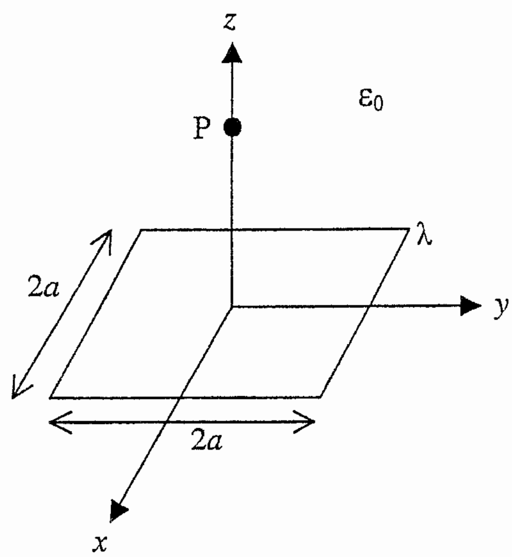
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.013 Electromagnetics and Applications
Quiz 1, 10/16/03

6.013 Formula sheet attached. You are also allowed to use a formula sheet on both sides of a 8½"x11" paper that you prepare.

Problem 1 (30 Points)



A uniformly distributed line charge in free space with constant density λ coulombs/meter is bent into the shape of a square with sides of length $2a$. The square shaped line charge is centered around the origin in the $z=0$ plane.

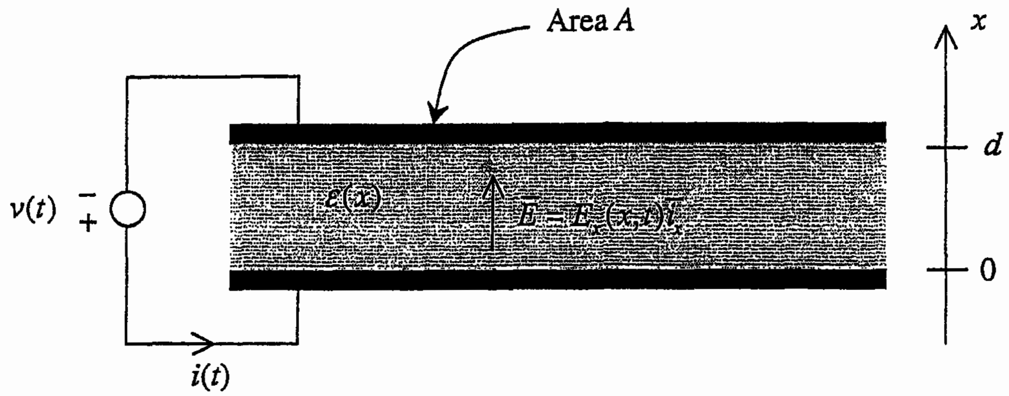
a) What is the approximate electric scalar potential and approximate electric field far from the line charge so that $r >> 2a$? Hint: No major computation is necessary for this part.

b) What is the electric scalar potential for any point P on the z axis?
Hint: One or more of the following indefinite integrals may be useful:

$$\int \frac{dx}{\left[x^2+c^2\right]^{1/2}} = \ln\left[x+\sqrt{x^2+c^2}\right], \qquad \int \frac{dx}{\left[x^2+c^2\right]} = \frac{1}{c}\tan^{-1}\left(\frac{x}{c}\right)$$

$$\int \frac{dx}{\left[x^2+c^2\right]^{3/2}} = \frac{x}{c^2\sqrt{x^2+c^2}}, \qquad \int \frac{xdx}{\left[x^2+c^2\right]^{3/2}} = \frac{-1}{\sqrt{x^2+c^2}}$$

Problem 2 (30 points)



Parallel plate electrodes with spacing d and area A enclose a lossless dielectric with space varying dielectric permittivity

$$\varepsilon(x) = \varepsilon_0 e^{\alpha x}$$

A voltage source, $v(t)$, is imposed across the electrodes. The lossless dielectric has no free volume charge density, $\rho_f = 0$, so that Gauss' law for this problem reduces to

$$\nabla \cdot \bar{D} = \nabla \cdot \left[\varepsilon(x)\bar{E}\right] = 0.$$

a) Find the electric field in the dielectric. Neglect fringing field effects so that the electric field can only be x directed and is only a function of x and t, $\bar{E} = E_x(x,t)\bar{i}_x$.

b) What are the free surface charge densities at each of the electrodes, $\sigma_s(x=0,t)$ and $\sigma_s(x=d,t)$?

c) What is the capacitance?

