**HERBERT GROSS:** Hi. Standing here waiting for today's lesson to begin, I was thinking of a story that came to mind. And that was the story of the foreman yelling down into an excavation, how many of you men down there? And the reply came back, three. And he said, OK, half of you come on up.

Now with that funny story, I would like to launch today's lesson, which could be, I guess, sub called, or whatever word that you'd like to pick, realness is in the eyes of the beholder, the question being, of course, that when you talk about taking one half of three people, ah-- well, let's put it this way. I was going to say, you can't do it. Let's say, if you could do it, you wouldn't like to see the answer.

On the other hand, to take three inches and say, let's divide that into two equal parts, there's a case where the answer does happen to make sense. Now today, you see we're going to talk about a new phase of our course called the complex numbers. The complex numbers happen to be a delightful topic, from the point of view that on the one hand, they offer a great deal of enrichment in pure mathematics, and on the other hand, they contribute a great deal to our physical understanding of reality.

Now to launch into this, let's get right into the topic. I call today's lesson, The Complex Numbers. And going back to my opening hilarious story, what we're saying is if the only numbers that we knew were the integers and we were given the equation 2x equals 3-- and notice that in terms of integers, the equation makes sense, because 2 and 3 happen to be integers-- we're looking for to solve this equation. And the question is, does this equation have a solution? The answer is it only has a solution provided that you want to invent numbers which are not integers.

In other words, this does not have a solution if we insist that the answer be an integer. But if we're willing to invent a new batch of numbers called the fractions or the rational numbers, then this equation will have a solution. Do we want this to have solutions? Well, sometimes there are meaningful situations in which it's meaningful to solve this equation, other times when it isn't. Again, the story of going into the bank and into the post office and asking for \$0.03 worth of \$0.02 stamps. And we can think of all sorts of hilarious ways of embellishing this story.

Going on further, though, let's take what ultimately became known as the "Real Numbers." And I'm going to put this in quotation marks. Because even though these are genuinely called the Real Numbers, they are no more real than the integers were real. By definition, a Real Number is simply any number whose square is non negative.

Assuming that to be the definition of real numbers, we come to the equation x squared equals minus 1. We say, does this equation have a solution? And the answer is if we insist that the solution has to be a real number, the answer, obviously, is that this does not have a solution. Because the definition of a real number is that it's square cannot be negative, and this says that x squared is minus 1. So either we must say that this equation does not have solutions. Or if we want it to have solutions, we must invent an extension of the number system. We must extend the real numbers in the same way that if we wanted this equation to have a solution, we had to extend the integers.

The extension of the integers that was necessary to solve this equation happened to be called the Rational Numbers. Let's talk about, first of all, whether we want an extension of the real numbers, and secondly, if we do, what shall we call it, and what properties shall it have? To motivate why we would like the equation, x squared plus 1 equals 0 to have solutions. And that, of course, is just equivalent to x squared equals minus 1. Let's go back to an example that we used when we first introduced exponents in part 1 of our course.

We observed that one of the properties of an equation like y equals e to the rx where r is a constant is that if we differentiate this with respect to x, the r comes down. But the fact, the e to the rx remains. What this told us was that in taking derivatives, we would get powers of r. But e to the rx would remain. And we use that technique for solving certain types of second, well, differential equations.

For example, looking at the equation y double prime plus y equals 0, one technique would be to replace y by e to the rx. When I differentiate twice, I have an r squared e to the rx here. Here is an e to the rx. The e to the rx cancels out. And I wind up with a polynomial equation for r. And we then solve for the values of r, such that this would be a solution of this.

Now back in part 1, because we didn't know non-real numbers. We always fixed it up so that the resulting equation that we got here would have real numbers for values of r. By the way, I have chosen this particular equation for two reasons. One is, the values of r can't be real when I get through here. The other is, I already know an answer to this problem. Remember that if you differentiate sine twice, you get minus sine. If you differentiate cosine twice, you get minus cosine.

