on the plunger at the top. For simplicity, we make a highly idealized model in which the magnetizable material surrounding the coil and filling its core, as well as that of the movable disk, is regarded as perfectly permeable. Moreover, the air gap spacing ξ is small compared to the radial dimension a, so the magnetic field intensity is approximated as uniform in the air gap. The wire is so fine that the magnetizable material removed to provide clearance for the wire can be disregarded.

With H_a and H_c defined as shown by the inset to Fig. 11.7.2, Ampère's integral law is applied to a contour passing upward through the center of the core, across the air gap at the center, radially outward in the disk, and then downward across the air gap and through the outer part of the stator to encircle the winding in the infinitely permeable material.

$$H_a\xi + H_c\xi = Ni \tag{14}$$

A second relation between H_a and H_c follows from requiring that the net flux out of the disk must be zero.

$$\mu_o H_a \pi(a^2 - b^2) = \mu_o H_c \pi c^2 \tag{15}$$

Using this last expression to replace H_c in (14) results in

$$H_c = \frac{Ni}{\xi\left(1 + \frac{c^2}{a^2-b^2}\right)} \tag{16}$$

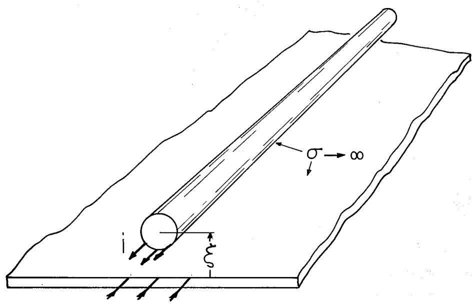

Fig. 11.7.3 Cross-section of perfectly conducting current-carrying wire over a perfectly conducting ground plane.

The magnetic flux linking every turn in the coil is $\mu_o H_c \pi c^2$. Thus, the total flux linked by the coil is

$$\lambda = Li; \qquad L = \frac{\mu_o N^2 \pi c^2}{\xi\left(1 + \frac{c^2}{a^2 - b^2}\right)} \tag{17}$$

Finally, the force follows from an evaluation of (13) (specialized to the single terminal pair system of this example).

$$f = \frac{1}{2} i^2 \frac{dL}{d\xi} = -\frac{1}{2} i^2 \frac{\mu_o N^2 \pi c^2}{\xi^2\left(1 + \frac{c^2}{a^2 - b^2}\right)} \tag{18}$$

As might have been expected by one who has observed magnetizable materials pulled into a magnetic field, the force is negative. Given the definition of ξ in Fig. 11.7.2, the application of a current will tend to close the air gap. To write a dot, the current is applied. To provide for a return of the plunger to its original position when the current is removed, a spring is inserted in the air gap.

In the magnetic transducer of the previous example, the force on the driver disk is due to magnetization. The next example illustrates the force associated with the current density.

Example 11.7.2. Force on a Wire over a Perfectly Conducting Plane

The cross-section of a perfectly conducting wire with its center a distance ξ above a perfectly conducting ground plane is shown in Fig. 11.7.3. The configuration is familiar from Demonstration 8.6.1. The current carried by the wire is returned in the ground plane. The distribution of this current on the surfaces of the wire and ground plane is consistent with the requirement that there be no flux density normal to the perfectly conducting surfaces. What is the force per unit length f acting on the wire?

The inductance per unit length is half of that for a pair of conductors having the center-to-center spacing 2ξ. Thus, it is half of that given by (8.6.12).

$$L = \frac{\mu_o}{2\pi} ln\left[\frac{\xi}{R} + \sqrt{\left(\frac{\xi}{R}\right)^2 - 1}\right] \tag{19}$$

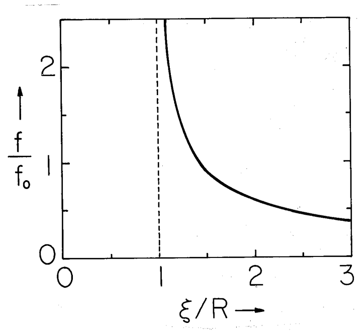

Fig. 11.7.4 The force tending to levitate the wire of Fig. 11.7.3 as a function of the distance to the ground plane normalized to the radius R of wire.

The force per unit length in the ξ direction then follows from an evaluation of (13) (again adapted to the single terminal pair situation).

$$f = \frac{1}{2} i^2 \frac{dL}{d\xi} = f_o \frac{1}{\sqrt{(\xi/R)^2 - 1}}; \qquad f_o = \frac{\mu_o i^2}{4\pi R} \tag{20}$$

The dependence of this force on the elevation above the ground plane is shown in Fig. 11.7.4. In the limit where the elevation is large compared to the radius of the conductor, (20) becomes

$$f \rightarrow \frac{1}{4} \frac{\mu_o i^2}{\pi \xi} \tag{21}$$

In Sec. 11.8, we will identify the force density acting on materials carrying a current density $\mathbf{J}$ as being $\mathbf{J} \times \mu_o \mathbf{H}$. Note that the upward force predicted by (20) is indeed consistent with the direction of this force density.

The force on a thin wire, (21), can be derived from this force density by recognizing that the contribution of the self-field of the wire to the total force per unit length is zero. Thus, the force per unit length can be computed using for $\mathbf{B}$ the flux density caused by the image current a distance 2ξ away. The flux density due to this image current has a magnitude that follows from Ampère's integral law as $\mu_o i/2\pi(2\xi)$. This field is essentially uniform over the cross-section of the wire, so the integral of the force density $\mathbf{J} \times \mathbf{B}$ over the cross-section of the wire amounts to an integration of the current density $\mathbf{J}$ over the cross-section. The latter is the total current i, and so we are led to a force per unit length of magnitude $\mu_o i^2/2\pi(2\xi)$, which is in agreement with (21).

The following is a demonstration of the force on current-carrying conductors exemplified previously. It also provides a dramatic demonstration of the existence of induced currents.

Demonstration 11.7.1. Steady State Magnetic Levitation

In the experiment shown in Fig. 11.7.5, the current-carrying wire of the previous example has been wound into a pancake shaped coil that is driven by about 20 amps of 60 Hz current. The conductor beneath is an aluminum sheet of 1.3 cm thickness. Even at 60 Hz, this conductor tends to act as a perfect conductor. This follows from

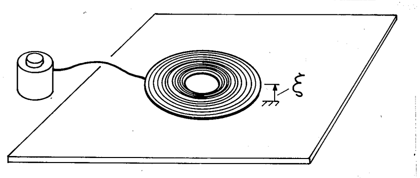

Fig. 11.7.5 When the pancake coil is driven by an ac current, it floats above the aluminum plate. In this experiment, the coil consists of 250 turns of No. 10 copper wire with an outer radius of 16 cm and an inner one of 2.5 cm. The aluminum sheet has a thickness of 1.3 cm. With a 60 Hz current i of about 20 amp rms, the height above the plate is 2 cm.

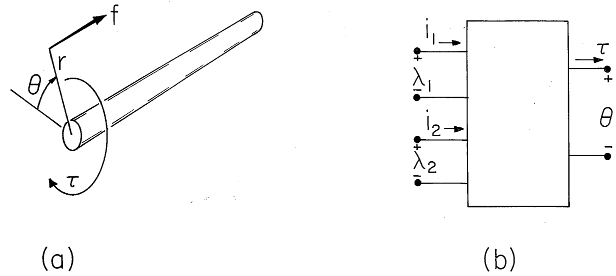

Fig. 11.7.6 (a) The force f, acting through a lever-arm of length r, produces a torque $\tau = rf$. (b) Mechanical terminal pair representing a rotational degree of freedom.

an evaluation of the product of the angular frequency ω and the time constant τ_m estimated in Sec. 10.2. From (10.2.17), $\omega\tau_m = \omega\mu_o\sigma\Delta a \approx 20$, where the average radius is $a = 9$ cm, Δ is the sheet thickness and the sheet conductivity is given by Table 7.1.1.

The time average force, of the type described in Example 11.7.2, is sufficiently large to levitate the coil. As the current is increased, its height above the aluminum sheet increases, as would be expected from the dependence of the force on the height for a single wire, Fig. 11.7.4.

The Torque of Electrical Origin. In some of the most important transducers, the mechanical response takes the form of a rotation rather than a translation. The shaft shown in Fig. 11.7.6a might be attached to the rotor of a motor or generator. A force f acting through a lever arm of length r that rotates the shaft through an incremental angle $d\theta$ causes a displacement $d\xi = rd\theta$. Thus, the incremental work done on the mechanical system $fd\xi$ becomes

$$fd\xi \rightarrow \tau d\theta \tag{22}$$

where τ is defined as the *torque.*

If the two terminal pair MQS system of Fig. 11.7.1 had a rotational rather than a displacement degree of freedom, the representation would be the same as has been outlined, except that $f \rightarrow \tau$ and $\xi \rightarrow \theta$. The mechanical terminal pair is now represented as in Fig. 11.7.6b.

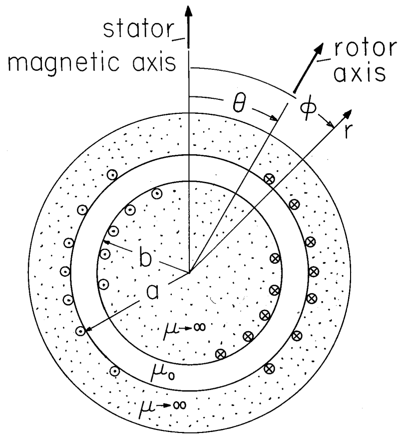

Fig. 11.7.7 Cross-section of rotating machine.

The torque follows from (13) as

$$\tau = \frac{1}{2} i_1^2 \frac{dL_{11}}{d\theta} + i_1 i_2 \frac{dL_{12}}{d\theta} + \frac{1}{2} i_2^2 \frac{dL_{22}}{d\theta} \tag{23}$$

Among the types of magnetic rotating motors and generators that could be used to exemplify the torque of (23), we now choose a synchronous machine. Although other types of motors are more common, it is a near certainty that if these words are being read with the aid of electrical illumination, the electricity used is being generated by means of a synchronous generator.

Example 11.7.3. A Synchronous Machine

The cross-section of a stator and rotor modeling a rotating machine is shown in Fig. 11.7.7. The rotor consists of a highly permeable circular cylindrical material mounted on a shaft so that it can undergo a rotation measured by the angle θ. Surrounding this rotor is a stator, composed of a highly permeable material. In slots, on the inner surface of the stator and on the outer surface of the rotor, respectively, are windings with sinusoidally varying turn densities. These windings, driven by the currents i_1 and i_2, respectively, give rise to current distributions that might be modeled by surface current densities

$$K_z = i_1 N_s \sin\phi \quad \text{at} \quad r = a \tag{24}$$

$$K_z = i_2 N_r \sin(\phi - \theta) \quad \text{at} \quad r = b \tag{25}$$

where N_s and N_r are constants descriptive of the windings. Thus, the current distribution shown on the stator in Fig. 11.7.7 is fixed and gives rise to a magnetic field having the fixed vertical axis shown in the figure. The rotor coil gives rise to a similar field except that its axis is at the angle θ of the rotor. The rotor magnetic axis, also shown in Fig. 11.7.7, therefore rotates with the rotor.

Electrical Terminal Relations. With the rotor and stator materials taken as infinitely permeable, the air gap fields are determined by using (24) and (25) to write boundary conditions on the tangential **H** and then solving Laplace's equation for the air gap magnetic fields (Secs. 9.6 and 9.7). The flux linked by the respective coils is then of the form

$$\begin{aligned}\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} &= \begin{bmatrix}L_{11} & L_{12}(\theta)\\ L_{12}(\theta) & L_{22}\end{bmatrix}\begin{bmatrix}i_1\\ i_2\end{bmatrix}\\ &= \begin{bmatrix}L_s & M\cos\theta\\ M\cos\theta & L_r\end{bmatrix}\begin{bmatrix}i_1\\ i_2\end{bmatrix}\end{aligned} \tag{26}$$

where the *self-inductances* L_s and L_r and peak *mutual inductance* M are constants.

The dependence of the inductance matrix on the angle of the rotor, θ, can be reasoned physically. The rotor is modeled as a smooth circular cylinder, so in the absence of a rotor current i_2, there can be no effect of the rotor angle θ on the flux linked by the stator winding. Hence, the stator self-inductance is independent of θ. Similar reasoning shows that the rotor self-inductance must be independent of rotor angle θ. The θ dependence of the mutual inductance is plausible because the flux λ_1 linked by the stator, due to the current in the rotor, must peak when the magnetic axes of the coils are aligned ($\theta = 0$) and must be zero when they are perpendicular ($\theta = 90$ degrees).

Torque Evaluation. The magnetic torque on the rotor follows directly from using (26) to evaluate (23).

$$\tau = i_1 i_2 \frac{d}{d\theta} M\cos\theta = -i_1 i_2 M \sin\theta \tag{27}$$

This torque depends on the currents and θ in such a way that the magnetic axis of the rotor tends to align with that of the stator. With $\theta = 0$, the axes are aligned and there is no torque. If θ is slightly positive and the currents are both positive, the torque is negative. This is as would be expected with the magnetic axis of the stator vertical and that of the rotor in the first quadrant (as in Fig. 11.7.7).

