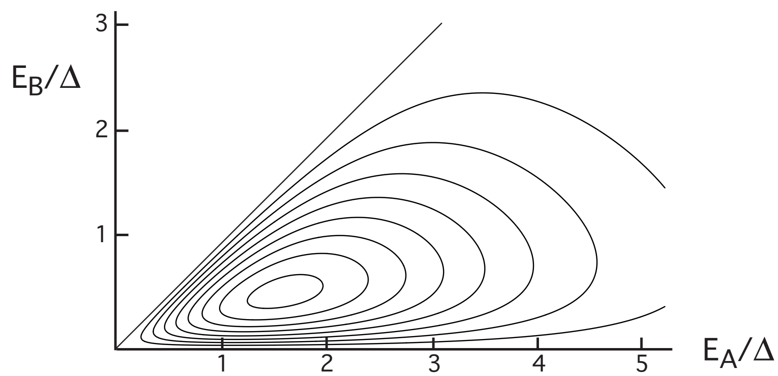
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044 Statistical Physics I — Spring Term 2013

Practice Exam #1

Problem 1 (30 points) Collision Products



A certain collision process in high energy physics produces a number of biproducts. When the biproducts include a pair of elementary particles A and B the energies of those particles, E_A and E_B, are distributed according to the joint probability density

$$
\begin{aligned}
p(E_A, E_B) &= \frac{4E_B(E_A - E_B)}{\Delta^4} \exp[-(E_A + E_B)/\Delta] \qquad \text{for } E_A > 0 \text{ and } E_A > E_B > 0 \\
&= 0 \qquad \text{elsewhere}
\end{aligned}
$$

Δ is a parameter with the units of energy. A contour plot of $p(E_A, E_B)$ is shown above. Note that the energy E_A is always positive and greater than the energy E_B.

a) Find $p(E_B)$. Sketch the result.

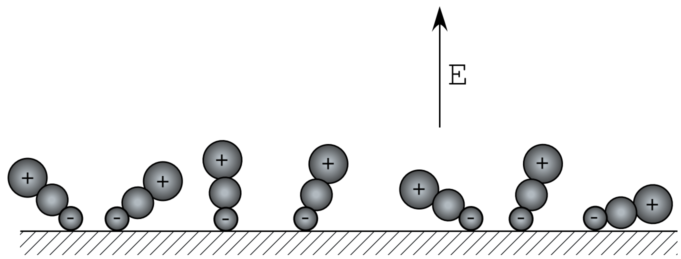
b) Find the conditional probability density $p(E_A \mid E_B)$. Sketch the result.

c) Are E_A and E_B statistically independent? Explain your reasoning.

The collisions are statistically independent random events that occur at some uniform rate in time. The pair A and B only occurs in a fraction f of the collisions. When the pair is produced, it is detected with 100% efficiency. When the pair is not produced, there are no competing background events.

d) If the overall collision rate is 10^6 per hour, how long must one run the experiment in order that the uncertainty in the determination of f is of the order of one part in 10^4 of the value of f measured in that run? Note: one does not need the answers to a), b), or c) to answer this question.

Problem 2 (30 points) Polar Molecules



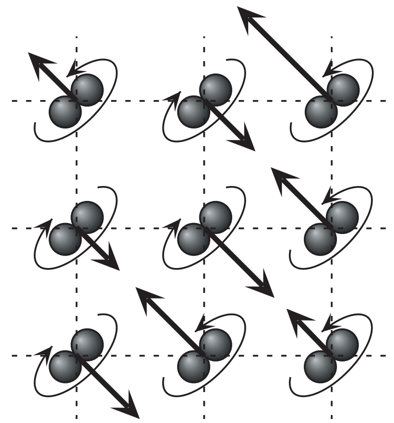
In a particular situation polar molecules (molecules possessing a permanent electric dipole moment) can be adsorbed on a surface creating a dipole layer with a total electric dipole moment $\mathcal{P}$ that remains finite even when the electric field perpendicular to the surface $\mathcal{E}$ goes to zero. Expressions for two important response functions in this system are given below.

$$\chi_T \equiv \left(\frac{\partial \mathcal{P}}{\partial \mathcal{E}}\right)_T = (a + \frac{b}{T})N + 3cN\mathcal{E}^2$$

$$\left(\frac{\partial T}{\partial \mathcal{E}}\right)_{\mathcal{P}} = \frac{aT^2 + bT + 3cT^2\mathcal{E}^2}{b\mathcal{E} - dT^2}$$

In these expressions a, b, c and d are constants and N is the number of molecules. One also knows that $\mathcal{P} = \mathcal{P}_0$ when $T = T_0$ and $\mathcal{E} = 0$. Find an analytic expression for the electric dipole moment $\mathcal{P}$.