Consequently, the second derivative of sine x plus sine x is 0. Second derivative of cosine x plus cosine x is 0. So I know at least two solutions to this equation, two real solutions to this equation, this equation has physical meaning because it essentially says, if you think of the parameter as being time here, look at the second derivative is equal to minus the function itself. That says that the acceleration is equal to minus the displacement. And that's a form of simple harmonic motion.

You see, what I'm trying to establish here is that this does have a real solution. And the problem itself has real physical significance. At any rate, if we now try to solve this problem using this technique, and remember, again, I can give you problems where we won't know the answer intuitively, and the values of r will still come out to be imaginary. I simply chose this problem so that we could correlate a so-called imaginary answer with a real answer.

The idea, again, is I differentiate each of the rx twice. I get this. So y double prime plus y is this. I factor out the e to the rx, which can't be 0. Therefore, it must be that r squared plus 1 is 0, or r is equal to plus or minus the square root of minus 1.

Now you see, so far, I don't know what this thing means. Because there certainly is no real number, which has this property. Because the square of real numbers have to be positive. They can't be minus 1. Let me, for the sake of argument, invent a number, which is equal to the square root of minus 1. I'll call that number i. So r is equal to plus or minus i.

And if I now remember what equation I'm solving-- it was y equals e to the rx-- I can replace r by either i or minus i. So the two solutions I would get would be y equals e to the ix or y equals e to the minus x. But this is bothering me. Because I really have no real feeling for what I means right now.

What I do know, however, is this, that mechanically, this is a solution to a real problem. y double prime plus y equals zero is a real problem, not an imaginary one. And by the way, notice mechanically here, taking i to be a constant, if we still assume that the derivative of e to the x is re to the rx, even when r happens to be non-real, notice that if I differentiate this thing twice, the first time I differentiate, I bring down an i, the second time I differentiate, I bring down an i. That gives me i squared, which is minus 1. This is going to be minus e to the ix. And if I add on e to the ix, I really do get 0.

Notice also, now, that I do know that a real solution of this equation is y equals cosine x, or also y equals sine x. So that somehow or other, I get the feeling that I should exist, and more to the point, that this expression e to the ix should somehow be related to sign x and cosine x. In other words, if I had never been lucky enough to invent the trigonometric functions, and I want to solve the problem, y double prime plus y equals 0 and wound up with this, then my feeling is, I must find a real interpretation, a physically real interpretation for each of the ix. Because I sense that my problem has a solution.

Again, I can take the coward's way out. I can say, if it's going to be this much work, I don't want the problem to have a solution. But once I want the problem to have a solution, I must extend the number system. And the upside of that whole thing is that we now invent something called the Complex Number System.

First of all, the complex numbers are defined to be the set of all symbols-- let's just call them Symbols for the time being-- all symbols of the form x plus iy, where x and y are real numbers. And just to refresh our memories here, i is the symbol which numerically represents the square root of minus 1, OK? Notice that the X and the y are real numbers, though, where by Real, we mean what? The squares are non negative.

Now you will recall, that in any mathematical system, the thing that you're dealing with is more than a set of numbers, it's a set of numbers or a set of objects together with a structure, certain rules that tell us how to work with these elements that make up the set. Remember, the difference between a set and a system is a set that's just a collection of objects. The system is the collection of objects together with how we combine these objects to form similar objects, et cetera.

And what is the structure in this case? Given the two complex numbers, x1 plus iy1 and x2 plus iy2, we define equality to be sort of component by component. The real parts x1 and x2 must be equal. And the imaginary parts-- and by the way, just a word of caution here, the imaginary part of a complex number is, by definition, the coefficient of i. Notice that the imaginary part is itself a real number. The coefficient of i is the real number y1. What we're saying is we define the definition of equality for complex numbers to be that two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

Secondly, we agree to add two complex numbers, component by component, so to speak. In other words, to add two complex numbers, we add the two real parts, and we add the two imaginary parts, OK? That's all this says.

And thirdly, to multiply a complex number by a real number, we multiply the real part by the given real number and the imaginary part by the given real number, which is what this third rule says. And all I want to see from this is that by the very definition of these three structural properties, we have made the complex numbers a two-dimensional vector space. In other words, it means that we can now visualize the complex numbers, geometrically, in the same way that we could visualize the real numbers geometrically.