Synchronous Operation. In the synchronous mode of operation, the stator current is constrained to be sinusoidal while that on the rotor is a constant. To avoid having to describe the mechanical system, we will assume that the shaft is attached to a mechanical load that makes the angular velocity Ω constant. Thus, the electrical and mechanical terminals are constrained so that

$$\begin{aligned} i_1 &= I_1\cos\omega t\\ i_2 &= I_2\\ \theta &= \Omega t - \gamma\end{aligned} \tag{28}$$

Under what circumstances can we derive a time average torque on the shaft, and hence a net conversion of energy with each rotation?

With the constraints of (28), the torque follows from (27) as

$$\tau = -I_1 I_2 M\cos\omega t\sin(\Omega t - \gamma) \tag{29}$$

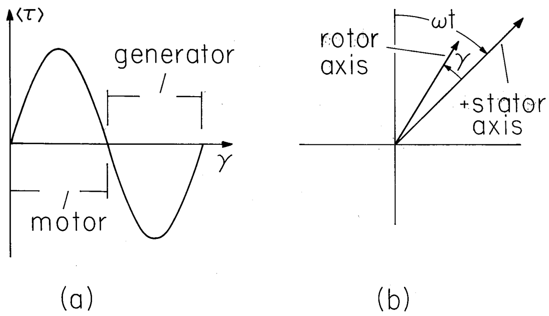

Fig. 11.7.8 (a) Time average torque as a function of angle γ. (b) The rotor magnetic axis lags the clockwise rotating component of the stator magnetic field axis by the angle γ.

A trigonometric identity[6] makes the implications of this result more apparent.

$$\tau = -\frac{I_1 I_2 M}{2}\Big\{\sin[(\omega+\Omega)t-\gamma]+\sin[-(\omega-\Omega)t-\gamma]\Big\} \tag{30}$$

There is no time average value of either of these sinusoidal functions of time unless one or the other of the frequencies, $(\omega+\Omega)$ and $(\omega-\Omega)$, is zero. For example, with the rotation frequency equal to that of the excitation,

$$\omega = \Omega \tag{31}$$

the time average torque is

$$\langle \tau \rangle = \frac{I_1 I_2 M}{2}\sin\gamma \tag{32}$$

The dependence of the time average torque on the phase angle γ is shown in Fig. 11.7.8a. With γ between 0 and 180 degrees, there is a positive time average torque acting on the external mechanical system in the direction of rotation Ω. In this range, the machine acts as a *motor* to convert energy from electrical to mechanical form. In the range of γ from 180 degrees to 360 degrees, energy is converted from mechanical to electrical form and operation is as a generator.

Stator Field Analyzed into Traveling Waves. From (30), it is clear that a time-average torque results from either a forward ($\Omega = \omega$) or a backward ($\Omega = -\omega$) rotation. This suggests that the field produced by the stator winding is the superposition of fields having magnetic axes rotating in the clockwise and counterclockwise directions. Formally, this can be seen by rewriting the stator surface current density, (24), using the electrical and mechanical constraints of (28). With the use once again of the double-angle trigonometric identity, the distribution of surface current density is separated into two parts.

$$K_z = N_s I_1 \cos\omega t \sin\phi = \frac{N_s I_1}{2}[\sin(\phi+\omega t)+\sin(\phi-\omega t)] \tag{33}$$

[6] $\sin(x)\cos(y) = \frac{1}{2}(\sin(x+y)+\sin(x-y))$

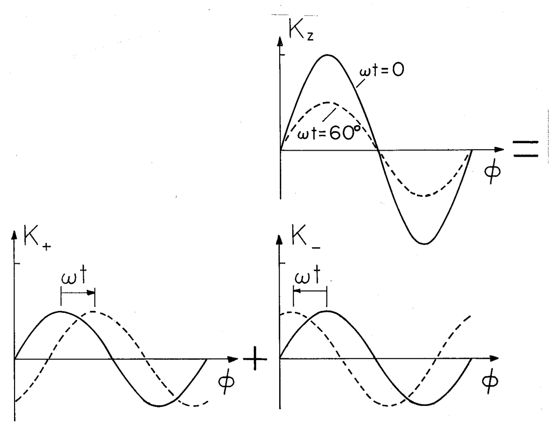

Fig. 11.7.9 With the sinusoidally distributed stator current excited by a current that varies sinusoidally with time, the surface current is a standing wave which can be analyzed into the sum of oppositely propagating traveling waves.

The sinusoidal excitation produces a standing-wave surface current with nodes at $\theta = 0$ and 180 degrees. This is the first distribution in Fig. 11.7.9. Analyzed as it is on the right in (33), and pictured in Fig. 11.7.9, it is the sum of two countertraveling waves. The magnetic axis of the wave traveling to the right is at $\phi = \omega t$.

We now have the following picture of the synchronous operation found to give rise to the time average torque. The field of the stator is composed of rotating parts, one with a magnetic axis that rotates in a clockwise direction at angular velocity ω, and the other rotating in the opposite direction. The frequency condition of (31) therefore represents a synchronous condition in which the "forward" component of the stator field and the magnetic axis of the rotor rotate at the same angular velocity.

In view of the definition of γ given in (28), if γ is positive, the rotor magnetic axis lags the stator axis by the angle γ, as shown in Fig. 11.7.8. When the machine operates as a motor, the forward component of the stator magnetic field "pulls" the rotor along. When the device operates as a generator, γ is negative and the rotor magnetic axis leads that of the forward component of the stator field. For generator operation, the rotor magnetic axis "pulls" the forward component of the stator field.

11.8 FORCES ON MICROSCOPIC ELECTRIC AND MAGNETIC DIPOLES

The energy principle was used in the preceding sections to derive the macroscopic forces on polarizable and magnetizable materials. The same principle can also be applied to derive the force distributions, the force densities. For this purpose, one needs more than a purely electromagnetic description of the system. In order to develop the simple model for the force density distribution, we need the expression for the force on an electric dipole for polarizable media, and on a magnetic dipole for magnetizable media. The force on an electric dipole will be derived simply from the Lorentz force law. We have not stated a corresponding force law for magnetic charges. Even though these are not found in nature as isolated charges but only

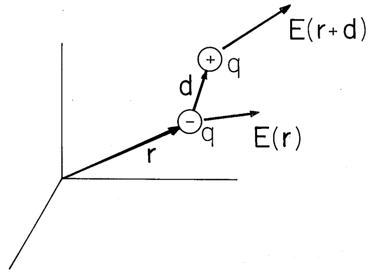

Fig. 11.8.1 An electric dipole experiences a net electric force if the positive charge q is subject to an electric field $\mathbf{E}(\mathbf{r}+\mathbf{d})$ that differs from $\mathbf{E}(\mathbf{r})$ acting on the negative charge q.

as dipoles, it is nevertheless convenient to state such a law. This will be done by showing how the electric force law follows from the energy principle. By analogy a corresponding law on magnetic charges will be derived from which the force on a magnetic dipole will follow.

Force on an Electric Dipole. The force on a stationary electric charge is given by the Lorentz law with $\mathbf{v} = 0$.

$$\mathbf{f} = q\mathbf{E} \tag{1}$$

A dipole is the limit of two charges of equal magnitude and opposite sign spaced a distance $\mathbf{d}$ apart, in the limit

$$\lim_{\substack{q\to\infty \\ |d|\to 0}} (q\mathbf{d}) = \mathbf{p}$$

with $\mathbf{p}$ being finite. Charges q of opposite polarity, separated by the vector distance $\mathbf{d}$, are shown in Fig. 11.8.1. The total force on the dipole is the sum of the forces on the individual charges.

$$\mathbf{f} = q[\mathbf{E}(\mathbf{r}+\mathbf{d}) - \mathbf{E}(\mathbf{r})] \tag{2}$$

Unless the electric field at the location $\mathbf{r}+\mathbf{d}$ of the positive charge differs from that at the location $\mathbf{r}$ of the negative charge, the separate contributions cancel.

In order to develop an expression for the force on the dipole in the limit where the spacing $\mathbf{d}$ of the charges is small compared to distances over which the field varies appreciably, (2) is written in Cartesian coordinates and the field at the positive charge expanded about the position of the negative charge. Thus, the x component is

$$\begin{aligned} f_x &= q[E_x(x+d_x,\ y+d_y,\ z+d_z) - E_x(x,y,z)] \\ &= q\Big[E_x(x,y,z) + d_x\frac{\partial E_x}{\partial x} + d_y\frac{\partial E_x}{\partial y} + d_z\frac{\partial E_x}{\partial z} \\ &\quad + \ldots - E_x(x,y,z)\Big] \end{aligned} \tag{3}$$

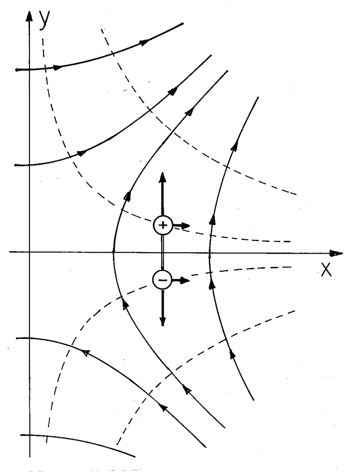

Fig. 11.8.2 Dipole having y direction and positioned on the x axis in field of (6) experiences force in the x direction.

The first and last terms cancel. In more compact notation, this expression is therefore

$$f_x = \mathbf{p} \cdot \nabla E_x \tag{4}$$

where we have identified the dipole moment $\mathbf{p} \equiv q\mathbf{d}$. The other force components follow in a similar fashion, with y and z playing the role of x. The three components are then summarized in the vector expression

$$\boxed{\mathbf{f} = \mathbf{p} \cdot \nabla \mathbf{E}} \tag{5}$$

The derivation provides an explanation of how $\mathbf{p} \cdot \nabla \mathbf{E}$ is evaluated in Cartesian coordinates. The i-th component of (5) is obtained by dotting $\mathbf{p}$ with the gradient of the i-th component of $\mathbf{E}$.

Illustration. Force on a Dipole

Suppose that a dipole finds itself in the field

$$\Phi = -V_o \frac{xy}{a^2}; \qquad \mathbf{E} = -\nabla \Phi = \frac{V_o}{a^2}(y\mathbf{i_x} + x\mathbf{i_y}) \tag{6}$$

which is familiar from Example 4.1.1. It follows from (5) that the force is

$$\mathbf{f} = \frac{V_o}{a^2}(p_y \mathbf{i_x} + p_x \mathbf{i_y}) \tag{7}$$

According to this expression, the y-directed dipole on the x axis in Fig. 11.8.2 experiences a force in the x direction. The y-directed force is zero because E_y is the same at the respective locations of the charges. The x-directed force exists because E_x goes from being positive just above the x axis to negative just below. Thus, the x-directed contributions to the force of each of the charges is in the same direction.

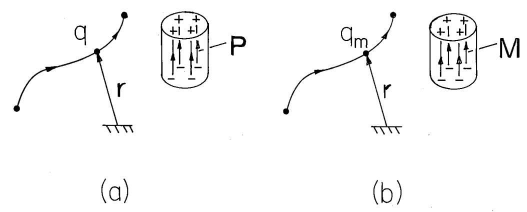

Fig. 11.8.3 (a) Electric charge brought into field created by permanent polarization. (b) Analogous magnetic charge brought into field of permanent magnet.

Force on Electric Charge Derived from Energy Principle. The force on an electric charge is stated in the Lorentz law. This law is also an ingredient in Poynting's theorem, and in the identification of energy and power flow. Indeed, $\mathbf{E} \cdot \mathbf{J}_u$ was recognized from the Lorentz law as the power density imparted to the current density of unpaired charge. The energy principle can be used to derive the force law on a microscopic charge "in reverse". This seems to be the hard way to obtain the Lorentz law of force on a stationary charge. Yet we go through the derivation for three reasons.

(a) The derivation of force from the EQS energy principle is shown to be consistent with the Lorentz force on a stationary charge.
(b) The derivation shows that the field can be produced by permanently polarized material objects, and yet the energy principle can be employed in a straightforward manner.
(c) The same principle can be applied to derive the microscopic MQS force on a magnetic charge.

Let us consider an EQS field produced by charge distributions and permanent polarizations $\mathbf{P}_p$ in free space as sketched in Fig. 11.8.3a. By analogy, we will then have found the force on a magnetic charge in the field of a permanent magnet, Fig. 11.8.3b. The Poynting theorem identifies the rate of energy imparted to the polarization per unit volume as

$$\frac{\text{rate of change of energy}}{\text{volume}} = \mathbf{E} \cdot \frac{\partial}{\partial t}\epsilon_o \mathbf{E} + \mathbf{E} \cdot \frac{\partial \mathbf{P}_p}{\partial t} \tag{8}$$

Because $\mathbf{P}_p$ is a permanent polarization, $\partial \mathbf{P}_p/\partial t = 0$, and the permanent polarization does not contribute to the change in energy associated with introducing a point charge. Hence, as charge is brought into the vicinity of the permanent polarization, the change of energy density is

$$\delta W_e = \mathbf{E} \cdot \delta(\epsilon_o \mathbf{E}) \tag{9}$$

where δ stands for the differential change of $\epsilon_o \mathbf{E}$. The change of energy is

$$\delta w_e = \int_V dv \mathbf{E} \cdot \delta(\epsilon_o \mathbf{E}) \tag{10}$$

where V includes all of space. The electroquasistatic E field is the negative gradient of the potential

$$\mathbf{E} = -\nabla \Phi \tag{11}$$

Introducing this into (10), one has

$$\delta w_e = -\int_V dv \nabla\Phi \cdot \delta(\epsilon_o \mathbf{E}) = -\int_V dv \nabla \cdot [\Phi\delta(\epsilon_o\mathbf{E})] + \int_V dv \Phi\nabla\cdot(\delta\epsilon_o\mathbf{E}) \tag{12}$$

where we have "integrated by parts," using an identity.[7] The first integral can be written as an integral over the surface enclosing the volume V. Since V is all of space, the surface is at infinity. Because $\mathbf{E}\Phi$ vanishes at infinity at least as fast as $1/r^3$ ($1/r^2$ for E, $1/r$ for Φ, where r is the distance from the origin of a coordinate system mounted within the electroquasistatic structure), the surface integral vanishes. Now

$$\nabla \cdot \delta(\epsilon_o\mathbf{E}) = \delta\rho_u \tag{13}$$

from Gauss' law, where $\delta\rho_u$ is the change of unpaired charge. Thus, from (12),

$$\delta w_e = \int_V \Phi\delta\rho_u dv \tag{14}$$

so the change of energy is equal to the charge increment $\delta\rho_u dv$ introduced at $\mathbf{r}$ times the potential Φ at $\mathbf{r}$, summed over all the charges.

