MIT Course 8.033, Fall 2006, Cosmology
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Topics

- The FRW metric
- Interpretation of FRW metric (curvature, expansion, comoving objects, geodesics, redshift)
- The Friedmann equation
- Cosmological parameters: Ω_γ, $\Omega_{\rm m}$, $\Omega_{\rm k}$, Ω_Λ, Ω_b, Ω_d, h
- Age of the Universe

Key formula summary

- FRW metric:

$$d\tau^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \right)$$

- Hubble parameter:

$$H \equiv \frac{\dot{a}}{a}$$

- Dimensionless current Hubble parameter:

$$h \equiv H_0/(100\text{km s}^{-1}\text{Mpc}^{-1}) \approx H_0 \times 9.7846\text{Gyr}$$

- Friedmann equation:

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \\ &= H_0^2 \left[\Omega_\gamma (1+z)^4 + \Omega_\text{m} (1+z)^3 + \Omega_\text{k} (1+z)^2 + \Omega_\Lambda \right] \end{aligned}$$

- Cosmological parameter measurements (2006):
  - $\Omega_b \approx 0.04$,
  - $\Omega_d \approx 0.21$,
  - $\Omega_\Lambda \approx 0.75$,
  - $\Omega_\text{k} \approx 0$,
  - $h \approx 0.7$,
  - $\Omega_\text{m} \equiv \Omega_b + \Omega_d \approx 0.25$,

- Age of the Universe at redshift z:

$$t(z) = \int_z^\infty \frac{dz'}{(1+z')H(z')}$$

The FRW metric

- Observations indicate that the Universe is homogeneous and isotropic on large scales, *i.e.*, that if we smooth out the density field sufficiently, we can approximate it as a independent of position: $\rho(\mathbf{r}, t) = \rho(t)$.

- One can prove that the most general metric that is homogeneous and isotropic is the FRW metric

$$d\tau^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right)$$

  for a function $a(t)$ called the *cosmic scale factor* (describing how the Universe expands over time) and a constant k that equals 1, 0 or -1.

Interpretation of the FRW metric

- Lines through spacetime where r, θ and ϕ are all constant are geodesics. You will prove this for the $k = 0$ case on PS8. Objects moving along such geodesics are called "comoving" and the position vector defined by r, θ and ϕ in polar coordinates is called the "comoving position" of an object. To first approximation, galaxies are comoving, *i.e.*, stay at a fixed comoving position.

- The physical distance between any two comoving objects changes over time, proportional to $a(t)$. One can therefore think of all galaxies as being at rest in (comoving) space, with physical space simply stretching uniformly over time proportionally to $a(t)$.

- If $k = 1$, then the FRW metric can be interpreted as that of the surface of a 4-dimensional hypersphere of radius $a(t)$, and the angles of a triangle will add up to more than 180°.

- If $k = -1$, then the FRW metric can be interpreted as that of the surface of a 4-dimensional hyperboloid of radius $a(t)$, and the angles of a triangle will add up to less than 180° (like on a saddle or Pringles' potato chip).

- If $k = 0$, then the FRW metric can be interpreted as that of an expanding flat 3-dimensional space, and can be rewritten in Cartesian coordinates as

$$d\tau^2 = dt^2 - a(t)^2 \left(dx^2 + dy^2 + dz^2 \right).$$

- In any metric, light rays have $d\tau = 0$. For a photon moving radially out from the origin $r = 0$ (with constant θ and ϕ), the FRW metric thus gives

$$\frac{dr}{\sqrt{1 - kr^2}} = \frac{dt}{a(t)}.$$

  (In one of the optional problems on PS8, you can integrate this to calculate a photon trajectory.)

- In cosmology, a is often used as a convenient time-variable in place of t.

- **Cosmological redshift:** Light emitted at time a_1 and observed at a later time a_2 has a redshift given by

$$\frac{\lambda_2}{\lambda_1} \equiv \frac{a_2}{a_1}.$$

  Interpretation: as it travels to us, light has its wavelength stretched out together with space itself.

- The redshift z is defined by the relation

$$1 + z \equiv \frac{\lambda_2}{\lambda_1},$$

  so if the emission occurs at some early epoch with scale factor a and the observation occurs today with scale factor a_0, then

$$1 + z \equiv \frac{a_0}{a}.$$

- In observational cosmology, the most commonly used time-variable of all is z, since it is the quantity that is directly observable when we look out in space and back in time When cosmologists say that something happened "at redshift z", they mean that it happened at the time when light would have had to be emitted to reach us with a redshift z. $z = 0$ is today, $z = \infty$ is the Big Bang and $z = -1$ is the infinite future. The most distant objects observed (2005) have z just above six. Cosmic microwave background images depict $z \approx 10^3$.

The Friedmann equation

- The FRW metric satisfies the Einstein field equations (the 2nd part of GR, specifying how matter affects the metric) if and only if the function $a(t)$ satisfies the so-called Friedmann equation:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2},$$

where the *Hubble parameter* is defined as

$$H \equiv \frac{\dot{a}}{a}.$$

- Amazingly, the correct form of the Friedman equation can derived from Newtonian gravity (you'll do this on PS8), but in that case there's of course no spatial curvature whatever the value of k is.