Basic Equations of Electrodynamics

Mathematical Identities

$v(t) = R_e\{\underline{V}e^{j\omega t}\}$ where $\underline{V} = |V|e^{j\phi}$

$\nabla = \hat{x}\,\partial/\partial x + \hat{y}\,\partial/\partial y + \hat{z}\,\partial/\partial z$

$\overline{A} \bullet \overline{B} = A_xB_x + A_yB_y + A_zB_z$

$\nabla^2\phi = (\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)\phi$

$\sin^2\theta + \cos^2\theta = 1$

$\nabla \bullet (\nabla \times \overline{A}) = 0$

$\nabla \times (\nabla \times \overline{A}) = \nabla(\nabla \bullet \overline{A}) - \nabla^2 \overline{A}$

$\int_V(\nabla \bullet \overline{G})dv = \int_A \overline{G} \bullet d\overline{a}$

$\int_A (\nabla \times \overline{G}) \bullet d\overline{a} = \oint_c \overline{G} \bullet d\overline{s}$

$e^{j\omega t} = \cos\omega t + j\sin\omega t$

$\cos\alpha + \cos\beta = 2\cos[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2]$

$\underline{H}(f) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt$

$e^x = 1 + x + x^2/2! + x^3/3! + \ldots$

$\sin\alpha = (e^{j\alpha} - e^{-j\alpha})/2j$

$\cos\alpha = (e^{j\alpha} + e^{-j\alpha})/2$

Electromagnetic Variables

$\overline{E}$ = electric field (Vm^{-1})

$\overline{H}$ = magnetic field (Am^{-1})

$\overline{D}$ = electric displacement (Cm^{-2})

$\overline{B}$ = magnetic flux density (T)

Tesla (T) = Weber m^{-2} = 10,000 gauss

ρ = charge density (Cm^{-3})

$\overline{J}$ = current density (Am^{-2})

σ = conductivity (Siemens m^{-1})

$\overline{J}_s$ = surface current density (Am^{-1})

ρ_s = surface charge density (Cm^{-2})

Maxwell's Equations, Force

$\nabla \times \overline{E} = -\partial \overline{B}/\partial t$

$\oint_c \overline{E} \bullet d\overline{s} = -\frac{d}{dt}\int_A \overline{B} \bullet d\overline{a}$

$\nabla \times \overline{H} = \overline{J} + \partial \overline{D}/\partial t$

$\oint_c \overline{H} \bullet d\overline{s} = \int_A \overline{J} \bullet d\overline{a} + \frac{d}{dt}\int_A \overline{D} \bullet d\overline{a}$

$\nabla \bullet \overline{D} = \rho \rightarrow \int_A \overline{D} \bullet d\overline{a} = \int_V \rho dv$

$\nabla \bullet \overline{B} = 0 \rightarrow \int_A \overline{B} \bullet d\overline{a} = 0$

$\nabla \bullet \overline{J} = -\partial\rho/\partial t$

$\overline{f} = q(\overline{E} + \overline{v} \times \mu_o \overline{H})$ [N]

Constants

$\varepsilon_o = 8.85 \times 10^{-12}$ Fm^{-1}

$\mu_o = 4\pi \times 10^{-7}$ Hm^{-1}

$c = (\varepsilon_o\mu_o)^{-0.5} \cong 3 \times 10^8 ms^{-1}$

$h = 6.624 \times 10^{-34}$ Js

$e = -1.60 \times 10^{-19}$ C

$k = 1.38 \times 10^{-23}$ JK^{-1}

$\eta_o \cong 377$ohms $= (\mu_o/\varepsilon_o)^{0.5}$

$m_e = 9.1066 \times 10^{-31}$ kg

Waves

$(\nabla^2 - \mu\varepsilon\partial^2/\partial t^2)\overline{E} = 0$ [Wave Eq.]

