18.785: Analytic Number Theory, MIT, spring 2007 (K.S. Kedlaya)
Error bounds in the prime number theorem

In this unit, we introduce (without proof for now) a formula which relates the distribution of primes to the zeroes of the Riemann zeta function. Given a suitable zero-free region for $\zeta(s)$ in the critical strip, this can be used to prove the prime number theorem with an estimate for the error term.

1 Zeta zeroes and prime numbers

For $x \notin \mathbb{N}$, define the counting function

$$\psi(x) = \sum_{n \leq x} \Lambda(n),$$

where $\Lambda : \mathbb{N} \to \mathbb{R}$ is the von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & n = p^a, a \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $x \in \mathbb{N}$, it is convenient to modify the definition to

$$\psi(x) = \sum_{n < x} \Lambda(n) + \frac{1}{2}\Lambda(x).$$

Note that for the function ϑ we defined earlier as

$$\vartheta(x) = \sum_{p \leq x} \log p,$$

we have

$$\psi(x) - \vartheta(x) = O(x^{1/2} \log x) \qquad (x \to \infty)$$

so the prime number theorem is equivalent to

$$\psi(x) \sim x \qquad (x \to \infty).$$

The formula of von Mangoldt expresses the difference $\psi(x) - x$ in terms of the zeroes of $\zeta(s)$. We will prove this formula in a later unit.

Theorem 1 (von Mangoldt's formula)**.** *For $x \geq 2$ and $T > 0$,*

$$\psi(x) - x = -\sum_{\rho : |\operatorname{Im}(\rho)| < T} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - x^{-2}) + R(x, T)$$

with ρ running over the zeroes of $\zeta(s)$ in the region $\operatorname{Re}(s) \in [0, 1]$, and

$$R(x, T) = O\left(\frac{x \log^2(xT)}{T} + (\log x) \min\left\{1, \frac{x}{T\langle x \rangle}\right\}\right).$$

Here $\langle x \rangle$ denotes the distance from x to the nearest prime power other than possibly x itself.

The region $\mathrm{Re}(s) \in [0,1]$ is called the *critical strip* for ζ, because we can account for all of the zeroes outside this strip: they are the trivial zeroes $s = -2, -4, \dots$ forced by the functional equation and the fact that $\Gamma(s/2)$ has poles at nonpositive even integers. In fact, the last term in the formula is merely $-\sum_\rho \frac{x^\rho}{\rho}$ for ρ running over the trivial zeroes.

Incidentally, one can check by a numerical calculation that there are no real zeroes of ζ in the critical strip, by numerically approximating the integral representation of $\xi(s)$. This raises an interesting point: in general, direct numerical approximation can be used to prove that an analytic function does not vanish in a region, but not that it does vanish at a particular point. The best one can do is use a zero-counting formula to prove that there must be a zero near the proposed vanishing point.

Note that for x fixed, $R(x,T) = o(1)$ as $T \to \infty$, so we have

$$\psi(x) - x = -\sum_\rho \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - x^{-2})$$

as long as we interpret the sum over ρ to mean the limit of the partial sums over $|\mathrm{Im}(\rho)| < T$ as $T \to \infty$. This formula, while pretty, is not as useful in practice as the form with remainder; we will use the remainder form by taking T to be some (preferably large) function of x as $x \to \infty$.

2 How to use von Mangoldt's formula

In order to use von Mangoldt's formula to bound $\psi(x) - x$, we need to give an upper bound on the sum $\sum_\rho x^\rho/\rho$ for ρ running over nontrivial zeroes of ζ in the region $|\mathrm{Im}(s)| \leq T$.

Put $\beta = \mathrm{Re}(\rho), \gamma = \mathrm{Im}(\rho)$. Suppose we can prove that $\beta < 1 - f(|\gamma|)$ for some nonincreasing function $f : [0,\infty) \to (0, 1/2)$; then

$$|x^\rho| = x^\beta < x^{1-f(|\gamma|)} < x^{1-f(T)}$$

and $|\rho| \geq |\gamma|$. We thus have

$$\left| \sum_{\rho:|\gamma|<T} \frac{x^\rho}{\rho} \right| \leq x^{1-f(T)} \sum_{\rho:|\gamma|<T} \frac{1}{\gamma}.$$

Let $N(T)$ be the number of zeroes in the critical strip with $|\gamma| \leq T$. Then

$$\sum_{\rho:0<|\gamma|<T} \frac{1}{\gamma} = \int_0^T t^{-1} dN(t) = \frac{N(T)}{T} + \int_0^T t^{-2} N(t)\, dt.$$

At this point we need some information about $N(T)$; again, we will prove this (and a bit more) later.