Suppose that one introduces only a small test charge q, so that $\rho_u dv = q$ at point $\mathbf{r}$. Then

$$\delta w_e(\mathbf{r}) = q\Phi(\mathbf{r}) \tag{15}$$

The change of energy is the potential Φ at the point at which the charge is introduced times the charge. This form of the energy interprets the potential of an EQS field as the work to be done in bringing a charge from infinity to the point of interest.

If the charge is introduced at $\mathbf{r}+\Delta\mathbf{r}$, then the change in total energy associated with introducing that charge is

$$\delta w_e(\mathbf{r}+\Delta\mathbf{r}) = q[\Phi(\mathbf{r}) + \Delta\mathbf{r}\cdot\nabla\Phi(\mathbf{r})] \tag{16}$$

Introduction of a charge q at $\mathbf{r}$, subsequent removal of the charge, and introduction of the charge at $\mathbf{r}+\Delta\mathbf{r}$ is equivalent to the displacement of the charge from $\mathbf{r}$ to $\mathbf{r}+\Delta\mathbf{r}$. If there is a net energy decrease, then work must have been done by the force $\mathbf{f}$ exerted by the field on the charge. The work done by the field on the charge is

$$-[\delta w_e(\mathbf{r}+\Delta\mathbf{r}) - \delta w_e(\mathbf{r})] = -q\Delta\mathbf{r}\cdot\nabla\Phi(\mathbf{r}) = \Delta\mathbf{r}\cdot\mathbf{f} \tag{17}$$

and therefore

$$\mathbf{f} = -q\nabla\Phi = q\mathbf{E} \tag{18}$$

Thus, the Lorentz law for a stationary charge is implied by the EQS laws.

Before we attack the problem of force on a magnetic charge, we explore some features of the electroquasistatic case. In (17), q is a small test charge. Electric test

[7] $(\nabla\psi)\cdot\mathbf{A} = \nabla\cdot(\mathbf{A}\psi) - (\nabla\cdot\mathbf{A})\psi$

charges are available as electrons. But suppose that in analogy with the magnetic case, no free electric charge was available. Then one could still produce a test charge by the following artifice. One could polarize a very long-thin rod of cross-section a, with a uniform polarization density $\mathbf{P}$ along the axis of the rod (Sec. 6.1). At one end of the rod, there would be a polarization charge $q = Pa$, at the other end there would be a charge of equal magnitude and opposite sign. If the rod were of very long length, while the end with positive charge could be used as the "test charge," the end of opposite charge would be outside the field and experience no force. Here the charge representing the polarization of the rod has been treated as unpaired.

We are now ready to derive the force on a magnetic charge.

Force on a Magnetic Charge and Magnetic Dipole. The attraction of a magnetizable particle to a magnet is the result of the force exerted by a magnetic field on a magnetic dipole. Even in this case, because the particle is macroscopic, the force is actually the sum of forces acting on the microscopic atomic constituents of the material. As pointed out in Secs. 9.0 and 9.4, the magnetization characteristics of macroscopic media such as the iron particle relate back to the magnetic moment of molecules, atoms, and even individual electrons. Given that a particle has a magnetic moment $\mathbf{m}$ as defined in Sec. 8.2, what is the force on the particle in a magnetic field intensity $\mathbf{H}$? The particle can be comprised of a macroscopic material such as a piece of iron. However, to distinguish between forces on macroscopic media and microscopic particles, we should consider here that the force is on an elementary particle, such as an atom or electron.

We have shown how one derives the force on an electric charge in an electric field from the energy principle. The electric field could have been produced by permanently polarized dielectric bodies. In analogy, one could produce a magnetic field by permanently magnetized magnetic bodies. In the EQS case, the test charge could have been produced by a long, uniformly and permanently polarized cylindrical rod. In the magnetic case, an "isolated" magnetic charge could be produced by a long, uniformly and permanently magnetized rod of cross-sectional area a. If the magnetization density is $\mathbf{M}$, then the analogy is

$$\nabla \cdot \mathbf{P} \leftrightarrow \nabla \cdot \mu_o \mathbf{M}, \qquad \Phi \leftrightarrow \Psi, \qquad q\mathbf{E} \leftrightarrow q_m \mathbf{H} \tag{19}$$

where, for the uniformly magnetized rod, and the magnetic charge

$$q_m = \mu_o M a \tag{20}$$

is located at one end of the rod, the charge $-q_m$ at the other end of the rod (Example 9.3.1). The force on a magnetic charge is thus, in analogy with (18),

$$\boxed{\mathbf{f} = q_m \mathbf{H}} \tag{21}$$

which is the extension of the Lorentz force law for a stationary electric charge to the magnetic case. Of course, the force on a dipole is, in analogy to (5) (see Fig. 11.8.4),

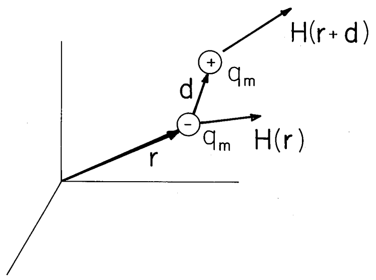

Fig. 11.8.4 Magnetic dipole consisting of positive and negative magnetic charges q_m.

$$\boxed{\mathbf{f} = q_m \mathbf{d} \cdot \mathbf{H} = \mu_o \mathbf{m} \cdot \nabla \mathbf{H}} \tag{22}$$

where $\mathbf{m}$ is the magnetic dipole moment.

We have seen in Example 8.3.2 that a magnetic dipole of moment $\mathbf{m}$ can be made up of a circulating current loop with magnitude $m = ia$, where i is the current and a the area of the loop. Thus, the force on a current loop could also be evaluated from the Lorentz law for electric currents as

$$\mathbf{f} = i \int d\mathbf{s} \times \mu_o \mathbf{H} \tag{23}$$

with i the total current in the loop. Use of vector identities indeed yields (22) in the MQS case. Thus, this could be an alternate way of deriving the force on a magnetic dipole. We prefer to derive the law independently via a Lorentz force law for stationary magnetic charges, because an important dispute on the validity of the magnetic dipole model rested on the correct interpretation of the force law[1–3]. While the details of the dispute are beyond the scope of this textbook, some of the issues raised are fundamental and may be of interest to the reader who wants to explore how macroscopic formulations of the electrodynamics of moving media based on magnetization represented by magnetic charge (Chap. 9) or by circulating currents are reconciled.

The analogy between the polarization and magnetization was emphasized by Prof. L. J. Chu[2], who taught the introductory electrical engineering course in electromagnetism at MIT in the fifties. He derived the force law for moving magnetic charges, of which (21) is the special case for a stationary charge. His approach was soon criticized by Tellegen[3], who pointed out that the accepted model of magnetization is that of current loops being the cause of magnetization. While this in itself would not render the magnetic charge model invalid, Tellegen pointed out that the force computed from (23) in a dynamic field does not lead to (22), but to

$$\mathbf{f} = \mu_o \mathbf{m} \cdot \nabla \mathbf{H} - \mu_o \epsilon_o \mathbf{m} \times \frac{\partial \mathbf{E}}{\partial t} \tag{24}$$

Because the force is different depending upon whether one uses the magnetic charge model or the circulating current model for the magnetic dipole, so his reasoning went, and because the circulating current model is the physically correct one, the magnetic charge model is incorrect. The issue was finally settled[4] when it was shown that the force (24) as computed by Tellegen was incorrect. Equation (23) assumes that i could be described as constant around the current loop and pulled out from under the integral. However, in a time-varying electric field, the charges induced in the loop cause a current whose contribution precisely cancels the second term in (24). Thus, both models lead to the same force on a magnetic dipole and it is legitimate to use either model. The magnetic model has the advantage that a stationary dipole contains no "moving parts," while the current model does contain moving charges. Hence the circulating current formalism is by necessity more complicated and more likely to lead to error.

Comparison of Coulomb's Force on an Electron to the Force on its Magnetic Dipole. Why is it possible to accurately describe the motions of an electron in vacuum by the Lorentz force law without including the magnetic force associated with its dipole moment? The answer is that the magnetic dipole effect on the electron is relatively small. To obtain an estimate of the magnitude of the magnetic dipole effect, we compare the forces produced by a typical (but large) electric field achievable without electrical breakdown in air on the charge e of the electron, and by a typical (but large) magnetic field gradient acting on the magnetic dipole moment of the electron. Taking for $\mathbf{E}$ the value 10^6 V/m, with $e \approx 1.6 \times 10^{-19}$ coulomb,

$$f_e = eE \approx 1.6 \times 10^{-13} N \tag{25}$$

A $\mathbf{B}$ of 1 tesla (10,000 gauss) is a typical large flux density produced by an iron core electromagnet. Let us assume that a flux density variation of this order can be produced over a distance of 1 cm, which is, in practice, a rather high gradient. Yet taking this value and a moment of one Bohr magneton (9.0.1), we obtain from (22) for the force on the electron

$$f_m = 1.8 \times 10^{-25} N \tag{26}$$

Note that the electric force associated with the net charge is much greater than the magnetic one due to the magnetic dipole moment. Because of the large ratio f_e/f_m for fields of realistic magnitudes, experiments designed to detect magnetic dipole effects on fundamental particles did not utilize particles having a net charge, but rather used neutral atoms (most notably, the Stern-Gerlach experiment[8]). Indeed, a stray electric field on the order of 10^{-6} V/m would deflect an electron as strongly as a magnetic field gradient of the very large magnitude assumed in calculating (26).

The small magnetic dipole moment of the electron can become very important in solid matter because macroscopic solids are largely neutral. Hence, the forces exerted upon the positive and negative charges within matter by an applied electric field more or less cancel. In such a case, the forces on the electronic magnetic dipoles in an applied magnetic field can dominate and give rise to the significant macroscopic force observed when an iron filing is picked up by a magnet.

[8] W. Gerlach and O. Stern, "Uber die Richtungsquantelung im Magnetfeld," *Ann. d. Physik*, 4th series, Vol. 74, (1924), pp. 673-699.

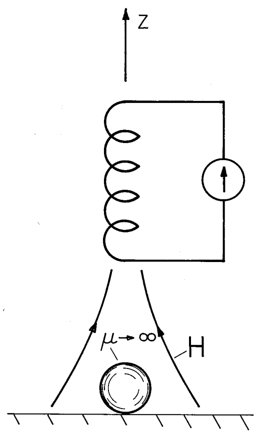

Fig. 11.8.5 By dint of its field gradient, a magnet can be used to pick up a spherical magnetizable particle.

Example 11.8.1. Magnetization Force on a Macroscopic Particle

Suppose that we wanted to know the force exerted on an iron particle by a magnet. Could the microscopic force, (22), be used? The energy method derivation shows that, provided the particle is surrounded by free space, the answer is yes. The particle is taken as being spherical, with radius R, as shown in Fig. 11.8.5. It is assumed to have such a large magnetizability that its permeability can be taken as infinite. Further, the radius R is much smaller than other dimensions of interest, especially those characterizing variations in the applied field in the neighborhood of the particle.

Because the particle is small compared to dimensions over which the field varies significantly, we can compute its *moment* by approximating the local field as uniform. Thus, the magnetic potential is determined by solving Laplace's equation in the region around the particle subject to the conditions that $\mathbf{H}$ be the uniform field H_o at "infinity" and Ψ be constant on the surface of the particle. The calculation is fully analogous to that for the electric potential surrounding a perfectly conducting sphere in a uniform electric field. In the electric analog, the dipole moment was found to be (6.6.5), $\mathbf{p} = 4\pi\epsilon_o R^3\mathbf{E}$. Therefore, it follows from the analogy provided by (19) that the magnetic dipole moment at the particle location is

$$\mu_o m = 4\pi\mu_o R^3 H \Rightarrow \mathbf{m} = 4\pi R^3\mathbf{H} \tag{27}$$

Directly below the magnet, $\mathbf{H}$ has only a z component. Thus, the dipole moment follows from (27) as

$$\mathbf{m} = 4\pi R^3 H_z\mathbf{i_z} \tag{28}$$

Evaluation of (22) therefore gives

$$f_z = 4\pi R^3 H_z\frac{\partial H_z}{\partial z} \tag{29}$$

where H_z and its derivative are evaluated at the location of the particle.