$(\nabla^2 + k^2)\underline{\overline{E}} = 0$, $\underline{\overline{E}} = \underline{\overline{E}}_o e^{-j\overline{k}\bullet\overline{r}}$

$k = \omega(\mu\varepsilon)^{0.5} = \omega/c = 2\pi/\lambda$

$k_x^2 + k_y^2 + k_z^2 = k_o^2 = \omega^2\mu\varepsilon$

$v_p = \omega/k$, $v_g = (\partial k/\partial\omega)^{-1}$

$E_y(z,t) = E_+(z-ct) + E_-(z+ct) = R_e\{\underline{E}_y(z)e^{j\omega t}\}$

$H_x(z,t) = \eta_o^{-1}[E_+(z-ct) - E_-(z+ct)]$ [or(ωt-kz) or (t-z/c)]

$\int_A(\overline{E} \times \overline{H}) \bullet d\overline{a} + (d/dt)\int_V(\varepsilon|\overline{E}|^2/2 + \mu|\overline{H}|^2/2)dv$

$= -\int_V \overline{E} \bullet \overline{J}\,dv$ (Poynting Theorem)

Media

$\overline{D} = \varepsilon_o\overline{E} + \overline{P}$

$\nabla \bullet \overline{D} = \rho_f$, $\tau = \varepsilon/\sigma$

$\nabla \bullet \varepsilon_o\overline{E} = \rho_f + \rho_p$

$\nabla \bullet \overline{P} = -\rho_p$, $\overline{J} = \sigma\overline{E}$

$\overline{B} = \mu\overline{H} = \mu_o(\overline{H} + \overline{M})$

$\varepsilon = \varepsilon_o(1 - \omega_p^2/\omega^2)$

$\omega_p = (Ne^2/m\varepsilon_o)^{0.5}$

$\varepsilon_{eff} = \varepsilon_o(1 - j\sigma/\omega\varepsilon)$

skin depth $\delta = (2/\omega\mu\sigma)^{0.5}$ [m]

Boundary Conditions

$\overline{E}_{1//} - \overline{E}_{2//} = 0$

$\overline{H}_{1//} - \overline{H}_{2//} = \hat{n} \times \overline{K}_s$

$\overline{B}_{1\perp} - \overline{B}_{2\perp} = 0$

$\overline{D}_{1\perp} - \overline{D}_{2\perp} = \rho_s$

0 = ↵ if $\sigma = \infty$

Planar Interfaces

$\theta_r = \theta_i$

$\sin\theta_t/\sin\theta_i = k_i/k_t = n_i/n_t$

$\theta_c = \sin^{-1}(n_t/n_i)$

$\theta > \theta_c \Rightarrow \underline{\overline{E}}_t = \underline{\overline{E}}_i\underline{T}\,e^{+\alpha x - jk_z z}$

$\underline{\overline{k}} = \overline{k}' - j\,\overline{k}''$

$\underline{\Gamma} = \underline{T} - 1$

$\underline{T}_{TE} = 2/(1 + [\eta_o\cos\theta_t/\eta_t\cos\theta_i])$

$\underline{T}_{TM} = 2/(1 + [\eta_t\cos\theta_t/\eta_i\cos\theta_i])$

$\theta_B = \tan^{-1}(\varepsilon_t/\varepsilon_i)^{0.5}$ for TM

$P_d \cong |\underline{\overline{J}}_s|^2/2\sigma\delta$ [Wm^{-2}]

Radiation

$\overline{E} = -\nabla\phi - \partial\overline{A}/\partial t$, $\overline{B} = \nabla \times \overline{A}$

$\underline{\phi}(r) = \int_{V'}(\underline{\rho}(\overline{r})e^{-jk|\overline{r}'-\overline{r}|}/4\pi\varepsilon_o|\overline{r}' - \overline{r}|)dv'$

$\underline{\overline{A}}(r) = \int_{V'}(\mu_o\underline{\overline{J}}(\overline{r})e^{-jk|\overline{r}'-\overline{r}|}/4\pi|\overline{r}' - \overline{r}|)dv'$

