CONVOLUTION, ADDING GAMMA VARIABLES, AND CHI-SQUARES

This will be a review of some facts from probability that might or might not be familiar. If \( f \) and \( g \) are two real-valued functions of a real variable, having in mind probability density functions, then their convolution is defined by

\[
(f * g)(t) = \int_{-\infty}^{\infty} f(t-y)g(y)dy = \int_{-\infty}^{\infty} f(x)g(t-x)dx,
\]

where if either integral exists for a given \( t \), so does the other one with the same value, by the substitutions \( x = t - y \) or \( y = t - x \). Convolution gives us the density of the sum of two independent random variables having densities:

**Theorem 1.** If \( X \) and \( Y \) are independent random variables having densities \( f \) and \( g \) respectively, then \( X + Y \) has density \( f * g \).

**Proof.** By independence, \((X,Y)\) has bivariate density \( f(x)g(y) \). Thus for any \( t \),

\[
P(X + Y \leq t) = \int \int_{x+y\leq t} f(x)g(y) \, dy \, dx.
\]

Since \( x + y \leq t \) is equivalent to \( y \leq t - x \), we get

\[
\int_{-\infty}^{\infty} f(x) \int_{-\infty}^{t-x} g(y) \, dy \, dx.
\]

Making the substitution \( u = y + x \) in the inner integral for each fixed value of \( x \), so that \( du = dy \), we get

\[
\int_{-\infty}^{\infty} f(x) \int_{-\infty}^{t} g(u-x) \, du \, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(u-x) \, dx \, du
\]

where the integrals were interchanged (justifiably since \( f \geq 0 \) and \( g \geq 0 \)). To find the density of \( X + Y \) we just have to differentiate with respect to \( t \), which by the fundamental theorem of calculus gives \((f * g)(t)\) as stated, Q.E.D.

Next, convolution will be applied to gamma densities. Recall that for any \( \alpha > 0 \) and \( \lambda > 0 \) a \( \Gamma(\alpha, \lambda) \) density is given by \( f_{\alpha,\lambda}(x) = 0 \) for \( x \leq 0 \) and for \( x > 0 \) it equals \( \lambda^{\alpha}x^{\alpha-1}e^{-\lambda x}/\Gamma(\alpha) \) where \( \Gamma(\alpha) \) is defined as \( \int_{0}^{\infty} x^{\alpha-1}e^{-x} \, dx \). The next fact can be called the “Addition theorem” for gamma variables. Also recall that \( X \sim D \) means that \( X \) has distribution or density \( D \).

**Theorem 2.** If \( X \) and \( Y \) are independent, \( X \sim \Gamma(\alpha, \lambda) \) and \( Y \sim \Gamma(\beta, \lambda) \) (with the same \( \lambda \)), then \( X + Y \sim \Gamma(\alpha + \beta, \lambda) \).

**Proof.** Applying Theorem 1 with \( f \) the \( \Gamma(\alpha, \lambda) \) density and \( g \) the \( \Gamma(\beta, \lambda) \) density, noting that \( f(u) = 0 \) for \( u \leq 0 \) and \( g(y) = 0 \) for \( y \leq 0 \), we have \((f * g)(x) = 0 \) for \( x \leq 0 \) while for \( x > 0 \) we have

\[
(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy = \int_{0}^{x} f(x-y)g(y)dy
\]
because for the integrand to be non-zero we need \( y > 0 \) and \( x - y > 0 \) so \( y < x \). Then plugging in the definitions of the gamma densities, we have for the constant \( c = \lambda^{\alpha+\beta}/(\Gamma(\alpha)\Gamma(\beta)) \),

\[
(f \ast g)(x) = c \int_0^x (x - y)^{\alpha-1}e^{-\lambda(x-y)}y^{\beta-1}e^{-\lambda y} dy = ce^{-\lambda x} \int_0^x (x - y)^{\alpha-1}y^{\beta-1} dy.
\]

Making the substitution \( u = y/x \), noting that \( y \) and \( u \) are variables of integration and that integrals are evaluated for each fixed value of \( x > 0 \), we get

\[
(f \ast g)(x) = ce^{-\lambda x} x^{\alpha+\beta-1} \int_0^1 u^{\beta-1}(1 - u)^{\alpha-1} du.
\]

Now, \( B(\beta,\alpha) \), defined as \( \int_0^1 u^{\beta-1}(1 - u)^{\alpha-1} du \), and called the beta function of \( \beta \) and \( \alpha \), doesn’t depend on \( x \). So, since \( f \ast g \) is a probability density by Theorem 1, with \( \int_0^\infty (f \ast g)(x)dx = 1 \), it must be the \( \Gamma(\alpha + \beta, \lambda) \) density, as stated, Q.E.D.

Moreover, matching up the constants at the end of the last proof, we can express the beta function in terms of gamma functions:

\[
B(\beta,\alpha) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta).
\]

From this it follows that \( B(\alpha,\beta) \equiv B(\beta,\alpha) \). A family of probability densities called beta densities on the interval \((0,1)\) is defined by \( b_{\alpha,\beta}(x) = x^{\alpha-1}(1 - x)^{\beta-1}/B(\alpha,\beta) \) for \( 0 < x < 1 \) and \( b_{\alpha,\beta}(x) = 0 \) otherwise for any \( \alpha \) and \( \beta \) such that \( 0 < \alpha < \infty \) and \( 0 < \beta < \infty \).

For any positive integer \( k \), a \( \chi_k^2 \) variable, or a chi-squared variable with \( k \) degrees of freedom, is defined as a variable given by

\[
\chi_k^2 = Z_1^2 + \cdots + Z_k^2
\]

where \( Z_1,\ldots,Z_k \) are i.i.d. \( N(0,1) \) variables. This is the definition given in Rice, section 6.2, first for \( k = 1 \) and then for general \( k = 2,3,\ldots \). As Rice mentions, a \( \chi_k^2 \) variable has a \( \Gamma(k/2,1/2) \) distribution. To show this, for any \( x > 0 \), \( P(Z_1^2 \leq x) = P(|Z_1| \leq \sqrt{x}) \) and by the fundamental theorem of calculus and the chain rule we see that \( Z_1^2 \) does have a \( \Gamma(1/2,1/2) \) distribution. Then, the fact for all \( k = 2,3,\ldots, \), follows by induction, applying the addition theorem for gamma variables (Theorem 2) at each step. We have \( \lambda = 1/2 \) for all these distributions.