A typical axial distribution of H_z is shown in Fig. 11.8.6 together with two pictures aimed at gaining insight into the origins of the magnetic dipole force. In

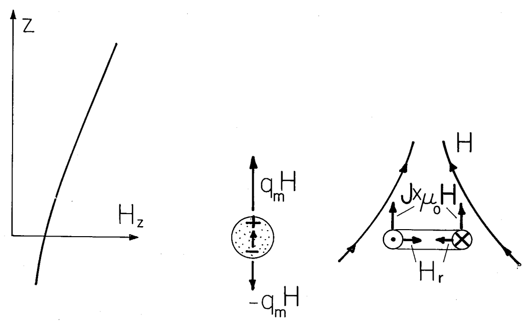

Fig. 11.8.6 In an increasing axial field, the force on a dipole is upward whether the dipole is modeled as a pair of magnetic charges or as a circulating current.

the first, the dipole is again depicted as a pair of magnetic monopoles, induced to form a moment collinear with the **H**. Because the field is more intense at the north pole of the particle than at the south pole, there is then a net force.

Alternatively, suppose that the dipole is actually a circulating current, so that the force is given by (23). Even though the energy argument makes it clear that the force is again given by (22), the physical picture is different. Because **H** is solenoidal, an intensity that increases with z implies that the field just off axis has a component that is directed radially inward. It is this radial component of the flux density crossed with the current density that results in an upward force on each segment of the loop.

REFERENCES

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[2] R. M. Fano, L. J. Chu, and R. B. Adler, **Electromagnetic Fields, Energy, and Forces**, John Wiley and Sons, New York (1960).

[3] D. B. H. Tellegen, "Magnetic-dipole models," *Am. J. Phys.* Vol. 30 (Sept. 1962), pp. 650-652.

[4] H. A. Haus and P. Penfield, Jr., "Force on a current loop," *Phys. Lett.*, Vol. 26A (March 1968), pp. 412-413.

11.9 MACROSCOPIC FORCE DENSITIES

A macroscopic force density $\mathbf{F}(\mathbf{r})$ is the force per unit volume acting on a medium in the neighborhood of $\mathbf{r}$. Fundamentally, the electromagnetic force density is the result of forces acting on those microscopic particles embedded in the material that are charged, or that have electric or magnetic dipole moments. The forces acting on these individual particles are passed along through interparticle forces to the macroscopic material as a whole. In the limit where that volume becomes small, the force density can then be regarded as the sum of the microscopic forces over a volume element ΔV.

$$\mathbf{F} = \lim_{\Delta V \to 0} \sum_{\Delta V} \mathbf{f} \tag{1}$$

Of course, the linear dimensions of ΔV are large compared to the microscopic scale.

Strictly, the forces in this sum should be evaluated using the microscopic fields. However, we can gain insight concerning the form taken by the force density by using the macroscopic fields in this evaluation. This is the basis for the following discussions of the force densities associated with unpaired charges and with conduction currents (the Lorentz force density) and with the polarization and magnetization of media (the Kelvin force density).

To be certain that the usage of macroscopic fields in describing the force densities is consistent with that implicit in the constitutive laws already introduced to describe conduction, polarization, and magnetization, the electromagnetic force densities should be derived using energy arguments. These derivations are extensions of those of Secs. 11.6 and 11.7 for forces. We end this section with a discussion of the results of such derivations and of circumstances under which they will predict the same total forces or even material deformations as those derived here.

The Lorentz Force Density. Without restricting the generality of the resulting force density, suppose that the electrical force on a material is due to two species of charged particles. One has N_+ particles per unit volume, each with a charge q_+, while the other has density N_- and a charge equal to $-q_-$. With $\mathbf{v}$ denoting the velocity of the macroscopic material and $\mathbf{v}_\pm$ representing the respective velocities of the carriers relative to that material, the Lorentz force law gives the force on the individual particles.

$$\mathbf{f}_+ = q_+[\mathbf{E} + (\mathbf{v}_+ + \mathbf{v}) \times \mu_o \mathbf{H}] \tag{2}$$

$$\mathbf{f}_- = -q_-[\mathbf{E} + (\mathbf{v}_- + \mathbf{v}) \times \mu_o \mathbf{H}] \tag{3}$$

Note that q_- is a positive number.

In typical solids and fluids, the charged particles are either bonded to the material or migrate relative to the material, suffering many collisions with the neutral material during times of interest. In either case, the inertia of the particles is inconsequential, so that on the average, the forces on the individual particles are passed along to the macroscopic material. In either situation, the force density on the material is the sum of (2) and (3), respectively, multiplied by the charged particle densities.

$$\mathbf{F} = N_+\mathbf{f}_+ + N_-\mathbf{f}_- \tag{4}$$

Substitution of (2) and (3) into this expression gives the *Lorentz force density*

$$\boxed{\mathbf{F} = \rho_u \mathbf{E} + \mathbf{J} \times \mu_o \mathbf{H}} \tag{5}$$

where ρ_u is the unpaired charge density (7.1.6) and $\mathbf{J}$ is the current density.

$$\rho_u = N_+q_+ - N_-q_-; \qquad \mathbf{J} = N_+q_+(\mathbf{v} + \mathbf{v}_+) - N_-q_-(\mathbf{v} + \mathbf{v}_-) \tag{6}$$

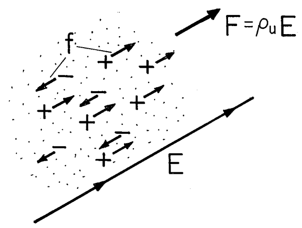

Fig. 11.9.1 The electric Lorentz force density ρ_u is proportional to the *net* charge density because the charges individually pass their force to the material in which they are embedded.

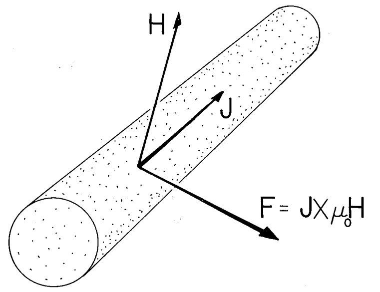

Fig. 11.9.2 The magnetic Lorentz force density $\mathbf{J} \times \mu_o \mathbf{H}$.

Because the material is in motion, with velocity $\mathbf{v}$, the current density $\mathbf{J}$ has not only the contribution familiar from Sec. 7.1 (7.1.4) due to the migration of the carriers relative to the material, but one due to the net charge carried by the moving material as well.

In EQS systems, the first term in (5) usually outweighs the second, while in MQS systems (where the unpaired charge density is negligible), the second term tends to dominate.

The derivation and Fig. 11.9.1 suggest why the electric term is proportional to the *net* charge density. In a given region, the force density resulting from the positively charged particles tends to be canceled by that due to the negatively charged particles, and the net force density is therefore proportional to the difference in absolute magnitudes of the charge densities. We exploited this fact in Chap. 7 to let electrically induced material motions evidence the distribution of the unpaired charge density. For example, in Demonstration 7.5.1, the unpaired charge density was restricted to an interface, and as a result, the motion of the fluid was suppressed by constraining the interface. A more recent example is the force on the upper electrode in the capacitor transducer of Example 11.6.1. Here again the force density is confined to a thin region on the surface of the conducting electrode.

The magnetic term in (6), pictured in Fig. 11.9.2 as acting on a current-carrying wire, is also familiar. This force density was responsible for throwing the metal disk into the air in the experiment described in Sec. 10.2. The force responsible for the levitation of the pancake coil in Demonstration 11.7.1 was also the net effect of the Lorentz force density, acting either over the volume of the coil conductors or over that of the conducting sheet below. In MQS systems, where the contribution of the "convection" current $\rho_u \mathbf{v}$ is negligible, the current density is typically due

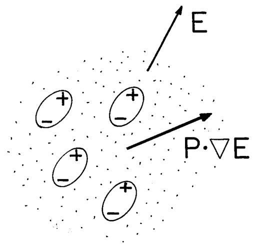

Fig. 11.9.3 The electric Kelvin force density results because the force on the individual dipoles is passed on to the neutral medium.

to conduction. Note that this means that the velocity of the charge carriers is determined by the electric field they experience in the conductor, and not simply by the motion of the conductor. The current density $\mathbf{J}$ in a moving conductor is generally not in the direction of motion.[9]

The Kelvin Polarization Force Density. If microscopic particles carrying a net charge were the only contributors to a macroscopic force density, it would not be possible to explain the forces on polarized materials that are free of unpaired charge. Example 11.6.2 and Demonstration 11.6.2 highlighted the polarization force. The experiment was carried out in such a way that the dielectric material did not support unpaired charge, so the force is not explained by the Lorentz force density.

In EQS cases where $\rho_u = 0$, the macroscopic force density is the result of forces on the microscopic particles with dipole moments. The resulting force density is fundamentally different from that due to unpaired charges; the forces $\mathbf{p} \cdot \nabla \mathbf{E}$ on the individual microscopic particles are passed along by interparticle forces to the medium as a whole. A comparison of Fig. 11.9.3 with Fig. 11.9.1 emphasizes this point. For a single species of particle, the force density is the force on a single dipole multiplied by the number of dipoles per unit volume N_p. By definition, the polarization density $\mathbf{P} = N_p \mathbf{p}$, so it follows that the force density due to polarization is

$$\boxed{\mathbf{F} = \mathbf{P} \cdot \nabla \mathbf{E}} \tag{7}$$

This is often called the *Kelvin polarization force density.*

Example 11.9.1. Force on a Dielectric Material

In Fig. 11.9.4, the cross-section of a pair of electrodes that are dipped into a liquid dielectric is shown. The picture might be of a cross-section from the experiment

[9] Indeed, it is fortunate that the carriers do not have the same velocity as the material, for if they did, it would not be possible to use the magnetic Lorentz force density for electromechanical energy conversion. If we recognize that the rate at which a force $\mathbf{f}$ does work on a particle that moves at the velocity $\mathbf{v}$ is $\mathbf{v} \cdot \mathbf{f}$, then it follows from the Lorentz force law, (1.1.1), that the rate of doing work on individual particles through the agent of the magnetic field is $\mathbf{v} \cdot (\mathbf{v} \times \mu_o \mathbf{H})$. The cross-product is perpendicular to $\mathbf{v}$, so this rate of doing work must be zero.

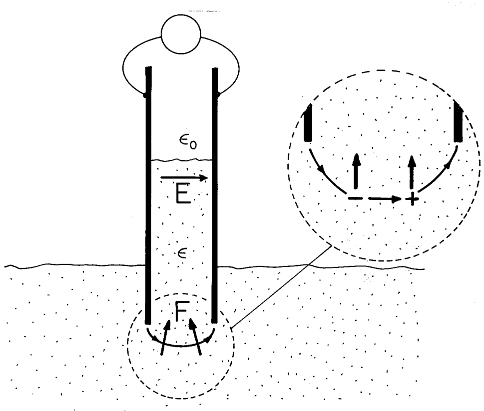

Fig. 11.9.4 In terms of the Kelvin force density, the dielectric liquid is pushed into the field region between capacitor plates because of the forces on individual dipoles in the fringing field.

of Demonstration 11.6.2. With the application of a potential difference to the electrodes, the dielectric rises between the electrodes. According to (7), what is the distribution of force density causing this rise?

For the liquid dielectric, the polarization constitutive law is taken as linear [(6.4.2) and (6.4.4)]

$$\mathbf{P} = (\epsilon - \epsilon_o)\mathbf{E} \tag{8}$$

so that with the understanding that ϵ is a function of position (uniform in the liquid, ϵ_o in the gas, and taking a step at the interface), the force density of (7) becomes

$$\mathbf{F} = (\epsilon - \epsilon_o)\mathbf{E} \cdot \nabla \mathbf{E} \tag{9}$$

By using a vector identity[10] and invoking the EQS approximation where $\nabla \times \mathbf{E} = 0$, this expression is written as

$$\mathbf{F} = \frac{1}{2}(\epsilon - \epsilon_o)\nabla(\mathbf{E} \cdot \mathbf{E}) \tag{10}$$

A second identity[11] converts this expression into one that will now prove useful in picturing the distribution of force density.

$$\mathbf{F} = \frac{1}{2}\nabla[(\epsilon - \epsilon_o)\mathbf{E} \cdot \mathbf{E}] - \frac{1}{2}\mathbf{E} \cdot \mathbf{E}\nabla(\epsilon - \epsilon_o) \tag{11}$$

Provided that the interface is well removed from the fringing fields at the top and bottom edges of the electrodes, the electric field is uniform not only in the dielectric and gas above and below the interface between the electrodes, but through the interface as well. Thus, throughout the region between the electrodes, there is no gradient of **E**, and hence, according to (7), no Kelvin force density. The Kelvin force density is therefore confined to the fringing field region where the fluid surrounds the lower edges of the electrodes. In this region, ϵ is uniform, so the force density reduces to the first term in (11). Expressed by this term, the direction and

[10] $\mathbf{A} \cdot \nabla \mathbf{A} = (\nabla \times \mathbf{A}) \times \mathbf{A} + \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A})$

[11] $\nabla(\psi\phi) = \psi\nabla\phi + \phi\nabla\psi$

magnitude of the force density is determined by the gradient of the scalar $\mathbf{E} \cdot \mathbf{E}$. Thus, where $\mathbf{E}$ is varying in the fringing field, it is directed generally upward and into the region of greater field intensity, as suggested by Fig. 11.9.4. The force on the dipole shown by the inset lends further credence to the dipolar origins of the force density.

