6.641 Quiz 2 Solutions

4/21/04

1. a) $K_x = \frac{i}{D} \Rightarrow H_z = -K_x = -\frac{i}{D}$

$$\Phi = -\mu_0 H_z s \xi = \frac{\mu_0 s \xi}{D} i$$

$$L(\xi) = \frac{\Phi}{i} = \frac{\mu_0 s \xi}{D}$$

b) $\Phi_0 = L(\xi_0) I_0 = \frac{\mu_0 s \xi_0}{D} I_0 = \frac{\mu_0 s \frac{\xi_0}{2} i}{D} \Rightarrow i = 2I_0,\ \Phi_0 = \frac{\mu_0 s \xi_0 I_0}{D}$

c) $f_x(\xi) = -\frac{1}{2} \Phi_0^2 \frac{d}{d\xi}\left(\frac{1}{L(\xi)}\right) = -\frac{1}{2} \frac{\Phi_0^2 D}{\mu_0 s} \frac{d}{d\xi}\left(\frac{1}{\xi}\right) = +\frac{1}{2} \frac{\Phi_0^2 D}{\mu_0 s \xi^2}$

d) $f_T = -K\xi + \frac{1}{2} \frac{\Phi_0^2 D}{\mu_0 s \xi^2}$

e) $f_{eq} = f_T = 0 = -K\xi_{eq} + \frac{1}{2} \frac{\Phi_0^2 D}{\mu_0 s \xi_{eq}^2} \Rightarrow \xi_{eq}^3 = \frac{1}{2} \frac{\Phi_0^2 D}{\mu_0 s K}$

$$\xi_{eq} = \left[\frac{1}{2} \frac{\Phi_0^2 D}{\mu_0 s K}\right]^{1/3}$$

f) $\left.\frac{\partial f_T}{\partial \xi}\right|_{\xi_{eq}} = -K - \frac{\Phi_0^2 D}{\mu_0 s \xi_{eq}^3} = -3K < 0 \Rightarrow$ stable

g) $M\frac{d^2\xi'}{dt^2} = f_T(\xi_{eq} + \xi') + f_p(t) \approx \cancelto{0}{f_T(\xi_{eq})} + \left.\frac{\partial f_T}{\partial \xi}\right|_{\xi_{eq}} \xi' + f_p(t) = -3K\xi' + f_0 u(t-T)$

$$\frac{d^2\xi'}{dt^2} + \omega_0^2 \xi' = \frac{f_0 u(t-T)}{M},\quad \omega_0^2 = \frac{3K}{M}$$

$$\xi'(t) = \frac{f_0}{M\omega_0^2} + A_1 \sin\omega_0(t-T) + A_2 \cos\omega_0(t-T) \qquad t > T$$

$\xi'(t=T) = 0 = \frac{f_0}{M\omega_0^2} + A_2 \Rightarrow A_2 = -\frac{f_0}{M\omega_0^2}$

$\left.\frac{d\xi'}{dt}\right|_{t=T} = 0 = \omega_0 A_1 \Rightarrow A_1 = 0$

$\Rightarrow \xi'(t) = \frac{f_0}{M\omega_0^2}\left(1 - \cos\omega_0(t-T)\right) = \frac{f_0}{3K}\left(1 - \cos\omega_0(t-T)\right) \qquad t > T$

2. a) $B_y = \dfrac{\mu N i}{h}$, $\bar{B} = B_y \bar{i}_y$

b) $\lambda = N B_y w D = \dfrac{\mu N^2 w D\, i}{h} \Rightarrow L = \dfrac{\lambda}{i} = \dfrac{\mu N^2 w D}{h}$

c) $J_x = \dfrac{i}{hD} = \sigma(E_x - V B_y)$ $\left(\bar{J} = \sigma(\bar{E} + \bar{v}\times\bar{B})\right)$

$E_x = \dfrac{i}{\sigma h D} + V B_y$

$v_{MHD} = E_x w = \dfrac{i\, w}{\sigma h D} + \dfrac{V \mu N w\, i}{h} = i\left[\dfrac{w}{\sigma h D} + \dfrac{\mu N w V}{h}\right]$

d) $v_{MHD} + v_C + v_L = 0 \Rightarrow i\left[\dfrac{w}{\sigma h D} + \dfrac{\mu N w V}{h}\right] + \dfrac{1}{C}\int i\,dt + L\dfrac{di}{dt} = 0$

$$L\frac{d^2 i}{dt^2} + \left[\frac{w}{\sigma h D} + \frac{\mu N w V}{h}\right]\frac{di}{dt} + \frac{i}{C} = 0$$

e) $\omega_0^2 = \dfrac{1}{LC}$, $R = \dfrac{w}{\sigma h D}$, $G = \dfrac{\mu N w}{h}$

$$\frac{d^2 i}{dt^2} + \frac{1}{L}\left[R + GV\right]\frac{di}{dt} + \omega_0^2 i = 0$$

$i = Ae^{st}$

$$s^2 + s\frac{[R+GV]}{L} + \omega_0^2 = 0$$

$$s = -\frac{[R+GV]}{2L} \pm \left[\left[\frac{R+GV}{2L}\right]^2 - \omega_0^2\right]^{1/2}$$

For self-excitation: $R + GV < 0 \Rightarrow V < -\dfrac{R}{G} \Rightarrow V < -\dfrac{1}{\sigma \mu D N}$

$|V| > \dfrac{1}{\sigma \mu D N}$ with direction in $-z$ direction

f) $\omega_0^2 > \left[\dfrac{R+GV}{2L}\right]^2 \Rightarrow C < \dfrac{4\mu N^2 D h}{w\left[\frac{1}{\sigma D} + \mu N V\right]^2}$