$\underline{\overline{E}}_{ff} = \hat{\vartheta}\,(j\eta_o k\underline{I}d/4\pi r)\,e^{-jkr}\sin\theta$

$\nabla^2\underline{\Phi} + \omega^2\mu_o\varepsilon_o\underline{\Phi} = -\rho/\varepsilon_o$

$\nabla^2\underline{\overline{A}} + \omega^2\mu_o\varepsilon_o\underline{\overline{A}} = -\mu_o\underline{\overline{J}}$

Antennas

$G(\theta,\phi) = P_r/(P_R/4\pi r^2)$

$P_R = \int_{4\pi} P_r(\theta,\phi,r)r^2\sin\theta d\theta d\phi$

$P_{rec} = P_r(\theta,\phi)A_e(\theta,\phi)$

$A_e(\theta,\phi) = G(\theta,\phi)\lambda^2/4\pi$

$R_r = P_R/\langle i^2(t)\rangle$

$E_{ff}(\theta \cong 0) = (je^{jkr}/\lambda r)\int_A E_t(x,y)\,e^{jk_x x + jk_y y}\,dxdy$

$P_{rec} = P_R(G\lambda/4\pi r^2)^2\sigma_s/4\pi$

$\underline{\overline{E}} = \Sigma_i a_i \underline{\overline{E}}_i e^{-jkr_i}$ =

(element factor)(array f)

$E_{bit} \geq \sim 4 \times 10^{-20}$ [J]

$\underline{Z}_{12} = \underline{Z}_{21}$ if reciprocity

12/23/02

Circuits

KCL: $\Sigma_i I_i(t) = 0$ at node

KVL: $\Sigma_i V_i(t) = 0$ around loop

$C = Q/V = A\varepsilon/d$ [F]

$L = \Lambda/I$

$i(t) = C\, dv(t)/dt$

$v(t) = L\, di(t)/dt = d\Lambda/dt$

$C_{parallel} = C_1 + C_2$

$C_{series} = (C_1^{-1} + C_2^{-1})^{-1}$

$w_e = Cv^2(t)/2$; $w_m = Li^2(t)/2$

$L_{solenoid} = N^2\mu A/W$

$\tau = RC$, $\tau = L/R$

$\Lambda = \int_A \overline{B} \bullet d\overline{a}$ (per turn)

Waveguides

$\underline{\overline{E}}_{TE} = \hat{y}E_o \sin k_x x \cdot e^{-jk_z z}$

$\underline{\overline{E}}_{TE} = \hat{y}E_o \sin k_x x \cdot e^{-\alpha z}$

$k_x^2 + k_z^2 = k_o^2 = \omega^2(\mu_o\varepsilon_o)$

$\lambda_g = \lambda_z = (\lambda_o^{-2} - \lambda_x^{-2})^{-0.5}$

$v_g = c \sin\theta_i = (\partial k_z/\partial\omega)^{-1}$

$v_p = c/\sin\theta_i = \omega/k_z$

Kinematics

$f = ma = d(mv)/dt$

$x = x_o + v_o t + at^2/2$

$P = fv$ [W] $= T\omega$

$w_k = mv^2/2$

$T = I\, d\omega/dt$

$I = \Sigma_i m_i r_i^2$

TEM Transients

$\partial v(z)/dz = -L\partial i(z)/dt$

$\partial i(z)/dz = -C\partial v(z)/dt$

$\partial^2 v/dz^2 = LC\, \partial^2 v/dt^2$

$v(z,t) = f_+(t - z/c) + f_-(t + z/c)$

$\quad = g_+(z - ct) + g_-(z + ct)$

$i(z,t) = Y_o[f_+(t - z/c) - f_-(t + z/c)]$

$c = (LC)^{-0.5} = (\mu\varepsilon)^{-0.5}$

$Z_o = Y_o^{-1} = (L/C)^{0.5}$

$\Gamma_L = f_-/f_+ = (R_L - Z_o)/(R_L + Z_o)$

$v(z,t) = g_+(z - ct) + g_-(z + ct)$

$V_{Th} = 2f_+(t)$, $R_{Th} = Z_o$

Electromagnetic Forces

$\overline{f} = q(\overline{E} + \overline{v} \times \mu_o \overline{H})$ [N]