Remember, geometrically-- when we define the real numbers to be those numbers whose squares are non negatives, we do not need any picture to visualize that. It just happens that the number line, the x-axis, is a very convenient way of visualizing real numbers pictorially. What we're saying is that since complex numbers seem to indicate a two-dimensional vector space, namely, real and imaginary parts which are independent, it would appear that the analog for visualizing complex numbers pictorially would be to use the plane. And that's exactly what we do.

The diagram is called the Argand Diagram. The idea is this. Given the complex number x plus iy, which we'll call z, we visualize z as either the point x comma y in the xy plane-- in other words, notice that the x-axis is the real axis, meaning it denotes the real part of the complex number. The y-axis is called the imaginary axis, because it denotes the coefficient of i, the imaginary part, OK? And what we're saying is we can visualize the complex number x plus iy to be the point x comma y. Or for that matter, the vector that goes from the origin to the point z, whose components are x and y, meaning what? x plus iy. The components are the real and the imaginary parts here.

We can translate this into polar coordinates, meaning that we can measure the point z by its r value and its theta value. But one important thing to remember is that in polar coordinates, it's always assumed that r is positive. Unlike the usual polar coordinates, where r could be either positive or negative, the idea is that we would like to identify the absolute value of a complex number with being its distance from the origin. Since we want absolute value to be non-negative, we simply say that r is the positive square root of x squared plus y squared. And that names the magnitude of the complex number.

The angle theta is called the Argument, abbreviated Arg, not to be confused with this funny coincidence of Argand diagram. This comes from the word Argument, argument. The angle is called the Argument, all right?

And if you elect to write the complex number x plus iy as r comma theta, because of the connection between polar and Cartesian coordinates, r-- see, x is what? x is r cosine theta. y is r sine theta. So the complex number is r cosine theta plus ir sine theta. That's what r comma theta denotes.

Now using this as background-- so we now have what? We've invented, at least abstractly, a system of numbers called the Complex Numbers. And by the way, let me point out here that don't use the word Imaginary too strongly. There is certainly nothing imaginary about this geometric interpretation. What may be imaginary is I may not have a place where I want to use these kind of numbers. But believe me, we wouldn't have made a whole block of material about these numbers if there weren't real interpretations of the so-called complex numbers.

This is a perfectly real geometric interpretation. So far, we have established the fact that this interpretation gives us a vector space. But the complex numbers have an additional structure as well, namely, if I saw the two complex numbers, a plus bi times-- and c plus di, and I multiplied them-- remember now, a, b, c, and d are real-- I would like to believe that the rules of ordinary algebra still apply here. To multiply this, I would want this times this, this times this, this times this, and this times this.

Notice that a times c as ac. bi times di should be bd times i squared, if the ordinary rules of arithmetic are to apply. Since i squared is minus 1, this term becomes minus bd. Since a, c, b, and d are real numbers, this is a real number, right? Correspondingly, the coefficient of-- the multiplier of i, the coefficient of i is what? bc plus ad, which is also a real number. So if we make up this definition of multiplication for complex numbers, we have made sure that the rules of arithmetic for complex numbers parallel the rules for real numbers. And by the way, it's very important here to make the aside that this is a crucial result.

Notice that among the complex numbers, the real numbers are included, namely, thinking of it geometrically, if you look at the plane, the real numbers are those points in the plane which lie along the x-axis. It's the same as saying that when you extended the integers to form the rational numbers, the rational numbers included the integers.

In other words, if I look at the number 3, I can think of it as the integer 3. I could also think of it as the ratio 3 divided by 1, you see? And therefore, I would like whatever rules I have for the complex numbers to remain valid if the complex numbers happen to be real. It would be terrible to have two different rules for multiplication, one for real numbers and one for complex numbers, and then given two real numbers, when I multiply them, I get one answer if I think of them as being real, and another answer if I think of them as being complex.

At any rate, the important point is with this as motivation, I have a way of defining the product of two complex numbers to be a complex number. I don't mean that these things are equal to r, so they simply stand for Real. This is real plus i times a real, which is a complex number.

