Problem 2 – Solution

(a) The 4 equations are those we wrote in recitation, just expressed in terms of $z$. That is

1. Conservation of momentum: \[ \frac{\partial \sigma_{zz}^{tot}}{\partial z} = 0 \]

2. Darcy’s law: \[ U_z = -k \frac{\partial p}{\partial z} \]

3. Constitutive law: \[ \sigma_{zz}^{tot} = (2G + \lambda) \varepsilon_{zz} - p \]
   where \( H = 2G + \lambda \)

4. Mass conservation: \[ U_z = -\frac{\partial u_z}{\partial t} + U_{z0} \]
   where \( U_{z0} \) is the area-averaged velocity at a point where the solid is not moving, \( = U_0 \)

(b) Combining (1)-(4) we obtain, for the general case:

\[ \frac{\partial \sigma_{zz}^{tot}}{\partial z} = 0 = H \frac{\partial \varepsilon_{zz}}{\partial z} - \frac{\partial p}{\partial z} \]

And \[ H \frac{\partial \varepsilon_{zz}}{\partial z} = H \frac{\partial^2 u_z}{\partial z^2} \] [from definition of strain]

\[ \frac{\partial p}{\partial z} = \frac{U_z}{k} \]

\[ = \frac{1}{k} \left( \frac{\partial u_z}{\partial t} - U_{z0} \right) \] [from (4)]

Or \[ \frac{\partial u_z}{\partial t} - U_{z0} = H k \frac{\partial^2 u_z}{\partial z^2} \]

Note that the term \( U_{z0} \) is only \( = 0 \) in cases for which there is one non-porous boundary (as in the handout from class) but not in general.

- This is where we went wrong in recitation!

(c) Now we can assume steadiness \( (\partial / \partial t = 0) \) and obtain

\[ -U_{z0} = -U_0 = H k \frac{\partial^2 u_z}{\partial z^2} \] (1)

Or \[ -\frac{U_0}{H k} = \frac{\partial^2 u_z}{\partial z^2} \] (2)

Multiply by \( dz \) and integrate once:

\[ -\frac{U_0}{H k} z + c_1 = \frac{\partial u_z}{\partial z} \] (3)

And integrate again:

\[ -\frac{U_0}{H k} \frac{z^2}{2} + c_1 z + c_2 = u_z \] (4)
Boundary conditions are that:

\[ u_z(z = 0) = 0 \quad \Rightarrow \quad c_2 = 0 \]

And

\[ \sigma_{zz}^{\text{tot}}(z = -L) = -p_0 \quad \Rightarrow \quad \varepsilon_{zz} = \frac{du_z}{dz} \bigg|_{z=-L} = 0 \]

Using (3):

\[ - \frac{U_0}{Hk} (-L) + c_1 = 0 \]

\[ c_1 = - \frac{U_0 L}{Hk} \]

So

\[ u_z(z) = - \frac{U_0}{Hk} \left( \frac{z^2}{2} + Lz \right) \]

This solution satisfies the boundary conditions we discussed in recitation, and has the form

\[ \frac{\partial u_1}{\partial t} = Hk \frac{\partial^2 u_1}{\partial x_1^2} \]

rather than the more general expression in (b) above!!!