Theorem 2 (Hadamard)**.** *We have $N(T) = O(T \log T)$ as $T \to \infty$.*

This implies that

$$\left| \sum_{\rho:|\gamma|<T} \frac{1}{\gamma} \right| = O(\log^2 T),$$

so

$$\left| \sum_{\rho:|\gamma|<T} \frac{x^\rho}{\rho} \right| = O(x^{1-f(T)} \log^2 T).$$

For x an integer, we now take $T = T(x)$ to be a suitable function of x, and invoke von Mangoldt's formula with remainder to deduce that

$$\psi(x) - x = O\left(x^{1-f(T)} \log^2 T(x) + \frac{x \log^2 x}{T(x)} + \frac{x \log^2 T(x)}{T(x)} \right). \tag{1}$$

3 The Riemann Hypothesis

Riemann calculated a few of the zeroes of ζ and, based on this evidence, made the following remarkable conjecture (whose resolution is worth $1,000,000 from the Clay Mathematics Institute).

Conjecture 3 (Riemann Hypothesis)**.** *The nontrivial zeroes of* ζ *all lie on the line* $\mathrm{Re}(s) = \frac{1}{2}$.

This is a best-case scenario in terms of deducing error bounds on $\psi(x) - x$. Namely, suppose every nontrivial zero ρ of ζ satisfies $c \leq \mathrm{Re}(\rho) \leq 1 - c$ for some $c \in (0, 1/2)$; then we can take $f(T) = c$ in (1), yielding

$$\psi(x) - x = O\left(x^{1-c} \log^2 T(x) + \frac{x \log^2 x}{T(x)} + \frac{x \log^2 T(x)}{T(x)} \right).$$

By taking $T(x) = x$, we obtain

$$\psi(x) - x = O(x^{1-c} \log^2 x).$$

If I can take c to be any value less than 1/2, that means

$$\psi(x) - x = O(x^{1/2+\epsilon}) \qquad (\epsilon > 0),$$

and similarly one gets a strong estimate on $\pi(x)$ (see exercises).

Unfortunately, for *no* value of $c > 0$ are we able at present to prove that every nontrivial zero ρ satisfies $\mathrm{Re}(\rho) \leq 1 - c$. We will give a much smaller zero-free region in a later unit.

4 Variants for L-functions

For χ a Dirichlet character, define

$$\psi(x, \chi) = \sum_{n \leq x} \chi(n)\Lambda(n),$$

where again we multiply the $n = x$ term by $1/2$ if it is present.

Theorem 4. *For χ a nonprincipal Dirichlet character of level N,*

$$\psi(x, \chi) = -\sum_{\rho:|\gamma|<T} \frac{x^\rho}{\rho} - (1-a)\log x - b(\chi) + \sum_{m=1}^{\infty} \frac{x^{a-2m}}{2m-a} + R(x, T),$$

where $b(\chi)$ is an explicit constant, $a = 1$ for χ even and $a = 0$ for χ odd, and

$$R(x, T) = O\left(\frac{x \log^2(NxT)}{T} + (\log x) \min\left\{1, \frac{x}{T\langle x\rangle}\right\}\right).$$

For a fixed N, one can use this formula together with a zero-free region for all of the $L(s, \chi)$ with χ of level N, to obtain a prime number theorem for arithmetic progressions of difference N with an estimate for the error term.

However, one would also like to be able to establish a prime number theorem with error term for arithmetic progressions where the difference is allowed to vary. In this case, one of course must have a zero-free region for all of the relevant characters. But there are two extra complications.

- One must understand how the constant $b(\chi)$ varies with χ.
- One must deal with possible roots of $L(s, \chi)$ that are very close to $s = 0$ or $s = 1$ (so-called *Siegel zeroes*).

Dealing with these goes beyond the level of detail I have in mind for this course; see Davenport §14–22 for a systematic exposition.

Exercises

1. Assume that $\psi(x) = x + o(x^{1-\epsilon})$ for some given $\epsilon \in (0, 1/2)$. Deduce a corresponding upper bound for $\pi(x) - \text{li}(x)$, where $\text{li}(x)$ is the logarithmic integral function

$$\text{li}(x) = \int_2^x \frac{dt}{\log t}.$$

Then deduce that

$$\pi(x) - \frac{x}{\log x} \neq o(x^{1-\delta})$$

for any $\delta > 0$. (This last statement can be proved unconditionally, but don't worry about that for now.) This is the sense in which $\text{li}(x)$ is a better approximation than $x/(\log x)$ of the count of primes.