$f_z = -dw_T/dz$

$\overline{F} = \overline{I} \times \mu_o \overline{H}$ [Nm^{-1}]

$\overline{E}_e = -\overline{v} \times \mu_o \overline{H}$ inside wire

$P = \omega T = W_T\, dV_{olume}/dt$ [W]

Max $f/A = B^2/2\mu_o$, $D^2/2\varepsilon_o$ [Nm^{-2}]

$$vi = \frac{dw_T}{dt} + f\,\frac{dz}{dt}$$

TEM Sinusoidal Steady State

$(d^2/dz^2 + \omega^2 LC)\underline{V}(z) = 0$

$\underline{V}(z) = \underline{V}_+ e^{-jkz} + \underline{V}_- e^{+jkz}$

$\underline{I}(z) = Y_o[\underline{V}_+ e^{-jkz} - \underline{V}_- e^{+jkz}]$

$k = 2\pi/\lambda = \omega/c = \omega(\mu\varepsilon)^{0.5}$

$\underline{Z}(z) = \underline{V}(z)/\underline{I}(z) = Z_o\underline{Z}_n(z)$

$\underline{Z}_n(z) = [1+\underline{\Gamma}(z)]/[1-\underline{\Gamma}(z)] = R_n + jX_n$

$\underline{\Gamma}(z) = (\underline{V}_-/\underline{V}_+)e^{2jkz} = [\underline{Z}_n(z)-1]/[\underline{Z}_n(z)+1]$

$\underline{Z}(z) = Z_o(\underline{Z}_L - jZ_o \tan kz)/(Z_o - j\underline{Z}_L \tan kz)$

$VSWR = |\underline{V}_{max}|/|\underline{V}_{min}| = R_{nmax}$

Acoustics

$P = P_o + p$, $\overline{U} = \overline{U}_o + u$

$\nabla p = -\rho_o \partial \overline{u}/\partial t$

$\nabla \bullet \overline{u} = -(1/\gamma P_o)\partial p/\partial t$

$(\nabla^2 - k^2\partial^2/\partial t^2)p = 0$

$k^2 = \omega^2/c_s^2 = \omega^2\rho_o/\gamma P_o$

$c_s = v_p = v_g = (\gamma P_o/\rho_o)^{0.5}$ or $(K/\rho_o)^{0.5}$

$\eta_s = p/u = \rho_o c_s = (\rho_o\gamma P_o)^{0.5}$ gases

$\eta_s = (\rho_o K)^{0.5}$ solids, liquids

p, $\overline{u}_\perp$ continuous at boundaries

$p = p_+ e^{-jkz} + p_- e^{+jkz}$

$u_z = \eta_s^{-1}(p_+ e^{-jkz} - p_- e^{+jkz})$

$\int_A \overline{u}p \bullet d\overline{a} + (d/dt)\int_V(\rho_o|\overline{u}|^2/2 + p^2/2\gamma P_o)dV = 0$

RLC Resonators

$\underline{Z}_{series} = R + j\omega L + 1/j\omega C$

$\underline{Y}_{par} = G + j\omega C + 1/j\omega L$

$Q = \omega_o w_T/P_{diss} = \omega_o/\Delta\omega$

$\omega_o = (LC)^{-0.5}$

EM Resonators

At ω_o, $\langle w_e \rangle = \langle w_m \rangle$

$\langle w_e \rangle = \int_V (\varepsilon|\underline{\overline{E}}|^2/4)\, dv$

$\langle w_m \rangle = \int_V (\mu|\underline{\overline{H}}|^2/4)\, dv$

$Q_n = \omega_n w_{Tn}/P_n = \omega_n/2\alpha_n$

$f_{mnp} = (c/2)([m/a]^2 + [n/b]^2 + [p/d]^2)^{0.5}$

$s_n = j\omega_n - \alpha_n$

Quantum Phenomena

$E = hf$, photons or phonons

hf/c = momentum [$kg\, ms^{-1}$]

$dn_2/dt = -[An_2 + B(n_2 - n_1)]$