6.641 Electromagnetic Fields, Forces, and Motion
Quiz 2 Solutions
4/23/03

1. a)

$$\Phi(r,\phi,t=0) = \begin{cases} A(t=0)\, r^3 \sin 3\phi & 0 \le r \le R \\ \dfrac{B(t=0)}{r^3} \sin 3\phi & r \ge R \end{cases}$$

$$E_r(r,\phi,t=0) = -\frac{\partial \Phi(r,\phi,t=0)}{\partial r} = \begin{cases} -3A(t=0)\, r^2 \sin 3\phi & 0 \le r < R \\ \dfrac{3B(t=0)}{r^4} \sin 3\phi & r > R \end{cases}$$

$$\epsilon_2 E_r(r=R_+,\phi,t=0) - \epsilon_1 E_r(r=R_-,\phi,t=0) = \sigma_s(t=0)\Big|_{r=R} = \sigma_{s0} \sin 3\phi$$

$$\frac{3\epsilon_2 B(t=0)}{R^4}\cancel{\sin 3\phi} + 3\epsilon_1 A(t=0) R^2 \cancel{\sin 3\phi} = \sigma_{s0} \cancel{\sin 3\phi}$$

$$\Phi(r=R_+,\phi,t=0) = \Phi(r=R_-,\phi,t=0) \Rightarrow A(t=0) R^3 = \frac{B(t=0)}{R^3}$$

$$B(t=0) = A(t=0) R^6$$

$$3R^2 A(t=0)\left[\epsilon_1 + \epsilon_2\right] = \sigma_{s0}$$

$$A(t=0) = \frac{B(t=0)}{R^6} = \frac{\sigma_{s0}}{3R^2(\epsilon_1+\epsilon_2)}$$

$$\Phi(r,\phi,t=0) = \begin{cases} \dfrac{\sigma_{s0}}{3R^2(\epsilon_1+\epsilon_2)} r^3 \sin 3\phi & 0 \le r \le R \\ \dfrac{\sigma_{s0} R^4}{3(\epsilon_1+\epsilon_2)} \dfrac{\sin 3\phi}{r^3} & r \ge R \end{cases}$$

b)

$$\Phi(r,\phi,t) = \begin{cases} A(t)\, r^3 \sin 3\phi & 0 \le r \le R \\ \dfrac{B(t)}{r^3} \sin 3\phi & r \ge R \end{cases}$$

$$\Phi(r=R_+,\phi,t) = \Phi(r=R_-,\phi,t) \Rightarrow A(t) R^3 = \frac{B(t)}{R^3}$$

$$B(t) = A(t) R^6$$

$$\sigma_1 E_r(r=R_-,\phi,t) + \epsilon_1 \frac{\partial E_r(r=R_-,\phi,t)}{\partial t} = \sigma_2 E_r(r=R_+,\phi,t) + \epsilon_2 \frac{\partial E_r(r=R_+,\phi,t)}{\partial t}$$

$$E_r(r,\phi,t) = -\frac{\partial \Phi(r,\phi,t)}{\partial r} = \begin{cases} -3A(t)\, r^2 \sin 3\phi & 0 \le r < R \\ \dfrac{3B(t)}{r^4} \sin 3\phi & r > R \end{cases}$$

$$-3\sigma_1 R^2 A(t) - 3R^2 \epsilon_1 \frac{dA}{dt} = \frac{3\sigma_2}{R^4} B(t) + \frac{3\epsilon_2}{R^4}\frac{dB}{dt}$$
$$= 3R^2\left(\sigma_2 A(t) + \epsilon_2 \frac{dA}{dt}\right)$$

$$-\sigma_1 A(t) - \epsilon_1 \frac{dA}{dt} = \sigma_2 A(t) + \epsilon_2 \frac{dA}{dt}$$

$$(\epsilon_1+\epsilon_2)\frac{dA}{dt} + (\sigma_2+\sigma_1)A(t) = 0$$

$$\frac{dA}{dt} + \frac{A(t)}{\tau} = 0 \;; \quad \tau = \frac{\epsilon_1+\epsilon_2}{\sigma_1+\sigma_2}$$

$$A(t) = A(t=0)e^{-t/\tau} = \frac{\rho_{s0}}{3R^2(\epsilon_1+\epsilon_2)} e^{-t/\tau}$$

$$\Phi(r,\phi,t) = \begin{cases} \dfrac{\rho_{s0}}{3R^2(\epsilon_1+\epsilon_2)} r^3 \sin 3\phi\, e^{-t/\tau} & 0 \le r \le R \\ \dfrac{\rho_{s0} R^4}{3(\epsilon_1+\epsilon_2)} \dfrac{\sin 3\phi}{r^3} e^{-t/\tau} & r \ge R \end{cases}$$