- As space expands, the contents of space generally gets diluted and so ρ drops. Different types of matter dilute differently as space expands. Here are four popular examples:
  - $\rho_\gamma(a) \propto a^{-4} \propto (1+z)^4$ (photons),
  - $\rho_{\rm m}(a) \propto a^{-3} \propto (1+z)^3$ (ordinary matter, dark matter),
  - $\rho_{\rm k}(a) \propto a^{-2} \propto (1+z)^2$ (spatial curvature, *i.e.*, the k-term above),
  - $\rho_\Lambda(a)$ constant (vacuum energy, *i.e.*, cosmological constant)

- With these four components, we can therefore rewrite the Friedmann equation as

$$H(z)^2 = \frac{8\pi G}{3}\left[\rho_\gamma(0)(1+z)^4 + \rho_{\rm m}(0)(1+z)^3 + \rho_{\rm k}(0)(1+z)^2 + \rho_\Lambda\right],$$

where we have absorbed the k-term above by defining $\rho_{\rm k} \equiv -\frac{3kc^2}{8\pi G a^2}$. This means that H^2 equals a quadratic polynomial in $(1+z)$ whose coefficients specify the current ($z=0$) densities of various types. (Dark energy warning: There are also models for dark energy where $\rho_\Lambda(a)$ is not constant but evolves, measurements so far are consistent with "vanilla" dark energy where ρ_Λ is simply a constant.)

Cosmological parameters

- The current *critical density* is defined as

$$\rho_{\rm crit} \equiv \frac{3H}{8\pi G}.$$

  Note that since $\rho_{\rm crit}(0) = \rho_\gamma(0) + \rho_{\rm m}(0) + \rho_{\rm k}(0) + \rho_\Lambda$, space will be flat ($\rho_{\rm k} = 0$) if the total density of "stuff" ($\rho_\gamma + \rho_{\rm m} + \rho_\Lambda$) equals the critical density.

- The "Omega" for something is simply its current density over the current critical density:

$$\Omega_\gamma = \frac{\rho_\gamma(0)}{\rho_{\rm crit}(0)}, \quad \Omega_{\rm m} = \frac{\rho_{\rm m}(0)}{\rho_{\rm crit}(0)}, \quad \Omega_{\rm k} = \frac{\rho_{\rm k}(0)}{\rho_{\rm crit}(0)}, \quad \Omega_\Lambda = \frac{\rho_\Lambda(0)}{\rho_{\rm crit}(0)}.$$

- The total Omega of "stuff" (everything except curvature) is

$$\Omega_{\rm tot} \equiv \Omega_\gamma + \Omega_{\rm m} + \Omega_\Lambda.$$

- This means that, by definition, $1 = \Omega_\gamma + \Omega_{\rm m} + \Omega_{\rm k} + \Omega_\Lambda = \Omega_{\rm tot} + \Omega_{\rm k}$, *i.e.*, the curvature is given by

$$\Omega_{\rm k} = 1 - \Omega_{\rm tot}.$$

  For an FRW Universe with $\Omega_{\rm tot} > 1$, space is finite like the surface of a 4-dimensional hypersphere ($k = 1$) and for $\Omega_{\rm tot} \leq 1$, space is infinite (at all times, even right after the Big Bang).

- Determining $\Omega_{\rm tot}$ (and whether space is infinite or finite) has been one of the key challenges in cosmology for decades. There's been a sudden breakthrough using measuremends of the cosmic microwave background combined with other measurements like galaxy clustering, and the latest (2005) constraint is

$$\Omega_{\rm tot} = 1.01 \pm 0.02,$$

  i.e., beautifully consistent with perfectly flat space, $\Omega_{\rm tot} = 1$.

- The best (2005) measurements of the cosmological parameters above are
  - $\Omega_\gamma \approx 0.0001$,
  - $\Omega_{\rm m} = 0.30 \pm 0.05$,
  - $\Omega_\Lambda = 0.71 \pm 0.05$,
  - $\Omega_{\rm k} = -0.01 \pm 0.01$,
  - $h = 0.70 \pm 0.05$.

  Here h is the dimensionless current Hubble parameter ("Hubble constant"), defined as

$$h \equiv H_0/(100\text{km s}^{-1}\text{Mpc}^{-1}) \approx H_0 \times 9.7846\text{Gyr}$$

- The matter density is made of both ordinary baryonic matter (atoms) and dark matter (presumably some yet to be discovered particle that has no strong or electromagnetic interaction):

$$\Omega_{\rm m} = \Omega_b + \Omega_d,$$

  with the best (2005) measurements indicating
  - $\Omega_b \approx 0.05$,
  - $\Omega_d \approx 0.25$.

Age of the Universe

- With this notation, we can rewrite the Friedmann equation as

$$H(z)^2 = H_0^2 \left[\Omega_\gamma (1+z)^4 + \Omega_\mathrm{m}(1+z)^3 + \Omega_\mathrm{k}(1+z)^2 + \Omega_\Lambda\right].$$

- Because $H = a^{-1}da/dt$ and $a = (1+z)^{-1}$ implies $dt = da/aH = -dz/(1+z)H$, the age of the Universe at redshift z is

$$t(z) = \int_z^\infty \frac{dz'}{(1+z')H} = \int_z^\infty \frac{H_0^{-1}dz}{(1+z')\left[\Omega_\gamma(1+z')^4 + \Omega_\mathrm{m}(1+z')^3 + \Omega_\mathrm{k}(1+z')^2 + \Omega_\Lambda\right]^{1/2}}.$$

- The nice Javascript calculator at `http://www.astro.ucla.edu/~wright/CosmoCalc.html` performs this integral for cosmological parameters of your choice.