A very special case occurs when you take the complex number a plus bi and multiply it by a minus bi, the same number only changing the plus to a minus. Notice that we would get what? The sum and difference here gives us the sum of difference of two squares. It's going to be what? a squared minus b squared i squared. But since i squared is minus 1, minus b squared i squared is minus b squared times minus 1, which is plus b squared. And notice, therefore, if you multiply a plus bi, which is a complex number, by this other complex number, a minus bi, you get a squared plus b squared, which is a non-negative real number.

In particular, if you think back to the geometric interpretation of this, the point a plus bi in the plane is a comma b. And the distance of a comma b from the origin is the square root of a squared plus b squared. So this product is actually the square of the magnitude, the absolute value of a plus bi.

This little gizmo is so important that it's given a special name. By definition, given any complex number z, written in the form x plus iy, the complex conjugate of z, called z bar-- not to be confused with a z with a bar underneath that we were using to denote vectors-- but the complex conjugate z bar is what you get by just changing the plus sign here to a minus.

In other words, the complex conjugate of x plus iy is x minus iy. And geometrically, all this means is, you see, when you change the sign of the imaginary part, remember, the imaginary part is the y-axis. You're just reflecting the number symmetrically with respect to the x-axis. You're leaving the x-coordinate alone. And you're changing the y-coordinate to minus y.

In polar coordinates, what you're saying is a complex number and its conjugate are the same distance from the origin. But in one case, the angle is theta. And in the other case, the angle is minus theta. One of the many applications of complex conjugates-- I'll give you some of these other applications in the exercises. But for now, what I think is important is a simple one that essentially shows us how we find the quotient of two complex numbers.

Quite simply, sparing you the details, if I have c plus di divided by a plus bi, I simply multiply numerator and denominator by the complex conjugate of the denominator, OK? What will that do for me? Well, when I multiply the two factors in the numerator, I'm going to get a real part, namely, ac plus bd. In other words, it's minus bd i squared. But i squared is minus 1.

So I'm going to get ac plus bd. I'm going to get an imaginary part, which is ad minus bc. But the denominator will just be a squared plus b squared, which is a real number. In other words, what this tells me is I can now write the quotient of two complex numbers in the form what-- a real number plus i times a real number. In fact, the only time I can't do this-- I have to be careful. a or b must be unequal to 0. Because if both and b are 0, this is a 0 denominator, which I don't allow myself to divide by.

And the only way both a and b can be 0 is if the complex number is 0 plus 0i, which is the number 0. So I'm saying, again, just like in the real case, I still can't divide by 0.

Anyway, to give you a more practical illustration to see this with numbers-- not practical, but, at least, concrete-- 3 plus 2i divided by 4 plus i, I simply multiply numerator and denominator by 4 minus i. You see, what that does is downstairs, I get 4 squared, which is 16, minus i squared, which is plus 1, 16 plus 1 is 17. The real part is going to be what? Here's 12. 2i times minus i is plus 2 is 14. 8i minus 3i is 5i. So this quotient, 3 plus 2i times 4 plus i, is 14/17 plus 5/17 i. And by the way, as a trivial check, and you can do this if you wish for an exercise, simply take the answer that we got here, multiply it by the denominator, 4 plus i here, and actually check that that product comes out to be 3 plus 2i.

At any rate, this use of complex conjugates shows us that when you divide two complex numbers, the result will be a complex number, except for division by 0. You see, that's what went wrong with our integer problem, 2x equals 3. The reason that we couldn't solve it is that to solve 2x equals 3 required that the process of taking the quotient of two integers, and the quotient of two integers did not have to be an integer. Notice on the other hand, that the quotient of two real numbers is a real number, as long as the denominator is not 0. And the quotient of two complex numbers is a complex number, except for division by 0.