Although there is no physical basis for doing so, it might seem reasonable to take the force density caused by polarization as being $\rho_p \mathbf{E}$. After all, it is the polarization charge density ρ_p that was used in Chap. 6 to represent the effect of the media on the macroscopic electric field intensity $\mathbf{E}$. The experiment of Demonstration 11.6.2, pictured in Fig. 11.6.7, makes it clear that this force density is not correct. With the interface well removed from the fringing fields, there is no polarization charge density anywhere in the liquid, either at the interface or in the fringing field. If $\rho_p \mathbf{E}$ were the correct force density, it would be zero throughout the fluid volume except at the interfaces with the conducting electrodes. There, the forces are perpendicular to the surface of the electrodes. Such a force distribution could not cause the fluid to rise.

The Kelvin Magnetization Force Density. Forces caused by magnetization are probably the most commonly experienced electromagnetic forces. They account for the attraction between a magnet and a piece of iron. In Example 11.7.1, this force density acts on the disk of magnetizable material.

Given that the magnetizable material is made up of microscopic dipoles, each experiencing a force of the nature of (11.8.22), and that the magnetization density $\mathbf{M}$ is the number of these per unit volume multiplied by $\mathbf{m}$, it follows from the arguments of the preceding section that the force density due to magnetization is

$$\boxed{\mathbf{F} = \mu_o \mathbf{M} \cdot \nabla \mathbf{H}} \tag{12}$$

This is sometimes called the *Kelvin magnetization force density.*

Example 11.9.2. Force Density in a Magnetized Fluid

With the dielectric liquid replaced by a ferrofluid having a uniform permeability μ, and the electrodes replaced by the pole faces of an electromagnet, the physical configuration shown in Fig. 11.9.4 becomes the one of Fig. 11.9.5, illustrating the magnetization force density. In such fluids[1], the magnetization results from an essentially permanent suspension of magnetized particles. Each particle comprises a magnetic dipole and passes its force on to the liquid medium in which it is suspended. Provided that the magnetization obeys a linear law, the discussion of the distribution of force density given in Example 11.9.1 applies equally well here.

Alternative Force Densities. We now return to comments made at the beginning of this section. The fields used to express the Lorentz and Kelvin force densities are macroscopic. To assure consistency between the averages implied by these force densities and those already inherent in the constitutive laws, an energy principle can be used. The approach is a continuum version of that exemplified

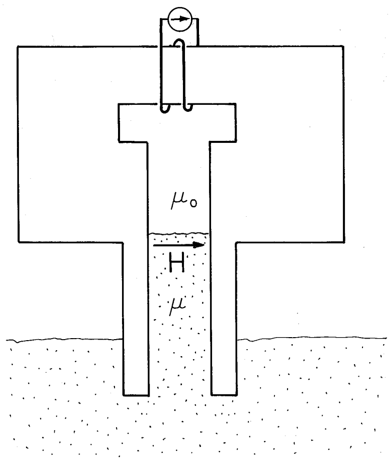

Fig. 11.9.5 In an experiment that is the magnetic analog of that shown in Fig. 11.9.4, a magnetizable liquid is pushed upward into the field region between the pole faces by the forces on magnetic dipoles in the fringing region at the bottom.

for lumped parameter systems in Secs. 11.7 and 11.8. In the lumped parameter systems, electrical terminal relations were used to determine a total energy, and energy conservation was used to determine the force. In the continuum system[2], the electrical constitutive law is used to find an energy density, and energy conservation used, in turn, to find a force density. This energy method, like the one exemplified in Secs. 11.7 and 11.8 for lumped parameter systems, describes systems that are loss free. In making practical use of the result, it is assumed that it will be applicable even if there are losses. A more general method, which invokes a *principle of virtual power*[3], allows for dissipation but requires more empirical information than the polarization or magnetization constitutive law as a starting point.

Force densities derived from more rigorous arguments than given here can have very different distributions from the superposition of the Lorentz and Kelvin force densities. We would expect that the arguments break down when the microscopic particles become so densely packed that the field experienced by one is significantly altered by its nearest neighbor. But surely the difference between the magnetic force density of Lorentz and Kelvin (LK)

$$\mathbf{F}_{\text{LK}} = \mathbf{J} \times \mu_o \mathbf{H} + \mu_o \mathbf{M} \cdot \nabla \mathbf{H} \tag{13}$$

we have derived here and the *Korteweg-Helmholtz force density* (KH) for incompressible media

$$\mathbf{F}_{\text{KH}} = \mathbf{J} \times \mathbf{B} - \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \nabla \mu \tag{14}$$

cited in the literature[2] is not due to interactions between microscopic particles. This latter force density is often obtained for an *incompressible* material from energy arguments. [Note that with $-\nabla\mu$ and $\mathbf{H} \cdot \mathbf{H}$, respectively, playing the roles of $dL/d\xi$ and i^2, the magnetization term in (14) takes a form found for the force on a magnetizable material in Sec. 11.7.]

In Example 11.9.2 (where $\mathbf{J} = \mathbf{0}$), we found the force density of (13) to be confined to the fringing field. By contrast, (14) gives no force density in the fringing region (where μ is uniform), but rather puts it all at the interface. According to this latter equation, through the agent of a surface force density (a force density that is a spatial impulse at the interface), the field pulls upward on the interface.

The question may then be asked whether, and how, the two force density expressions can be reconciled. The answer is that if $\mu_o\mathbf{M} = (\mu - \mu_o)\mathbf{H}$, they predict the same motion for any volume-conserving material deformations such as those of an incompressible fluid. We shall demonstrate this for the case of a liquid, such as shown in Fig. 11.9.5, but allowing for the action of a current $\mathbf{J}$ as well. As the first step in the derivation, we shall show that (13) and (14) differ by the gradient of a scalar, $\pi(\mathbf{r})$.

To see this, use a vector identity[12] to write (13) as

$$\mathbf{F}_{\mathrm{LK}} = \mathbf{J} \times \mu_o\mathbf{H} + (\mu - \mu_o)[(\nabla \times \mathbf{H}) \times \mathbf{H} + \frac{1}{2}\nabla(\mathbf{H} \cdot \mathbf{H})] \tag{15}$$

The MQS form of Ampère's law makes it possible to substitute $\nabla \times \mathbf{H}$ for $\mathbf{J}$ in this expression, which then becomes

$$\mathbf{F}_{\mathrm{LK}} = \mathbf{J} \times \mathbf{B} + \frac{1}{2}(\mu - \mu_o)\nabla(\mathbf{H} \cdot \mathbf{H}) \tag{16}$$

The second term in this expression is then expanded using a second vector identity[13]

$$\mathbf{F}_{\mathrm{LK}} = \mathbf{J} \times \mathbf{B} - \frac{1}{2}\mathbf{H} \cdot \mathbf{H}\nabla\mu + \nabla\big[\frac{1}{2}(\mu - \mu_o)\mathbf{H} \cdot \mathbf{H}\big] \tag{17}$$

This expression differs from (14) by the last term, which indeed takes the form $\nabla\pi$ where

$$\pi = \frac{1}{2}(\mu - \mu_o)\mathbf{H} \cdot \mathbf{H} \tag{18}$$

Now consider Newton's force law for an elemental volume of material. Using the Korteweg-Helmholtz force density, (14), it takes the form

$$\mathbf{F}_{\mathrm{inertial}} = \mathbf{F}_m - \nabla p + \mathbf{F}_{\mathrm{KH}} \tag{19}$$

where p is the internal fluid pressure and $\mathbf{F}_m$ is the sum of all other mechanical contributions to the force density. Alternatively, using (13) written as (17) as the force density, this same law is represented by

$$\mathbf{F}_{\mathrm{inertial}} = \mathbf{F}_m - \nabla p + \mathbf{F}_{\mathrm{KH}} + \nabla\pi = \mathbf{F}_m - \nabla p' + \mathbf{F}_{\mathrm{KH}} \tag{20}$$

For an incompressible material, none of the other laws needed to describe the continuum (such as mass conservation) involve the pressure.[14] Thus, if (19) is used, p

[12] $\mathbf{A} \cdot \nabla\mathbf{A} = (\nabla \times \mathbf{A}) \times \mathbf{A} + \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A})$

[13] $\nabla(\psi\phi) = \psi\nabla\phi + \phi\nabla\psi$

[14] For example, for a compressible fluid, the pressure depends on mass density and temperature, so the pressure does appear in the physical laws. Indeed, in the constitutive law relating these values, the pressure has a well-defined value. However, in an incompressible fluid, the constitutive law relating the pressure to mass density and temperature is not relevant to the prediction of material motion.

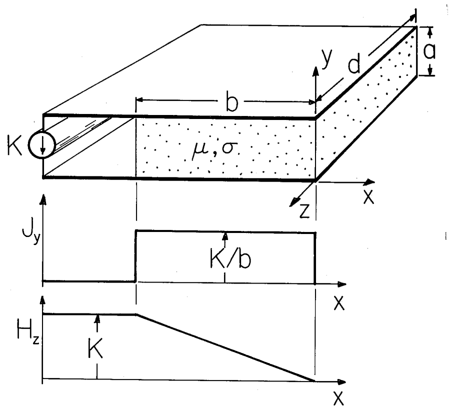

Fig. 11.9.6 The block having uniform permeability and conductivity carries a uniform current density in the y direction which produces a z-directed magnetic field intensity. Although the force densities of (13) and (14) have very different distributions in the block, they predict the same net force.

appears only in that equation and if (20) is used, $p' \equiv p - \pi$ appears only in that expression. This means that p and p' play identical roles in predicting the deformation. In an incompressible material, it is the role of the pressure to adjust itself so that only volume conserving deformations are allowed.[15] The two formulations would differ in what one would call the pressure, but would result in the same material deformation and velocity. An example is the height of rise of the fluid between the parallel plates in Fig. 11.9.5.

Included in the class of incompressible deformations are rigid body motions. If used self-consistently, force densities that differ by the gradient of a "π" will predict the same motions of rigid bodies. Thus, the net force on a body surrounded by free space will be the same whether found using the Lorentz-Kelvin or the Korteweg-Helmholtz force density. The following example illustrates this concept.

Example 11.9.3. Magnetic Force on a Magnetizable Current-Carrying Material

A block of conducting material having permeability μ is shown in Fig. 11.9.6 sandwiched between perfectly conducting plates. A current source, distributed over the left edges of these electrodes, drives a constant surface current density K in the $+x$ direction along the left edge of the lower electrode. This current passes through the block in the y direction as a current density

$$\mathbf{J} = \frac{K}{b}\mathbf{i_y} \tag{21}$$

and is returned to the source in the $-x$ direction at the left edge of the upper electrode. The thickness a of the block is small compared to its other two dimensions, so the magnetic field between the electrodes is z directed and dependent only on x.

[15] Like the "perfectly permeable material" of magnetic circuits, in which $\mathbf{B}$ remains finite as $\mathbf{H}$ goes to zero, the "perfectly incompressible" material is one in which the pressure remains finite even as the material becomes infinitely "stiff" to all but those deformations that conserve volume.

From Ampère's law it follows that

$$\frac{\partial H_z}{\partial x} = -J_y \Rightarrow \mathbf{H} = -\mathbf{i_z}\frac{K}{b}x \tag{22}$$

in the conducting block.

The alternative force densities, (13) and (14), have very different distributions in the block. Yet we must find that the net force on the block, found by integrating each over its volume, is the same. To see that this is so, consider first the sum of the Lorentz and Kelvin force densities, (13).

There is no x component of the magnetic field intensity, so for this particular configuration, the magnetization term makes no contribution to (13). Evaluation of the first term using (19) and (20) then gives

$$(F_x)_{\text{LK}} = -\frac{K^2}{b^2}\mu_o x \tag{23}$$

Integration of this force density over the volume amounts to a multiplication by the cross-sectional area ad, and integration on x. Thus, the net force predicted by using the force density of Lorentz and Kelvin is

$$(f_x)_{\text{LK}} = ad\int_{-b}^{0} -\frac{K^2\mu_o x}{b^2}dx = \frac{1}{2}adK^2\mu_o \tag{24}$$

Now, the Korteweg-Helmholtz force density given by (14) is evaluated. The permeability μ is uniform throughout the interior of the block, so the magnetization term is again zero there. However, μ is a step function at the ends of the block, where $x = -b$ and $x = 0$. Thus, $\nabla\mu$ is an impulse there and we must take care to include the contributions from the surface regions in our integration. Evaluation of the x component of (14) using (21) and (22) gives

$$(F_x)_{\text{KH}} = -\frac{\mu K^2}{b^2}x - \frac{1}{2}H_z^2\frac{\partial\mu}{\partial x} \tag{25}$$

Integration of (25) over the volume of the block therefore gives

$$(f_x)_{\text{KH}} = ad\left[\int_{-b}^{0}\frac{-K^2\mu}{b^2}xdx - \int_{-b^-}^{-b^+}\frac{1}{2}H_z^2\frac{\partial\mu}{\partial x}dx\right] \tag{26}$$

Note that H_z is constant through the interface at $x = -b$. Thus, the integration of the last term can be carried out. Simplification of this expression gives the same total force as found before, (24).

The distributions of the force densities given by (13) and (14) are generally different, even very different. It is therefore natural to ask which of the two is the "right" one. In general, until the "other" force densities acting on the medium in question are specified, this question cannot be answered. Here, where a discussion of continuum mechanics is beyond our purview, we have identified a class of mechanical deformations (namely, those that are volume conserving or "incompressible"), where these force densities are equally valid. In fact, any other force density differing from these by a term having the form $\nabla\pi$ would also be valid. The combined

Lorentz and Kelvin force densities have the advantage of a satisfying physical interpretation. However, the derivation has the weakness of making an *ad hoc* use of the macroscopic fields. Force densities resulting from an energy argument have the advantage of dealing rigorously with the macroscopic fields.