Problem 3 (40 points) Molecular Solid



In a particular molecular solid the individual molecules are localized at specific lattice sites and possess no center of mass motion. However, each of the N molecules is free to rotate about a fixed direction in space which we will designate as the z direction. As far as the rotational motion is concerned the molecules can be considered to be non-interacting. The classical microscopic state of each molecule is specified by a rotation angle $0 \leq \theta < 2\pi$ and a canonically conjugate angular momentum $-\infty < l < \infty$ about the z axis. The energy of a single molecule is independent of θ and depends quadratically on l. Thus the Hamiltonian for the system is given by

$$\mathcal{H} = \sum_{i=1}^{N} \frac{l_i^2}{2I}$$

where I is the moment of inertia of a molecule about the z axis.

a) Represent the system by a microcanonical ensemble where the energy lies between E and $E + \Delta$. Find an expression for the phase space volume Ω. Use Sterling's approximation to simplify your result. [It may be helpful to consult the attached information sheet.]

b) Based on your calculations in a) find the probability density $p(\theta)$ for the orientation angle of a single molecule and explain your method.

c) The probability density $p(l)$ for the angular momentum of a single molecule can be written in the form $p(l) = \Omega'/\Omega$ where $\Omega = \Omega(E, N)$ is the quantity you found in a). Find Ω'. Do not try to simplify your answer. Do explain how to eliminate E from your expression for $p(l)$.

d) Find the energy of the system as a function of temperature, $E(T, N)$.

PARTIAL DERIVATIVE RELATIONSHIPS

Let x, y, z be quantities satisfying a functional relation $f(x,y,z) = 0$. Let w be a function of any two of x, y, z. Then

$$\left(\frac{\partial x}{\partial y}\right)_w \left(\frac{\partial y}{\partial z}\right)_w = \left(\frac{\partial x}{\partial z}\right)_w$$

$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{1}{\left(\frac{\partial y}{\partial x}\right)_z}$$

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

COMBINATORIAL FACTS

There are $K!$ different orderings of K objects. The number of ways of choosing L objects from a set of K objects is

$$\frac{K!}{(K-L)!}$$

if the order in which they are chosen matters, and

$$\frac{K!}{L!(K-L)!}$$

if order does not matter.

STERLING'S APPROXIMATION

When $K \gg 1$

$$\ln K! \approx K \ln K - K \quad \text{or} \quad K! \approx (K/e)^K$$

DERIVATIVE OF A LOG

$$\frac{d}{dx} \ln u(x) = \frac{1}{u(x)} \frac{du(x)}{dx}$$

VOLUME OF AN α DIMENSIONAL SPHERE OF RADIUS R

$$\frac{\pi^{\alpha/2}}{(\alpha/2)!} R^\alpha$$

LIMITS

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

$$\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$$

$$\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$$

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$$

WORK IN SIMPLE SYSTEMS

System	Intensive quantity	Extensive quantity	Work
Hydrostatic system	P	V	$-PdV$
Wire	$\mathcal{F}$	L	$\mathcal{F}dL$
Surface	$\mathcal{S}$	A	$\mathcal{S}dA$
Reversible cell	E	Z	EdZ
Dielectric material	$\mathcal{E}$	$\mathcal{P}$	$\mathcal{E}d\mathcal{P}$
Magnetic material	H	M	HdM

INTEGRALS

$$\int e^{ax}\,dx = \frac{e^{ax}}{a}$$

$$\int x\,e^{ax}\,dx = \frac{e^{ax}}{a^2}(ax - 1)$$

$$\int x^2\,e^{ax}\,dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2)$$

$$\int \frac{dx}{1 + e^x} = \ln\left[\frac{e^x}{1 + e^x}\right]$$

DEFINITE INTEGRALS

For integer n and m

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx = \sqrt{\pi}$$

$$(2\pi\sigma^2)^{-1/2} \int_{-\infty}^\infty x^{2n} e^{-x^2/2\sigma^2}\,dx = 1 \cdot 3 \cdot 5 \cdots (2n - 1)\,\sigma^n$$

$$\int_0^\infty x\,e^{-x^2}\,dx = \frac{1}{2}$$

$$\int_0^1 x^m (1 - x)^n\,dx = \frac{n!m!}{(m + n + 1)!}$$

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