While we're talking about multiplication, a very enlightening thing happens if we think of multiplication in terms of polar coordinates, namely, take the two numbers, which in polar coordinates, are r1 comma theta 1 and r2 comma theta 2. Now again, remember from our Argand diagram what it means to say that the complex number has polar coordinates r1 comma theta 1. Remember, the complex number is written in the form x plus iy. The x-coordinate is r1 cosine theta 1. The y-coordinate is r1 sine theta 1. At any rate, this product, in terms of the standard notation, the x plus iy form, says we're multiplying these two numbers together.

And using our rule for multiplication, multiplying term by term, and remembering that i squared is minus 1, observe that if we collect the terms here-- and I'll leave the details for you to verify at your leisure-- we wind up with r1 r2 being a common factor. The real part is r1 r2 times cosine theta 1 cosine theta 2 minus sine theta 1 sine theta 2. The minus coming, because i squared is minus 1. The imaginary part is r1 r2 sine theta 1 cosine theta 2 sine theta 2 cosine theta 1.

Remember, theta is a real number. If we recall our geometric-- trigonometric definitions, cosine theta 1 cosine theta 2 minus sine theta 1 sine theta 2 is cosine theta 1 plus theta 2. This expression this sine theta 1 plus theta 2. And remembering now that in polar form, what we're saying is what? This is the complex number whose magnitude is r1 r2 and whose argument is theta 1 plus theta 2. And putting these two steps together, what it says is that to multiply two complex numbers, the same definition of multiplication that we were using before, if we interpret this in terms of polar coordinate, it says, look at, to multiply two complex numbers [INAUDIBLE] lengths, you simply do what? Multiply the two lengths to get the resulting length of the product and add the two arguments, the two angles.

That's a rather interesting result. You see, for example, if we now want to think of the complex numbers as being vectors, this gives us a third vector product that we had never talked about before and which I'll reinforce in the learning exercises. The idea is that using this as a model, and here's another real application, why not define a new product of two vectors obtained by multiplying the two lengths and adding the two angles? And the point was, that in our physical examples, there was really no motivation to invent this vector definition, in the same way that we could define the dot product and the cross product for complex numbers.

Because after all, they are viewed as vectors in the plane. But again, we have no great practical application for this. So we don't bother doing it. By the way, just for kicks here, I thought you might enjoy the aside that this little interesting result explains why the product of two negative numbers happen to be positive. Remember a negative number, if r has to-- see, a negative number lies on the real axis. And if r has to be positive, that means you're measuring in the direction of the negative x-axis, which means that the polar angle is 180 degrees. If you multiply two numbers whose angles are 180 degrees, if you add the two angles, you get what? 360 degrees.

You see what I'm driving at here? In other words, if I multiply two numbers in polar form, each of whose angles is 180, the product will have the angle equal to 360, you see? And that puts you back on the positive real axis. And so you have such real interpretations as y in terms of complex numbers, we can explain very nicely what it means for the product of two negatives to be a positive.

By the way, we can carry this result further if we had n factors written in polar form. Then to multiply these n factors, we would simply, by induction, so to speak, multiply the n magnitudes together and add the n angles. And a very interesting special case is if all of the factors happen to be equal-- in other words, if we want to raise the complex number written in polar form as r comma theta to the n-th power, a very interesting thing is that r comma theta to the n-th power is what? You multiply the magnitude. So the magnitude of the product will be r to the n. You add the angles. So the angle of the product, the argument will be n times theta.

A special case of the special case is if r equals 1. And if r equals 1, that says that the complex number whose polar form is 1 comma theta, when raised to the n-th power, is that complex number in polar form 1 comma n theta. And by the way, again, remember what this thing means? To say that the complex number is 1 comma theta means what? That the distance from the origin is 1 and the angle is theta. See, that was what we mean by 1 comma theta over here.

And notice that in Cartesian form, that makes this length cosine theta and this length sine theta. So i comma theta is cosine theta plus i sine theta. At any rate, translating both of these into Cartesian form, we wind up with a very famous result, called De Moivre's theorem. In my high school, it was called "De Moyvre's" theorem. But it's De Moivre's theorem. And it simply says that cosine theta plus i sine theta to the n is cosine n theta plus i sine n theta. That result may not seem that remarkable to you. But let me give you another example using this result that shows how we can get real results using imaginary numbers.