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11.10 SUMMARY

Far reaching as they are, the laws summarized by Maxwell's equations are directly applicable to the description of only one of many physical subsystems of scientific and engineering interest. Like those before it, this chapter has been concerned with the electromagnetic subsystem. However, by casting the electromagnetic laws into statements of power flow, we have come to recognize how the electromagnetic subsystem couples to the thermodynamic subsystem through the power dissipation density and to the mechanical subsystem through forces and force densities of electromagnetic origin.

The basis for a self-consistent macroscopic description of any continuum subsystem is a power flow statement having the forms identified in Sec. 11.1. Describing the energy and power flow in and into a volume V enclosed by a surface S, the integral conservation of energy statement takes the form (11.1.1).

$$-\oint_S \mathbf{S}\cdot d\mathbf{a} = \frac{d}{dt}\int_V W dv + \int_V P_d dv \tag{1}$$

The differential form of the conservation of energy statement is implied by the above.

$$-\nabla\cdot\mathbf{S} = \frac{\partial W}{\partial t} + P_d \tag{2}$$

Poynting's theorem, the subject of Sec. 11.2, is obtained starting from the laws of Faraday and Ampère to obtain an expression of the form of (2). For materials that are Ohmic ($\mathbf{J} = \sigma\mathbf{E}$) and that are linearly polarizable and magnetizable ($\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$), the *power flux density* $\mathbf{S}$ (or *Poynting's vector*), *energy density* W, and *power dissipation density* P_d were shown in Sec. 11.3 to be

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} \tag{3}$$

$$W = \frac{1}{2}\epsilon \mathbf{E} \cdot \mathbf{E} + \frac{1}{2}\mu \mathbf{H} \cdot \mathbf{H} \tag{4}$$

$$P_d = \sigma \mathbf{E} \cdot \mathbf{E} \tag{5}$$

Of course, taking the free space limit where ϵ and μ assume their free space values and $\sigma = 0$ gives the free space conservation statement discussed in Sec. 11.2.

In Sec. 11.3, we found that in EQS systems, an alternative to Poynting's vector is (11.3.24).

$$\mathbf{S} = \Phi\big(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\big) \tag{6}$$

This expression is of practical importance, because it can be evaluated without determining $\mathbf{H}$, which is generally not of interest in EQS systems.

An important application of the integral form of the energy conservation statement is to lumped parameter systems. In these cases, the surface S of (1) encloses a system that is connected to the outside world through terminals. It is then convenient to describe the power flow in terms of the terminal variables. It was shown in Sec. 11.3 (11.3.29), that the net power into the system represented by the left-hand side of (1) becomes

$$-\oint_S \mathbf{E} \times \mathbf{H} \cdot d\mathbf{a} = \sum_{i=1}^{n} v_i i_i \tag{7}$$

provided that the magnetic induction and the electric displacement current through the surface S are negligible.

This set the stage for the application of the integral form of the energy conservation theorem to lumped parameter systems.

In Sec. 11.4, attention focused on the energy storage term, the first terms on the right in (1) and (2). The energy density concept was broadened to include materials having constitutive laws relating the flux densities to the field intensities that were single valued and collinear. With E, D, H, and B representing the field magnitudes, the energy density was found to be the sum of electric and magnetic energy densities.

$$W = W_e + W_m; \qquad W_e = \int_0^D E(D')\delta D'; \qquad W_m = \int_0^B H(B')\delta B' \tag{8}$$

Integrated over the volume V of a system, this function leads to the total energy w.

For quasistatic lumped parameter systems, the total electric or magnetic energy is often conveniently found following a different route. First the terminal relations are determined and then the total energy is found by adding up the increments of energy put into the system as it is energized. In the case of an n terminal pair EQS system, where the relation between terminal voltage v_i and associated charge q_i is $v_i(q_1,\ q_2,\ \dots q_n)$, the increment of energy is $v_i dq_i$, and the total electric energy is (11.4.9).

$$w_e = \sum_{i=1}^{n} \int v_i dq_i \tag{9}$$

The line integration in an n-dimensional space representing the n independent q_i's was illustrated by Example 11.4.2.

Similarly, for an n terminal pair MQS system where the current i_i is related to the flux linkage λ_i by $i_i = i_i(\lambda_1,\ \lambda_2,\ \ldots \lambda_n)$, the total energy is (11.4.12).

$$w_m = \sum_{i=1}^{n} \int i_i d\lambda_i \tag{10}$$

Note the analogy between these expressions for the total energy of EQS and MQS lumped parameter systems and the electric and magnetic energy densities, respectively, of (8). The transition from the field picture afforded by the energy densities to the lumped parameter characterization is made by $E \rightarrow v,\ D \rightarrow q$ and by $H \rightarrow i,\ B \rightarrow \lambda$.

Especially in using the energy to evaluate forces of electrical origin, we found it convenient to define coenergy density functions.

$$W'_e = DE - W_e; \qquad W'_m = BH - W_m \tag{11}$$

It followed that these functions were natural when it was desirable to use E and H as the independent variables rather than D and B.

$$W'_e = \int_0^E D(E')\delta E'; \qquad W'_m = \int_0^H B(H')\delta H' \tag{12}$$

The total coenergy functions for lumped parameter EQS and MQS systems could be found either by integrating these densities over the volume or by again viewing the system in terms of its terminal variables. With the total coenergy functions defined by

$$w'_e = \sum_{i=1}^{n} q_i v_i - w_e; \qquad w'_m = \sum_{i=1}^{n} \lambda_i i_i - w_m \tag{13}$$

it followed that the coenergy functions could be determined from the terminal relations by again carrying out line integrations, but this time with the voltages and currents as the independent variables. For EQS systems,

$$w'_e = \sum_{i=1}^{n} \int q_i dv_i \tag{14}$$

while for MQS systems,

$$w'_m = \sum_{i=1}^{n} \int \lambda_i di_i \tag{15}$$

Again, note the analogy to the respective terms in (12).

The remaining sections of the chapter developed some of the possible implications of the "dissipation" term in the energy conservation statement, the last terms in (1) and (2). In Sec. 11.5, coupling to a thermal subsystem was discussed. In this section, the disparity between the power input and the rate of increase of the energy stored was accounted for by heating. In addition to Ohmic heating, caused by collisions between the migrating carriers and the neutral media, we considered losses associated with the dynamic polarization and magnetization of materials.

In Secs. 11.6–11.9, we considered coupling to a mechanical subsystem as a second mechanism by which energy could be extracted from (or put into) the electromagnetic subsystem. With the displacement of an object denoted by ξ, we used an energy conservation postulate to infer the total electric or magnetic force acting on the object from the energy functions [(11.6.9), and its magnetic analog]

$$f_e = -\frac{\partial w_e(q_1 \; \dots \; q_n, \; \xi)}{\partial \xi}; \qquad f_m = -\frac{\partial w_m(\lambda_1 \; \dots \; \lambda_n, \; \xi)}{\partial \xi} \tag{16}$$

or from the coenergy functions [(11.7.7) and the analogous expression for electric systems].

$$f_e = \frac{\partial w_e'(v_1 \; \dots \; v_n, \; \xi)}{\partial \xi}; \qquad f_m = \frac{\partial w_m'(i_1 \; \dots \; i_n, \; \xi)}{\partial \xi} \tag{17}$$

In Sec. 11.8, where the Lorentz force on a particle was generalized to account for electric and magnetic dipole moments, one objective was a microscopic picture that would lend physical insight into the forces on polarized and magnetized materials. The Lorentz force was generalized to include the force on stationary electric and magnetic dipoles, respectively.

$$\mathbf{f} = \mathbf{p} \cdot \nabla \mathbf{E}; \qquad \mathbf{f} = \mu_o \mathbf{m} \cdot \nabla \mathbf{H} \tag{18}$$

The total macroscopic forces resulting from microscopic forces had already been encountered in the previous two sections. The force density describes the interaction between a volume element of the electromagnetic subsystem and a mechanical continuum. The force density inferred by averaging over the forces identified in Sec. 11.8 as acting on microscopic particles was

$$\mathbf{F} = \rho_u \mathbf{E} + \mathbf{J} \times \mu_o \mathbf{H} + \mathbf{P} \cdot \nabla \mathbf{E} + \mu_o \mathbf{M} \cdot \nabla \mathbf{H} \tag{19}$$

A more rigorous approach to finding the force density could be based on a generalization of the energy method introduced in Secs. 11.6 and 11.7. As background for further pursuit of this subject, we have illustrated the importance of including the mechanical continuum with which the force density acts. Before there can be a meaningful answer to the question, "Which force density is correct?" the other force densities acting on the material must be specified. As an illustration, we found that very different electric or magnetic force densities would result in the same deformations of an incompressible material and in the same net force on an object surrounded by free space[1,2].

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PROBLEMS

11.1 Introduction

11.1.1* A capacitor C, an inductor L, and a resistor R are in series, driven by the voltage $v(t)$ and carrying the current $i(t)$. With v_c defined as the voltage across the capacitor, show that $vi = dw/dt + i^2R$ where $w = \frac{1}{2}Cv_c^2 + \frac{1}{2}Li^2$. Argue that w is the energy stored in the inductor and capacitor, while i^2R is the power dissipated in the resistor.

11.2 Integral and Differential Conservation Statements

11.2.1* Consider a system in which the fields are y and/or z directed and independent of y and z. Then $S = S_x(x,t)\mathbf{i_x}$, $W = W(x,t)$, and $P_d = P_d(x,t)$.

(a) Show that for a volume having area A in any $y - z$ plane and located between $x = x_1$ and $x = x_2$, (1) becomes

$$-[AS_x(x_1) - AS_x(x_2)] = \frac{d}{dt}A\int_{x_2}^{x_1} W dx + A\int_{x_2}^{x_1} P_d dx \qquad (a)$$

(b) Take the limit where $x_1 - x_2 = \Delta x \to 0$ and show that the one-dimensional form of (3) results.

(c) Based on (a), argue that S_x is the power flux density in the x direction.

11.3 Poynting's Theorem

11.3.1* The perfectly conducting plane parallel electrodes of Fig. 13.1.1 are driven at the left by a voltage source $V_d(t)$ and are "open circuit" at the right, as shown in Fig. 13.1.4. The system is EQS.

(a) Show that the power flux density is $\mathbf{S} = \mathbf{i_y}(-\epsilon_o y/a^2)V_d dV_d/dt$.

(b) Using $\mathbf{S}$, show that the power input is $d(\frac{1}{2}CV_d^2)/dt$, where $C = \epsilon_o bw/a$.

(c) Evaluate the right-hand side of (11.1.1) to show that if the magnetic energy storage is neglected, the same result is obtained.

(d) Show that the magnetic energy storage is indeed negligible if b/c is much shorter than times of interest.

11.3.2 The perfectly conducting plane parallel electrodes of Fig. 13.1.1 are driven at the left by a current source $I_d(t)$, as shown in Fig. 13.1.3. The system is MQS.

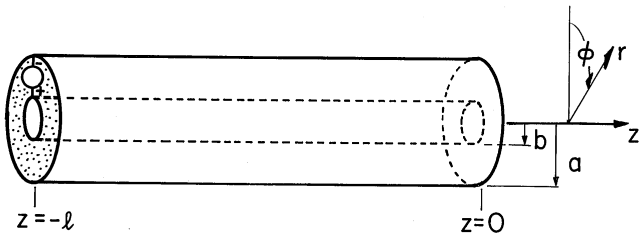

Fig. P11.3.2

(a) Determine $\mathbf{S}$.
(b) From $\mathbf{S}$, find the input power.
(c) Evaluate the right-hand side of (11.1.1) for a volume enclosing the region between the electrodes, and show that if the electric energy storage is neglected, it is indeed equal to the left-hand side.
(d) Under what conditions is the electric energy storage negligible?

11.4 Ohmic Conductors with Linear Polarization and Magnetization

11.4.1* In Example 7.3.2, a three-dimensional dipole current source drives circulating currents through a uniformly conducting material. This source is so slowly varying with time that time rates of change have a negligible effect. Consider first the power flow as pictured in terms of the Poynting flux density, (3).

(a) Show that

$$\mathbf{E} \times \mathbf{H} = \left(\frac{i_p d}{4\pi}\right)^2 \frac{1}{\sigma} \left| \left(\frac{-2\cos\theta\sin\theta}{r^5}\right)\mathbf{i}_\theta + \frac{\sin^2\theta}{r^5}\mathbf{i_r} \right| \qquad (a)$$

(b) Show that

$$P_d = \left(\frac{i_p d}{4\pi}\right)^2 \frac{1}{\sigma} \frac{(1 + 3\cos^2\theta)}{r^6} \qquad (b)$$

(c) Using these results, show that (11.1.3) is indeed satisfied.
(d) Now, using the alternative EQS power theorem, evaluate $\mathbf{S}$ as given by (23) and again show that (11.1.3) is satisfied.
(e) Observe that the latter evaluation is much simpler to carry out and that the latter power flux density is easier to picture.

11.4.2 Coaxial perfectly conducting circular cylindrical electrodes make contact with a uniformly conducting material of conductivity σ in the annulus $b < r < a$, as shown in Fig. P11.3.2. The length l is large compared to a. A voltage source v drives the system at the left, while the electrodes are "open" at the right. Assume that $v(t)$ is so slowly varying that the voltage can be regarded as independent of z.

