Review of Basic Concepts: Normal form

14.126 Game Theory
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Road Map

• Normal-form Games
• Dominance & Rationalizability
• Nash Equilibrium
  – Existence and continuity properties
• Bayesian Games
  – Normal-form/agent-normal-form representations
  – Bayesian Nash equilibrium—equivalence to Nash equilibrium, existence and continuity
Normal-form games

- A (normal form) game is a triplet $\langle N, S, u \rangle$:
  - $N = \{1, \ldots, n\}$ is a (finite) set of players.
  - $S = S_1 \times \ldots \times S_n$ where $S_i$ is the set of pure strategies of player $i$.
  - $u = (u_1,\ldots,u_n)$ where $u_i : S \to \mathbb{R}$ is player $i$'s vNM utility function.

- A normal form game is finite if $S$ and $N$ are finite.
- The game is common knowledge.

Mixed Strategies, beliefs

- $\Delta(X) =$ Probability distributions on $X$.
- $\Delta(S_i) = $ Mixed strategies of player $i$.
- Independent strategy profile:
  $\sigma = \sigma_1 \times \ldots \times \sigma_n \in \Delta(S_1) \times \ldots \times \Delta(S_n)$
- Correlated strategy profile:
  $\sigma \in \Delta(S)$
- $\Delta(S_{-i}) = $ possible conjectures of player $i$ (beliefs about the other players’ strategies). $[\sigma_{-i} \in \Delta(S_{-i})]$ 
  - A player may believe that the other players’ strategies are correlated!
- Expected payoffs:
  $u_i(\sigma) = E_\sigma(u_i) = \Sigma_{s \in S} \sigma(s) u_i(s)$
Rationality & Dominance

• Player $i$ is rational if he maximizes his expected payoff given his belief.
• $s_i^*$ is a best reply to a belief $\sigma_i$ iff
  $\forall s_i \in S_i: u_i(s_i^*, \sigma_i) \geq u_i(s_i, \sigma_i)$.
• $B_i(\sigma_i)$ = best replies to $\sigma_i$.
• $\sigma_i$ strictly dominates $s_i$ iff
  $\forall s_j \in S_j: u_i(\sigma_i, s_j) > u_i(s_i, s_j)$.
• $\sigma_i$ weakly dominates $s_i$ iff
  $\forall s_j \in S_j: u_i(\sigma_i, s_j) \geq u_i(s_i, s_j)$ with a strict inequality.

Theorem: In a finite game, $s_i^*$ is never a best reply to a (possibly correlated) conjecture $\sigma_i$ iff $s_i^*$ is strictly dominated (by a possibly mixed strategy).

Proof of Theorem

• Let
  - $S_i = \{s_i^1, \ldots, s_i^m\}$
  - $u_i(s_i^k) = (u_i(s_i^k, s_{-i}^1), \ldots, u_i(s_i^k, s_{-i}^m))$
  - $U = \{u_i(s_i) | s_i \in S_i\}$
  - $Co(U)$ = convex hull of $U = \{u_i(\sigma_i) | \sigma_i \in \Delta(S_i)\}$
  - $(\Rightarrow)$ Assume $s_i^* \in B_i(\sigma_i)$.
    $\Rightarrow \forall s_i, u_i(s_i^*, \sigma_i) \geq u_i(s_i, \sigma_i)$
    $\Rightarrow \forall \sigma_i, u_i(s_i^*, \sigma_i) \geq u_i(\sigma_i, \sigma_i)$
    $\Rightarrow$ No $\sigma_i$ strictly dominates $s_i^*$.

• SHT: Let $C$ and $D$ be non-empty, disjoint subsets of $\mathbb{R}^m$ with $C$ closed. Then, $\exists r \in \mathbb{R}^m(0): \forall x \in \text{cl}(D) \forall y \in C, \ r \cdot x \geq r \cdot y$.

• $(\Leftarrow)$ Define
  $D = \{x \in \mathbb{R}^m | x_k > u_i(s_i^*, s_{-i}^k) \ \forall k\}$.
  Assume $s_i^*$ is not strictly dominated.
  $\Rightarrow Co(U)$ and $D$ are disjoint.
  $\Rightarrow$ By SHT, $\exists r: \forall \sigma_i, u_i(s_i^*, \sigma_i) \geq u_i(\sigma_i, \sigma_i)$
  $\Rightarrow u_i(s_i^*, \sigma_i) \geq u_i(\sigma_i, \sigma_i)$
  where $\sigma_i(s_{-i}^k) = r^k/(r^1 + \ldots + r^m)$.
Iterated strict dominance & Rationalizability

• $S^0 = S$
• $S^k_i = B_i(\Delta(S^k_{-i}))$
• (Correlated) Rationalizable strategies:
  
  $S^\infty = \bigcap_{k=0}^\infty S^k_i$

• Independent rationalizability: $s_i \in S^k_i$ iff $s_i \in B_i(\prod_{j \neq i} \sigma_j)$ where $\sigma_j \in \Delta(S^k_{-i}) \forall j$.
• $\sigma_i$ is rationalizable iff $\sigma_i \in B_i(\Delta(S^\infty_{-i}))$.