c) $\rho_s(r=R,t) = \epsilon_2 E_r(R_+,\phi,t) - \epsilon_1 E_r(R_-,\phi,t)$

$$= \frac{3A(t)R^6}{R^4}\epsilon_2 \sin 3\phi + 3A(t)R^2 \epsilon_1 \sin 3\phi$$
$$= 3R^2 \sin 3\phi\, (\epsilon_1+\epsilon_2)\, A(t)$$
$$= \cancel{3R^2} \sin 3\phi\, \cancel{(\epsilon_1+\epsilon_2)} \frac{\rho_{s0}}{\cancel{3R^2(\epsilon_1+\epsilon_2)}} e^{-t/\tau}$$
$$= \rho_{s0} \sin 3\phi\, e^{-t/\tau}$$

2. a) $C(x) = \dfrac{\epsilon_0 A}{G-x}$, $f_x = \dfrac{1}{2} V^2 \dfrac{dC}{dx} = \dfrac{1}{2} \dfrac{V^2 \epsilon_0 A}{(G-x)^2}$

b) $$M\frac{d^2x}{dt^2} = -Kx + \frac{\epsilon_0 A V^2}{2(G-x)^2}$$

c) At equilibrium, $\dfrac{dx}{dt} = 0 \Rightarrow K\bar{X} = \dfrac{\epsilon_0 A V^2}{2(G-\bar{X})^2}$

d) $x = \bar{X} + x'$

$$M\frac{d^2x'}{dt^2} = -Kx' + \frac{\epsilon_0 A V^2}{\cancel{2}} \frac{(\cancel{2})\, x'}{(G-\bar{X})^3} = \left[-K + \frac{\epsilon_0 A V^2}{(G-\bar{X})^3}\right] x'$$

$$M\frac{d^2x'}{dt^2} = -K\left[1 - \frac{2\bar{X}}{(G-\bar{X})}\right] x'$$

$$= -K\left[G - 3\bar{X}\right] x'$$

e) Unstable if $G - 3\bar{X} < 0$

Incipience: $\bar{X} > \dfrac{G}{3}$

f) $$-K + \frac{\epsilon_0 A V^2}{(G-\bar{X})^3} = 0 = -K + \frac{\epsilon_0 A V^2}{(\frac{2}{3}G)^3} = -K + \frac{27}{8}\frac{\epsilon_0 A V^2}{G^3}$$

$$V = \left[\frac{8KG^3}{27\epsilon_0 A}\right]^{1/2}$$

3. a) $H_a a = NI$

$H_b b = NI$

$H_a = \frac{NI}{a}, \quad H_b = \frac{NI}{b}$

b) $\Phi = \mu_b H_b s_b d + H_a d(\mu_a x + \mu_0(s_0 - x))$

$= NId\left(\frac{\mu_b s_b}{b} + \frac{1}{a}(\mu_a x + \mu_0(s_0 - x))\right)$

$\lambda = N\Phi = N^2 I d\left[\frac{\mu_b s_b}{b} + \frac{1}{a}(\mu_a x + \mu_0(s_0 - x))\right]$

$L(x) = \frac{\lambda}{I} = N^2 d\left[\frac{\mu_b s_b}{b} + \frac{1}{a}(\mu_a x + \mu_0(s_0 - x))\right]$

c) $f_x = \frac{1}{2} I^2 \frac{dL}{dx} = \frac{1}{2} \frac{N^2 I^2 d}{a}(\mu_a - \mu_0)$

d) $M\frac{d^2 x}{dt^2} = -Kx + \frac{1}{2}\frac{N^2 I^2 d}{a}(\mu_a - \mu_0) = f_T(x)$

e) $f_T(x) = f_x - Kx_{eq} = 0 = \frac{1}{2}\frac{N^2 I^2 d}{a}(\mu_a - \mu_0) - Kx_{eq}$

$x_{eq} = \frac{1}{2}\frac{N^2 I^2 d}{Ka}$

f) $\left.\frac{df_T}{dx}\right|_{x=x_{eq}} = -K < 0$ (Stable)

g) $M\frac{d^2 x'}{dt^2} = \left.\frac{df_T}{dx}\right|_{x_{eq}} x' = -Kx' \Rightarrow \frac{d^2 x'}{dt^2} + \omega_0^2 x' = 0, \quad \omega_0^2 = K/M$

$x' = A\sin\omega_0 t + B\cos\omega_0 t$

$\frac{dx'}{dt} = \omega_0\left[A\cos\omega_0 t - B\sin\omega_0 t\right]$

$x'(t=0) = B = \Delta x$

$\left.\frac{dx'}{dt}\right|_{t=0} = \omega_0 A = v_0 \Rightarrow A = \frac{v_0}{\omega_0}$

$x'(t) = \frac{v_0}{\omega_0}\sin\omega_0 t + \Delta x\cos\omega_0 t$