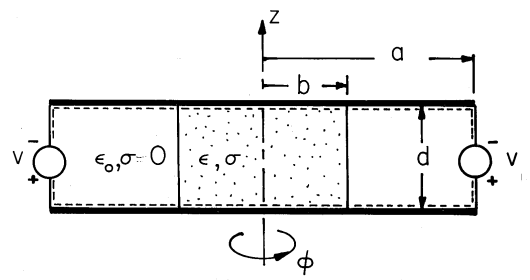

Fig. P11.3.3

(a) Determine **E**, Φ, and **H** in the annulus.
(b) Evaluate the Poynting power flux density **S** [as given by (3)] in the annulus.
(c) Use **S** to evaluate the total power dissipation by integration over the surface enclosing the annulus.
(d) Show that the same result is obtained by integrating P_d over the volume.
(e) Evaluate **S** as given by (23), and use that distribution of the power flux density to determine the total power dissipation.
(f) Make sketches of the alternative distributions of **S**.
(g) Show that the input power is vi, where i is the total current from the voltage source.

11.4.3* A pair of perfectly conducting circular plates having a spacing d form parallel electrodes in a system having cylindrical symmetry about the z axis and the cross-section shown by Fig. P11.3.3. The central region between the plates is filled out to the radius b by a uniformly conducting material having conductivity σ and uniform permittivity ϵ, while the surrounding region, where $b < r < a$, is free space. A distributed voltage source $v(t)$ constrains the potential difference between the outer edges of the electrodes. Assume that the system is EQS.

(a) Show that the Poynting power flux density is

$$\mathbf{S} = -\mathbf{i_r} \begin{cases} \frac{r}{2}\left(\frac{\sigma v}{d} + \frac{\epsilon}{d}\frac{dv}{dt}\right)\frac{v}{d}; & r < b \\ \frac{1}{2r}\left[\frac{1}{d}(\epsilon b^2 + \epsilon_o(r^2 - b^2))\frac{dv}{dt} + \frac{\sigma b^2}{d}v\right]\frac{v}{d}; & b < r < a \end{cases} \qquad (a)$$

(b) Integrate this flux density over a surface enclosing the region between the plates, and show that it is equal to the sum of the rate of change of electric energy storage and the power dissipation.
(c) Now show that the alternative power flux density given by (23) is

$$\mathbf{S} = -\frac{v}{d}(z - d)\mathbf{i_z} \begin{cases} \frac{\sigma v}{d} + \frac{\epsilon_o}{d}\frac{dv}{dt}; & r < b \\ \frac{\epsilon_o}{d}\frac{dv}{dt}; & b < r < a \end{cases} \qquad (b)$$

(d) Carry out part (b) using this distribution of **S**, and show that the result is the same.
(e) Show that the power input is equal to vi, where i is the total current from the voltage source.

11.4.4 In Example 7.5.1, the steady current distribution in and around a conducting circular cylindrical rod immersed in a conducting material was determined. Assume that E_o is so slowly varying that it can be regarded as static.

(a) Determine the distribution of Poynting power flux density $\mathbf{S}$, as given by (3).
(b) Determine the alternative $\mathbf{S}$ given by (23).
(c) Find the power dissipation density P_d in and around the rod.
(d) Show that the differential energy conservation law [(11.1.3) with $\partial W/\partial t = 0$] is satisfied at each point in and around the rod using either of these distributions of $\mathbf{S}$.

11.5 Energy Storage

11.5.1* In Example 8.5.1, the inductance L of a spherically shaped coil was found by "adding up" the flux linkages of the individual windings. Taking an alternative approach to finding L, use the fields found in that example to determine the total energy storage, w_m. Then use the fact that $w_m = \frac{1}{2}Li^2$ to show that L is as given by (8.5.20).

11.5.2 In Prob. 9.6.3, a coil has turns at the interface between a magnetizable material and a circular cylindrical core of free space, as shown in Fig. P9.6.3. Assume that the system has a length l in the z direction and determine the total energy, w_m. (Assume that the rotatable coil carries no current.) Use the fact that $w_m = \frac{1}{2}Li^2$ to find L.

11.5.3* In Example 8.6.4, the fields of a coil distributed throughout a volume were found. Using these fields to evaluate the total energy storage, show that the inductance is as given by (8.6.35).

11.5.4 The magnetic circuit described in Prob. 9.7.5 and shown in Fig. P9.7.5 has two electrical excitations. Determine the total magnetic coenergy, $w'_m(i_1, i_2, x)$.

11.5.5 The cross-section of a motor or generator is shown in Fig. 11.7.7.

(a) Determine the magnetic coenergy density W'_m, and hence the total coenergy w'_m.
(b) By writing w'_m in the form of (11.4.24), determine L_{11}, L_{12}, and L_{22}.

11.5.6* The material in the system of Fig. 11.4.3 has the constitutive law of (28). Show that the total coenergy is

$$w'_e = \left[\frac{\alpha_1}{\alpha_2}\left(\sqrt{1+\frac{\alpha^2 v^2}{a^2}}-1\right)+\frac{1}{2}\epsilon_o\frac{v^2}{a^2}\right]\xi ca + \frac{1}{2}\frac{\epsilon_o v^2}{a}(b-\xi)c \qquad (a)$$

11.5.7 Consider the system shown in Fig. P9.5.1 but with $\mu_a = \mu_o$ and the region where $\mathbf{B} = \mu_b\mathbf{H}$ now filled with a material having the constitutive law

$$\mathbf{B} = \big(\mu_o + \alpha_1/\sqrt{1+\alpha_2 H^2}\big)\mathbf{H} \qquad (a)$$

(a) Determine **B** and **H** in each region.
(b) Find the coenergy density in each region and hence the total coenergy w'_m as a function of the driving current i.

11.6 Electromagnetic Dissipation

11.6.1* In Example 7.9.2, the Maxwell capacitor has an area A (perpendicular to x), and the terminals are driven by a source $v = Re[\hat{v}\exp(j\omega t)]$. The sinusoidal steady state has been established. Show that the time average power dissipation in the lossy dielectrics is

$$\langle P_d\rangle = \frac{A}{2}\frac{[a\sigma_a(\sigma_b^2+\omega^2\epsilon_b^2) + b\sigma_b(\sigma_a^2+\omega^2\epsilon_a^2)]}{(b\sigma_a + a\sigma_b)^2 + \omega^2(b\epsilon_a + a\epsilon_b)^2}|\hat{v}|^2 \qquad (a)$$

11.6.2 In Example 7.9.3, the potential is found in the EQS approximation in and around a lossy dielectric sphere embedded in a lossy dielectric and stressed by a uniform field having a sinusoidal dependence on time (7.9.36).

(a) Find the time average power dissipation density in each region.
(b) What is the total time average power dissipated in the sphere?

11.6.3* Plane parallel perfectly conducting plates having the spacing d are shorted by a perfectly conducting sheet in the plane $x = 0$, as shown in Fig. P11.5.3. A sheet having thickness Δ and conductivity σ is in the plane $x = -b$ and makes contact with the perfectly conducting plates above and below. At their left edges, in the plane $x = -(a+b)$, a source of surface current density, $K(t)$, is connected to the plates. The regions to left and right of the resistive sheet are free space, and w is large compared to a, b, and d.

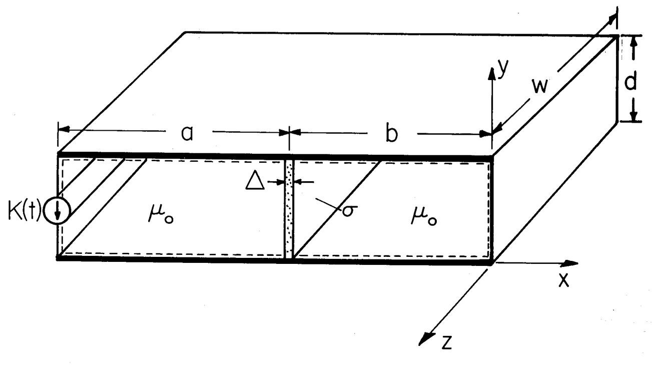

Fig. P11.5.3

(a) Show that the total power dissipation and magnetic energy stored as defined on the right in (11.1.1), are

$$\int_V P_d dv = \Delta\sigma w d\mu_o^2 b^2 \Big(\frac{dH^b}{dt}\Big)^2; \quad \int_V W dv = \frac{1}{2}\mu_o dw(bH_b^2 + aK^2) \qquad (a)$$

(b) Show that the integral on the left in (11.1.1) over the surface indicated by the dashed line in the figure gives the same result as found in part (a).

11.6.4 In Example 10.4.1, the applied field is $H_o(t) = H_m \cos(\omega t)$ and sinsuoidal steady state conditions prevail. Determine the time average power dissipation in the conducting sheet.

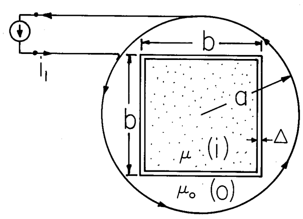

Fig. P11.5.5

11.6.5* The cross-section of an N-turn circular solenoid having radius a is shown in Fig. P11.5.5. It surrounds a thin cylindrical shell of square cross-section, with length b on a side. This shell has thickness Δ and conductivity σ, and is filled by a material having permeability μ. Both the shell and the solenoid have a length d perpendicular to the paper that is large compared to a.

(a) Given that the terminals of the solenoid are driven by the current $i_1 = i_o \cos\omega t$ and the sinusoidal steady state has been established, integrate the time average power dissipation density over the volume of the shell to show that the total time-average power dissipation is

$$p_d = \frac{2b}{\sigma\Delta d} N^2 i_o^2 \left[\frac{(\omega\tau_m)^2}{1 + (\omega\tau_m)^2}\right] \qquad (a)$$

(b) In the sinusoidal steady state, the time average Poynting flux through a surface enclosing the shell goes into the time average dissipation. Use this fact to obtain (a).

11.6.6 In describing the response of macroscopic media to fields in the sinusoidal steady state, it is convenient to use complex constitutive laws. The complex permittivity is introduced by (19). Here we introduce and illustrate the complex *permeability*. Suppose that field quantities take the form

$$\mathbf{E} = Re\hat{\mathbf{E}}(x, y, z)e^{j\omega t}; \qquad \hat{\mathbf{H}} = Re\hat{\mathbf{H}}(x, y, z)e^{j\omega t} \qquad (a)$$

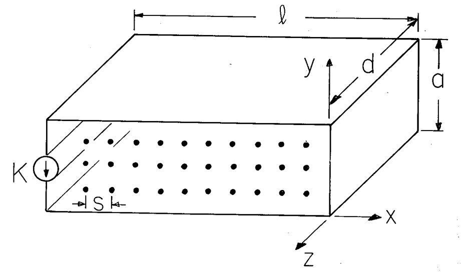

Fig. P11.5.6

(a) Show that in a region where there is no *macroscopic* current density, the MQS laws require that

$$\nabla \times \hat{\mathbf{E}} = -j\omega\hat{\mathbf{B}} \tag{b}$$

$$\nabla \times \hat{\mathbf{H}} = 0 \tag{c}$$

$$\nabla \cdot \hat{\mathbf{B}} = 0 \tag{d}$$

(c) Given that the spherical shell of Prob. 10.4.3 comprises each element in the cubic array of Fig. P11.5.6, each sphere with spacing s such that $s \gg R$, what is the complex permeability μ defined such that $\hat{\mathbf{B}} = \hat{\mu}\hat{\mathbf{H}}$?

(d) A macroscopic material composed of this array of spheres is placed in the one-turn solenoid of rectangular cross-section shown in Fig. P11.5.6. This configuration is long enough in the z direction so that fringing fields can be ignored. At their left edges, the perfectly conducting plates composing the top and bottom of the solenoid are driven by a distributed current source, $K(t)$. With the fringing fields in the neighborhood of the left end ignored, the resulting fields take the form $\mathbf{H} = H_z(x,t)\mathbf{i_z}$ and $\mathbf{E} = E_y(x,t)\mathbf{i_y}$. Use an evaluation of the Poynting flux to determine the total time average power dissipated in the length l, width d, and height a of the material.

11.6.7* In the limit where the skin depth δ is small compared to the length b, the magnetic field distribution in the conductor of Fig. 10.7.2 is given by (10.7.15). Show that (per unit $y - z$ area) the time average power dissipation associated with the current flowing in the "skin" region is $|K_s|^2/2\sigma\delta$ watts/m^2.

11.6.8 The conducting block shown in Fig. 10.7.2 has a length d in the z direction.

(a) Determine the total time average power dissipation.

(b) Show that in the case $\delta \ll b$ this expression reduces to that obtained in Prob. 11.5.7, while in the limit $\delta \gg b$, the result is i^2R where R is the dc resistance of the slab and i is the total current.

11.6.9* The toroid of Fig. 9.4.1 is filled with an insulating material having the magnetization constitutive law of Prob. 9.4.3. Show that from the terminals of the N_1-turn coil, the circuit is equivalent to one having an inductance $L = \mu_o N_1^2 w^2/8R$ in series with a resistance $R_m = \mu_o \gamma N_1^2 w^2/8R$.

11.6.10 The toroid of Fig. 9.4.1 is filled by a material having the magnetization characteristic shown in Fig. P11.5.10. A sinusoidal current is supplied with a particular amplitude, $i = (2H_c 2\pi R/N_1)\cos(\omega t)$.