Theorem (fixed-point definition): $S^\infty$ is the largest set $Z_1 \times \ldots \times Z_n$ s.t. $Z_i \subseteq B_i(\Delta(Z_{-i}))$ for each $i$. ($s_i$ is rationalizable iff $s_i \in Z_i$ for such $Z_1 \times \ldots \times Z_n$.)

Foundations of rationalizability

• If the game and rationality are common knowledge, then each player plays a rationalizable strategy.
• Each rationalizable strategy profile is the outcome of a situation in which the game and rationality are common knowledge.
• In any “adaptive” learning model the ratio of players who play a non-rationalizable strategy goes to zero as the system evolves.
Rationalizability in Cournot Duopoly

Simultaneously, each firm \( i \in \{1,2\} \) produces \( q_i \) units at marginal cost \( c \), and sells it at price \( P = \max\{0,1-q_1-q_2\} \).

\[
\begin{align*}
q_2 & \leq 1-c \\
1-c & \geq q_2 \\
\frac{1-c}{2} & \leq q_1 \\
\frac{1-c}{2} & \geq q_1 \\
1-c & \geq q_1 \\
1-c & \leq q_2 \\
\end{align*}
\]

As \( n \to \infty \), \( q^n \to (1-c)/3 \).
Nash Equilibrium

• The following are equivalent:
  – $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a Nash Equilibrium
  – $\forall i, \sigma_i^* \in B(\sigma_i^*)$, where $B_i$ contains mixed best replies
  – $\forall i, \forall s_i \in S_i : u(\sigma_i^*, \sigma_i^*) \geq u(s_i, \sigma_i^*)$,
  – $\forall i, \text{supp}(\sigma_i^*) \subseteq B(\sigma_i^*)$.

• Aumann & Brandenburger: In a 2-person game, if game, rationality, and conjectures are all mutually known, then the conjectures constitute a Nash equilibrium.

• For $n > 2$ players, we need common prior assumption and common knowledge of conjectures.

• Steady states of any adaptive learning process are Nash equilibria.

Existence and continuity

• For any correspondence $F : X \rightarrow 2^Y$, where $X$ compact and $Y$ bounded, $F$ is upper-hemicontinuous iff $F$ has closed graph:
  \[
  [x_m \rightarrow x & y_m \rightarrow y & y_m \in F(x_m)] \Rightarrow y \in F(x).
  \]

• Berge’s Maximum Theorem (existence and continuity of individual optimum): Assume $f : X \times Z \rightarrow Y$ is continuous and $X$, $Y$, $Z$ are compact. Let
  \[
  F(x) = \text{arg max}_{z \in Z} f(x, z).
  \]
  Then, $F$ is non-empty, compact-valued, and upper-hemicontinuous.

• Kakutani’s Fixed-point theorem: Let $X$ be a convex, compact subset of $\mathbb{R}^m$ and let $F : X \rightarrow 2^X$ be a non-empty, convex-valued correspondence with closed graph. Then, there exists $x \in X$ such that $x \in F(x)$. 
Existence of Nash Equilibrium

**Theorem:** Let each $S_i$ be a convex, compact subset of a Euclidean space and each $u_i$ be continuous in $s$ and quasi-concave in $s_i$. Then, there exists a Nash equilibrium $s \in S$.

**Corollary:** Each finite game has a (possibly mixed) Nash equilibrium $\sigma^*$.

**Proof of corollary:** Each $\Delta(S_i) \subseteq \mathbb{R}^m$ is convex and compact. Each $u_i(\sigma)$ is continuous, and linear in $\sigma_i$. Then, the game with strategy spaces $\Delta(S_i)$ has a NE $\sigma^* \in \Delta(S_1) \times \ldots \times \Delta(S_n)$.

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**Proof of Existence Theorem**

- Let $F : S \to 2^S$ be the “best reply” correspondence:
  \[ F_i(s) = B_i(s_{-i}) \]
- By the Maximum Theorem, $F$ is non-empty and has closed graph.
- By quasi-concavity, $F$ is convex valued.
- By Kakutani fixed-point theorem, $F$ has a fixed point: $s^* \in F(s^*)$.
- $s^*$ is a Nash equilibrium.
Upper-hemicontinuity of NE

- $X$, $S$ are compact metric spaces
- $u^x(s)$ is continuous in $x \in X$ and $s \in S$.
- $\text{NE}(x)$ is the set of Nash equilibria of $(N,S,u^x)$.
- $\text{PNE}(x)$ is pure Nash equilibria of $(N,S,u^x)$.