Let me take the special case n equals 2, just for the sake of argument. If I take n equals 2, this gives me cosine theta plus i sine theta squared equals cosine 2 theta plus i sine 2 theta. This says what? The real part of this is cosine squared theta minus sine squared theta. Again, it's i squared sine squared theta, which is minus sine squared theta. The imaginary part is 2 sine theta cosine theta.

So if I square this, I get cosine squared theta minus sine squared theta plus i2 sine theta cosine theta. By De Moivre's theorem, that equals cosine 2 theta plus i sine 2 theta. We saw that the only way that two complex numbers can be equal is if the real parts are equal and if the imaginary parts are equal. Comparing the real parts and comparing the imaginary parts, we get that sine 2 theta equals 2 sine theta cosine theta. Cosine 2 theta is cosine squared theta minus sine squared theta. Notice, by the way, that these are real results.

And also, notice, by the way, that even though these may look like old hat to you, I could just as easily, for example, I could have picked n 5 here or 6. And in fact, I will do that in the learning exercises. The point being that, what, I can raise this to the fifth power, compare this with cosine 5 theta plus i sine 5 theta, and wind up with real identities, in fact, for what?

Sine n theta and cosine n theta for any whole number value of n. See, I'm trying to hammer home the fact that as we're doing the complex number arithmetic, don't forget that this stuff does have real applications, and we haven't even started to scratch the surface yet. This is just our baby lecture introducing the arithmetic of complex numbers. We haven't even gotten to anything like algebra or calculus yet. Wait till that happens, and you're really going to see some nice applications.

At any rate, by the way, the polar form of multiplication that leads to a very interesting way of extracting roots of complex numbers. For example, suppose I want to find the sixth root of i. In other words, what complex number, raised to the sixth power, gives i? In fact, is there such a complex number? After all, we could raise real numbers to powers. But one of the reasons that we had to invent the complex numbers is that we couldn't extract the square root of minus 1. There was no real number whose square was minus 1. The question now is there a complex number, which when raised to the sixth power, equals i?

One way of doing this is to say, OK, let's assume there is a complex number. We'll call it x plus iy, which when raised to the sixth power, equals i. In other words, the sixth root the i is x plus iy. And let's see if we can solve for x and y. One way of doing this is to raise both sides here to the sixth power, in which case we see that i has to be x plus iy to the sixth power.

On the other hand, i is written as 0 plus 1i. If I raise this to the sixth power, I don't know if you've noticed this, every time I raise i to an even power, I get a real number. Why? Because i squared is minus 1. Therefore, i to the fourth is i squared squared, which is minus 1 squared, which is 1. i to the sixth is i to the fourth times i squared, which is 1 times i squared, which is minus 1.

And in the same way, if I take i cubed, that's what? i squared times i, which is minus i. In other words, the even powers of i are real, the odd powers of i give me back plus or minus i. So if I raise this to the sixth power and collect terms, I'll get a certain number of real terms and a certain number of purely imaginary terms.

In fact, using the binomial theorem and raising this to the sixth power and separating the terms for you in advance, I wind up with what? x to the sixth plus 6x to the fifth iy plus 15x to the fourth iy squared plus 20x cubed iy cubed plus 15x squared iy to the fourth plus 6x iy to the fifth plus iy to the sixth. I went through that rapidly. It's just using the binomial theorem, noticing that all of these terms will turn out to be real. All of these terms will be purely imaginary. In other words, getting rid of the i's to the best of my ability.

See, squaring over here, this is a minus y squared term. This is just y to the fourth. This is i to the sixth, which is the same as i squared. Because i to the fourth is 1. i to the sixth is i fourth times i squared, which is i squared. So this just comes out as minus 1, et cetera.

And making these translations, we wind up with the complicated algebraic system that to find x and y, we must be able to solve this system of equations. In other words, the root part must be 0, the imaginary part must be 1, all right? Now at this stage of the game, not only may it seem difficult to solve this, but it may be that there are no real values of x and y which solve this. And if I can't find x and y, if there are no values for x and y, it means that x plus iy doesn't exist.