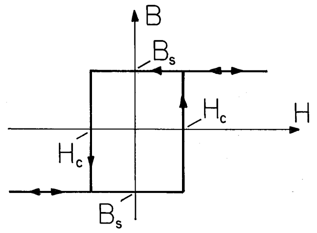

Fig. P11.5.10

(a) Draw a dimensioned plot of $B(t)$.
(b) Find the terminal voltage $v(t)$ and also make a dimensioned plot.
(c) Compute the time average power input, defined as

$$\langle vi \rangle = \frac{1}{T}\int_t^{t+T} vi dt \qquad (a)$$

where $T = 2\pi/\omega$.
(d) Show that the result of part (c) can also be found by recognizing that, during one cycle, there is an energy/unit volume dissipated which is equal to the area enclosed by the $B - H$ characteristic.

11.7 Electrical Forces on Macroscopic Media

11.7.1* A pair of perfectly conducting plates, the upper one fixed and the lower one free to move with the horizontal displacement ξ, have a fixed spacing a as shown in Fig. P11.6.1. Show that the force of electrical origin acting on the lower electrode in the ξ direction is $f = -\epsilon_o v^2 d/2a$.

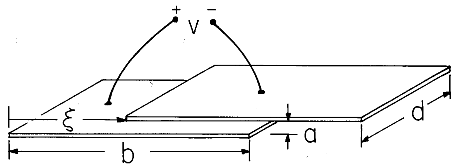

Fig. P11.6.1

11.7.2 In Example 4.6.3, the capacitance per unit length of the pair of parallel circular cylindrical conductors shown in Fig. 4.6.6 was found. Determine the force per unit length acting on the right cylinder in the x direction.

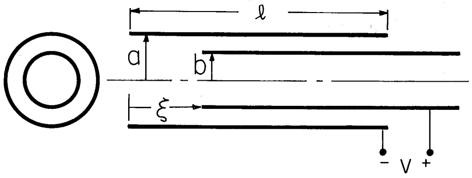

Fig. P11.6.4

11.7.3* The electric transducer shown in cross-section by Fig. P11.6.3 has cylindrical symmetry about the center line. A coaxial pair of perfectly conducting electrodes having length l are excited at the left end by a voltage source $v(t)$. A perfectly insulating dielectric material having permittivity ϵ is free to slide in and out of the annular region between electrodes.

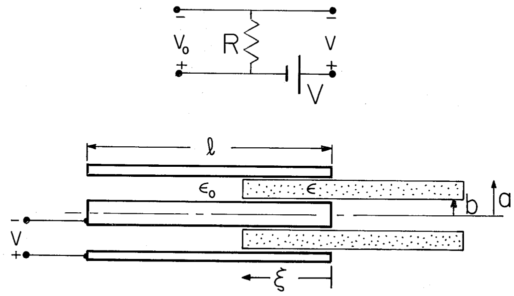

Fig. P11.6.3

(a) Show that the force of electric origin acting on the dielectric material in the axial direction is $f = v^2\pi(\epsilon - \epsilon_o)/ln(a/b)$.
(b) Show that if the electrical terminals are constrained by the circuit shown, R is very small and the plunger suffers the displacement $\xi(t)$ the output voltage is $v_o = -2\pi RV(\epsilon - \epsilon_o)(d\xi/dt)/ln(a/b)$.

11.7.4 The electrometer movement shown in Fig. P11.6.4 consists of concentric, perfectly conducting tubes, the inner one free to move in the axial direction.

(a) Ignore the fringing field and determine the force of electrical origin acting in the direction of ξ.
(b) For the energy conversion cycle of Demonstration 11.6.1, but for this transducer, make dimensioned plots of the cycle in the (q, v) and (f, ξ) planes (analogous to those of Fig. 11.6.5).
(c) By calculating both, show that the electrical energy input in one cycle is equal to the work done on the external mechanical system.

11.7.5* Show that the vertical force on the nonlinear dielectric material of Prob. 11.4.6 is

$$f = \left[\frac{\alpha_1}{\alpha_2}\left(\sqrt{1 + \frac{\alpha_2 v^2}{a^2}} - 1\right) + \frac{1}{2}\epsilon_o\frac{v^2}{a^2}\right]ca - \frac{\epsilon_o v^2 c}{2a} \qquad (a)$$

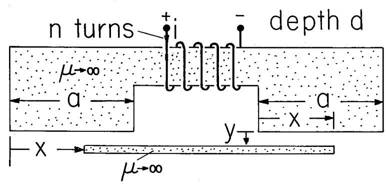

Fig. P11.7.3

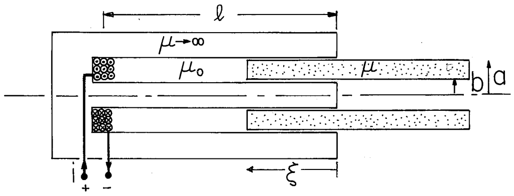

Fig. P11.7.4

11.8 Macroscopic Magnetic Fields

11.8.1* Show that the force acting in the x direction on the movable element of Prob. 9.7.5 (Note Prob. 11.4.4.) is

$$f = -\frac{\mu_o a w}{2x^2(1 + a/b)}(N_1^2 i_1^2 + 2N_1 N_2 i_1 i_2 + N_2^2 i_2^2) \tag{a}$$

11.8.2 Determine the force $f(i, \xi)$ acting in the x direction on the plunger of the magnetic circuit shown in Fig. P9.7.6.

11.8.3* The magnetic transducer shown in Fig. P11.7.3 consists of a magnetic circuit in which the lower element is free to move in the x and y directions. From the energy principle, ignoring fringing fields, show that the force on this element is

$$\mathbf{f} = \frac{\mu_o n^2 d i^2}{2a}\left[\frac{a - 2x}{y}\mathbf{i_x} - \frac{x(a - x)}{y^2}\mathbf{i_y}\right] \tag{a}$$

11.8.4 The magnetic circuit shown in cross-section by Fig. P11.7.4 has cylindrical symmetry. A plunger of permeability μ having outer and inner radii a and b can suffer a displacement ξ into the annular gap of a magnetic circuit otherwise made of infinitely permeable material. The coil has N turns. Assume that the left end of the plunger is well within the magnetic circuit, so that fringing fields can be ignored, and determine the force $f(i, \xi)$ acting to displace the plunger in the ξ direction.

11.8.5* The "variable reluctance" motor shown in cross-section in Fig. P11.7.5 consists of an infinitely permeable yoke and an infinitely permeable rotor

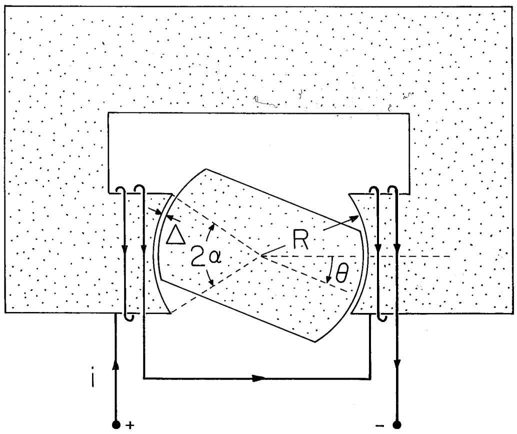

Fig. P11.7.5

element forming a magnetic circuit with two air gaps of length $\Delta \ll R$. The system has depth $d \gg \Delta$ into the paper. Assume that $0 < \theta < \alpha$, as shown, and show that the torque caused by passing a current i through the two N-turn coils is $\tau = -\mu_o RdN^2 i^2/\Delta$.

11.8.6 A "two-phase" synchronous machine is constructed having a cross-section like that shown in Fig. 11.7.7, except that there is an additional winding on the stator. This is identical to the one shown except that it is rotated 90 degrees in the clockwise direction. The current in the stator winding shown in Fig. 11.7.7 is denoted by i_a, while that in the additional winding is i_b. Thus, the magnetic axes of i_a and i_b, respectively, are upward and to the right. With L_s, L_r, and M given constants, the inductance matrix is

$$\begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_r \end{bmatrix} = \begin{bmatrix} L_s & 0 & M\cos\theta \\ 0 & L_s & M\sin\theta \\ M\cos\theta & M\sin\theta & L_r \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_r \end{bmatrix} \qquad (a)$$

(a) Determine the coenergy $w'_m(i_a, i_b, \theta)$.
(b) Find the torque on the rotor, $\tau(i_a, i_b, \theta)$.
(c) With $i_a = I\cos(\omega t)$ and $i_b = I\sin(\omega t)$, where I and ω are given constants, argue that the magnetic axis produced by the stator rotates with the angular velocity ω.
(d) Using these current constraints together with $i_r = I_r$ and $\theta = \Omega t - \gamma$, where I_r, γ and Ω are constants, show that under synchronous conditions (where $\omega = \Omega$), the torque is $\tau = MII_r\sin(\gamma)$.

11.9 Forces on Microscopic Electric and Magnetic Dipoles

11.9.1* In a uniform electric field $\mathbf{E}$, a perfectly conducting particle having radius R has a dipole moment $\mathbf{p} = 4\pi\epsilon_o R^3\mathbf{E}$. Provided that R is short compared to distances over which the field varies, this gives a good approximation to $\mathbf{p}$, even where the field is not uniform. Such a particle is shown at the location $x = X,\ y = Y$ in Fig. P11.8.1, where it is subject to the field produced by a periodic potential $\Phi = V_o \cos(\beta x)$ imposed in the plane $y = 0$.

(a) Show that the potential imposed in the region $0 < y$ is $V_o \cos(\beta x)$ exp $(-\beta y)$.
(b) Show that, provided that the particle has no net charge, the force on the particle is

$$\mathbf{f} = -4\pi\epsilon_o R^3 (V_o\beta)^2 \beta \mathbf{i_y} e^{-2\beta y} \qquad (a)$$

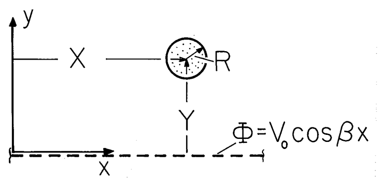

Fig. P11.8.1

11.9.2 The perfectly conducting particle described in Prob. 11.8.1, carrying no net charge but polarized by the imposed electric field, is subjected to the field of a charge Q located at the origin of a spherical coordinate system. In terms of its location R relative to the charged particle at the origin, determine the force on the particle.

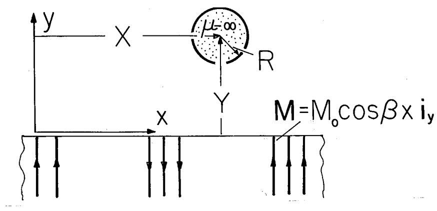

Fig. P11.8.3

11.9.3* In Fig. P11.8.3, permanent magnets in the lower half-space are represented by the magnetization density $\mathbf{M} = M_o \cos(\beta x)\mathbf{i_y}$, where M_o and β are given positive constants.

(a) Show that the resulting magnetic potential in the upper half-space is

$$\Psi = (M_o/2\beta)\cos(\beta x)\exp(-\beta y)$$

(b) A small infinitely permeable particle having the radius R is located at $x = X,\ y = Y$. Show that the magnetization force on the particle is as given by (a) of Prob. 11.8.1, with $V_o \to (M_o/2\beta)$ and $\epsilon_o \to \mu_o$.

11.9.4 A small "infinitely permeable" particle of radius R is a distance Z above an infinitely permeable plane, as shown in Fig. P11.8.4. A uniform field

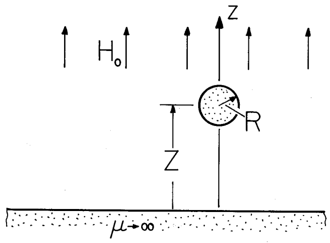

Fig. P11.8.4

$\mathbf{H} = H_o \mathbf{i_z}$ is imposed. Assume that $R \ll Z$, and use (27) to approximate the dipole moment induced in the particle. The effect of the infinitely permeable plane on the field induced by this dipole is equivalent to that of a second image dipole located at $z = -Z$. Thus, there is a force of attraction between the magnetized particle and the infinite plane that is equivalent to that attracting the dipole to its image. Determine the force in the z direction on the particle.

11.10 Macroscopic Force Densities

11.10.1 In Prob. 11.7.2, the total force on a magnetizable plunger is found (Fig. P9.7.6). Find this same force by integrating the force density, (14), over the volume of the plunger.

11.10.2* In Example 10.3.1, the transient current induced by applying a magnetic field intensity H_o to a conducting shell is determined.

(a) Show that there is a radial magnetic force per unit area acting on the shell $T_r = \mu_o K(H_o + H_i)/2$. (Note that the thin-shell model implies that $\mathbf{H}$ varies in an essentially linear fashion with R inside the shell.)
(b) Specifically, show that

$$T_r = -\frac{\mu_o H_o^2}{2}\left(2 - e^{-t/\tau_m}\right)e^{-t/\tau_m} \qquad (a)$$

11.10.3 In Example 10.4.1, the transient current induced in a conducting shell by the application of a transverse magnetic field is found. Suppose that the magnetizable core is absent.

(a) Show that the radial force per unit area acting on the shell is $T_r = \mu_o K(H_\phi^o + H_\phi^i)/2$. (Note that according to the thin-shell model, $\mathbf{H}$ has an essentially linear dependence on r within the shell.)
(b) Determine $T_r(\phi, t)$ and relate the result to Demonstration 10.4.1.