**Theorem**: NE and PNE are upper-hemicontinuous.

**Corollary**: If $S$ is finite, NE is non-empty, compact-valued, and upper-hemicontinuous.

**Proof**:
- $\Delta(S_i)$ is compact and $u^x(\sigma)$ is continuous in $(x,\sigma)$.
- Suppose: $x_m \to x$, $\sigma^m \in \text{NE}(x_m)$, $\sigma \not\in \text{NE}(x)$.
- $\exists i, s_i$: $u^x(s_i, \sigma_{-i}) > u^x(\sigma)$.
- $u^{x_m}(s_i, \sigma_{-i}^m) > u^{x_m}(\sigma^m)$ for large $m$.

Bayesian Games

- A **Bayesian game** is a list $(N, A, \Theta, T, u, p)$:
  - $N = \{1, \ldots, n\}$ is a (finite) set of players;
  - $A = A_1 \times \ldots \times A_n$; $A_i$ is the set of actions of $i$;
  - $\Theta$ is the set of payoff relevant parameters;
  - $T = T_1 \times \ldots \times T_n$; $T_i$ is the set of types of $i$;
  - $u = (u_1, \ldots, u_n)$; $u_i : \Theta \times A \to \mathbb{R}$ is $i$'s vNM utility function;
  - $p \in \Delta(\Theta \times T)$ is a common prior.
- A **Bayesian game** is a list $(N, A, \Theta, T, u, p)$ as above except $u_i : \Theta \times T \times A \to \mathbb{R}$.
- A **Bayesian game** is a list $(N, A, T, u, p)$ as above except $u_i : T \times A \to \mathbb{R}$.

**Fact**: All three formulations are equivalent (as long as you know what you are doing).

**Fact**: We can replace $p$ with $p_1, \ldots, p_n$, dropping CPA.
Normal-form representations

- Given a Bayesian game $\Gamma = (N, A, \Theta, T, u, p)$,
- Normal Form: $G(\Gamma) = (N, S, U)$:
  - $S_i = \{\text{functions } s_i: T_i \to A_i\}$
  - $U_i(s) = E_p[u_i(\theta, s_1(t_1), \ldots, s_n(t_n))]$.
- Agent-Normal Form: $AG(\Gamma) = (N, S, U)$:
  - $N = T_1 \cup \cdots \cup T_n$
  - $S_{t_i} = A_i$ for each $t_i \in T_i$
  - $U_{t_i}(s) = E_p[u_i(\theta, s_1(t_1), \ldots, s_n(t_n)) | t_i]$.

Bayesian Nash equilibrium

**Definition:** $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a Bayesian Nash Equilibrium iff for each $i$, $t_i$,

$$\sigma_i^*(a_i | t_i) > 0 \Rightarrow a_i \in \arg \max_{a_i} E_p[u_i(\theta, a_i, \sigma_{-i}^*(a_{-i} | t_{-i})) | t_i]$$

**Fact:** $\sigma^*$ is a Bayesian Nash equilibrium of $\Gamma$ iff the profile $\sigma^*(\cdot | t_i)$, $t_i \in T_i$, $i \in N$ is a Nash equilibrium of $AG(\Gamma)$.

**Fact:** If $\sigma^*$ is a Bayesian Nash equilibrium of $\Gamma$, then $\sigma^*$ is a Nash equilibrium of $G(\Gamma)$. If $p(t_i) > 0$ for each $t_i$, the converse is also true.
Existence of BNE

Consider $\Gamma=(N,A,\Theta,T,u,p)$ with finite $N$ and $T$.

**Theorem:** If

- each $A_i$ is compact and convex
- each $u_i$ is bounded, continuous in $a$, concave in $a_i$

then $\Gamma$ has a pure Bayesian Nash equilibrium.

**Proof:** $AG(\Gamma)$ has a pure Nash equilibrium.

**Corollary:** If $A$ is finite, $\Gamma$ has a (possibly mixed) Bayesian Nash equilibrium.

Upper-hemicontinuity of BNE

- $A$, $T$ finite and $\Theta$, $X$ compact.
- $u^x_i(\theta,a)$ continuous in $(x,\theta,a)$
- BNE$(x)$ Bayesian NE of $\Gamma^x=(N,A,\Theta,T,u^x,p)$.
- BNE$(p)$ Bayesian Nash equilibria of $(N,A,\Theta,T,u,p)$.

**Theorem:** BNE is upper-hemicontinuous.

**Proof:** BNE$(x) = NE(AG(\Gamma^x))$.

**Theorem:** Assume $p(t_i) > 0 \forall p \in P$, $\forall t_i \in T$, for compact $P \subseteq \Delta(\Theta \times T)$. BNE$(p)$ is upper-hemicontinuous on $P$.

**Proof:** $U_i(s;p) = E_p[u_i(\theta,s_1(t_1),...,s_n(t_n))]$ is continuous; BNE$(p) = NE(G((N,A,\Theta,T,u,p)))$. 