6.641 Quiz 1 Solutions
3/17/04

1\. a)

$$\chi(x,y) = \begin{cases} A\cos ay\, e^{ax} & x<0 \\ B\cos ay \cosh a(x-d) & 0<x<d \end{cases}$$

b) $H_y(x=0_+) - H_y(x=0_-) = K_z$

$\mu H_x(x=0_-) = \mu_0 H_x(x=0_+)$

$H_x(x=d) = 0$

c) $$H_y = -\frac{\partial \chi}{\partial y} = \begin{cases} aA\sin ay\, e^{ax} & x<0 \\ aB\sin ay\cosh a(x-d) & 0<x<d \end{cases}$$

$$H_x = -\frac{\partial \chi}{\partial x} = \begin{cases} -aA\cos ay\, e^{ax} & x<0 \\ -aB\cos ay\sinh a(x-d) & 0<x<d \end{cases}$$

$$\cancel{a}B\cancel{\sin ay}\cosh ad - \cancel{a}A\cancel{\sin ay} = \frac{K_0}{a}\cancel{\sin ay}$$

$$-\mu\cancel{a}A\cancel{\cos ay} = \mu_0\cancel{a}B\cancel{\cos ay}\sinh ad$$

$$A = -\frac{\mu_0}{\mu}B\sinh ad$$

$$B\left[\cosh ad + \frac{\mu_0}{\mu}\sinh ad\right] = \frac{K_0}{a}$$

$$B = \frac{K_0}{a\left[\cosh ad + \frac{\mu_0}{\mu}\sinh ad\right]}$$

$$A = -\frac{\mu_0 \sinh ad\, K_0}{\mu a\left[\cosh ad + \frac{\mu_0}{\mu}\sinh ad\right]}$$

$$\chi(x,y) = \begin{cases} \dfrac{-\mu_0 K_0 \sinh ad}{\mu a\left[\cosh ad + \frac{\mu_0}{\mu}\sinh ad\right]}\cos ay\, e^{ax} & x<0 \\[2ex] \dfrac{K_0 \cos ay \cosh a(x-d)}{a\left[\cosh ad + \frac{\mu_0}{\mu}\sinh ad\right]} & 0<x<d \end{cases}$$

d) $K_z(x=d) = -H_y(x=d) = -Ba\sin ay$

$$= \frac{-K_0 \sin ay}{\left[\cosh ad + \frac{\mu_0}{\mu}\sinh ad\right]}$$

e) $$\frac{\overline{f}}{\text{area}} = \frac{1}{2}\left[\overline{K} \times \mu_0 \overline{H}\right]\Big|_{x=d} = \frac{1}{2}\mu_0 K_z(x=d)\, H_y(x=d)\, \overline{i}_z \times \overline{i}_y$$

$$= -\overline{i}_x \frac{\mu_0}{2} K_z(x=d)\left(-K_z(x=d)\right)$$

$$= +\frac{\mu_0}{2} K_z^2(x=d)\, \overline{i}_x$$

$$= \frac{\mu_0}{2} \frac{K_0^2 \sin^2 ay}{\left[\cosh ad + \frac{\mu_0}{\mu}\sinh ad\right]^2}$$

2. (a) $$\Phi(r,\phi) = \begin{cases} A r^2 \sin 2\phi & r < R_1 \\ \left(Br^2 + \frac{C}{r^2}\right)\sin 2\phi & R_1 < r < R_2 \end{cases}$$

b) $\Phi(r=0,\phi)$ is finite

$\Phi(r=R_{1-},\phi) = \Phi(r=R_{1+},\phi) = V_0 \sin 2\phi$

$\Phi(r=R_2,\phi) = 0$

c) $A R_1^2 \cancel{\sin 2\phi} = V_0 \cancel{\sin 2\phi} \Rightarrow A = \frac{V_0}{R_1^2}$

$\left(BR_2^2 + \frac{C}{R_2^2}\right)\sin 2\phi = 0 \Rightarrow B = -\frac{C}{R_2^4}$

$\left(BR_1^2 + \frac{C}{R_1^2}\right)\cancel{\sin 2\phi} = V_0 \cancel{\sin 2\phi} \Rightarrow C\left(\frac{1}{R_1^2} - \frac{R_1^2}{R_2^4}\right) = V_0$

$$\Phi(r,\phi) = \begin{cases} \frac{V_0 r^2}{R_1^2}\sin 2\phi & r < R_1 \\ C\left(-\frac{r^2}{R_2^4} + \frac{1}{r^2}\right)\sin 2\phi = \frac{V_0 R_1^2 R_2^4}{R_2^4 - R_1^4}\sin 2\phi\left(\frac{1}{r^2} - \frac{r^2}{R_2^4}\right) & R_1 < r < R_2 \end{cases}$$

d) $$\sigma_s(r=R_1) = -\left[\epsilon_2 \frac{\partial \Phi}{\partial r}\Big|_{r=R_{1+}} - \epsilon_1 \frac{\partial \Phi}{\partial r}\Big|_{r=R_{1-}}\right] = \left[-\frac{\epsilon_2 V_0 R_1^2 R_2^4}{R_2^4 - R_1^4}\left(-\frac{2}{R_1^3} - \frac{2R_1}{R_2^4}\right) + \frac{\epsilon_1 V_0 2R_1}{R_1^2}\right]\sin 2\phi$$

$$= \frac{2V_0 \sin 2\phi}{R_1}\left[\epsilon_1 + \frac{\epsilon_2 (R_1^4 + R_2^4)}{(R_1^4 + R_2^4)}\right]$$

$$\sigma_s(r=R_2) = +\epsilon_2 \frac{\partial \Phi}{\partial r}\Big|_{r=R_2} = \frac{\epsilon_2 V_0 R_1^2 R_2^4}{R_2^4 - R_1^4}\left(-\frac{2}{R_2^3} - \frac{2R_2}{R_2^4}\right)$$

$$= -\frac{4\epsilon_2 V_0 R_1^2 R_2}{R_2^4 - R_1^4}$$