Well, here's where I wanted to show you the tremendous power of polar coordinates. You see, in polar coordinates, how would I write i? I is what? It's magnitude is 1. See, it's the 0.0 comma 1 in the Argand diagram. It's magnitude is 1. And it's argument is pi over 2.

By the way, again, I'm in this trouble with multiple angles. You see, it's only pi over 2. It could be 5 pi over 2, 9 pi over 2, et cetera. Every time I go through 2 pi, I come back to the same point. I'm going to mention that in a moment.

But the idea is, look at, this is what I want. This is i. And I want the [INAUDIBLE] divide. Let's assume that the answer can be written in polar coordinates. If I write it in polar coordinates, the answer is r comma theta. All I've got to do now is solve for r and theta. And I claim, believe it or not, that that's trivial, namely, given this, I raise both sides to the sixth power.

If I do that, I wind up here with i, which just, to get this all on one line, I just repeat this. This is 1 comma pi over 2 in polar coordinates. And this is r comma theta to the sixth power. But the beauty of multiplication in polar coordinates is that r comma theta to the sixth power is r to the sixth comma 6 theta. The magnitude is this-- so in other words, you multiply the magnitudes, and you add the angles.

Therefore, what this tells me is, remember, r must be real. There is only one real number, which when raised to the sixth power, is 1. And that's 1 itself. See, the n-th root of any positive number has exactly one real solution. That's why we can always find the r. In this case, I picked the simple example where r turns out to be 1. See, minus 1 also raised to the sixth power is 1. But that doesn't count. Because we agreed that r had to be positive. r was measuring the magnitude of the complex number, so r must be 1 by virtue of the fact that r is greater than or equal to 0. That eliminates minus 1.

Now 6 theta can either be pi over 2 or 5 pi over 2, et cetera. The important point being that we now tack on the 2 pi k. Because notice that as this changes by 360 degrees, theta only changes by 60 degrees. Because 6 theta is changing by 360. Therefore, you see, we're going to get a whole bunch of theta values that work this way. And again, I'll explain this in more detail as we go through the exercises on this unit.

But the idea is what? If I keep tacking on multiples of 2 pi, what I'm saying is r must be 1, and theta is what? It's pi over 12 plus what? 2 pi k over 6 pi over 3k, 60 [INAUDIBLE]. I essentially, in terms of angles, tack on 60 degree increments here.

To make a long story short, r must be 1. But theta could either be pi over 12, 5 pi over 12, 9 pi over 12, 13 pi over 12. See, I'm adding on 4 pi over 12 each time, pi over 3, 60 degrees, in degree measure. 17 pi over 12, 21 pi over 12. The next one would be 25 pi over 12. But I hope you can see that that's the same as, position-wise, 1 comma pi over 12 gives me the same thing as 1 comma 25 pi over 12. But all six of these angles give me different positions.

Just by way of illustration, pi over 12 turns out to be 15 degrees. What this says is that one of the six roots of i, geometrically, is what? It's that complex number, which is 1 unit from the origin. That means it's on the circle of radius 1, centered at the origin. And the angle must be 15 degrees. The argument is 15 degrees.

And by the way, just to check this out that this really is at 1 comma pi over 12 really is an answer here. How do you raise a complex number to a power if we view it as a length? To raise this to the sixth power, we must raise 1 to the sixth power. That will still be 1. So I'm still going to be on the circle. When I multiply, I add angles. So when I raise this to the sixth power, I'm taking, what, 15 degrees six times is 90 degrees. And that puts me right up where i is supposed to be.

In other words, to this thing backwards, so to speak, I know that 6 times the angle I'm looking for must be 90. So the angle itself must be 15. For example, if I was looking for the eighth root of i, I would do what? I would know that when I add the angle to itself 8 times, I want 90. So the angle would have had to been 90 over 8. And again, I'll leave this for you as exercises. The point is that geometrically, the sixth roots of i are all equally spaced points. The first one is the point 1 comma pi over 12. And the rest are spaced equally along the circle at 60 degree intervals. You see, it breaks the circle up into six equal parts. I come back to here.

Notice that, for example, if I take this one, which is 75 degrees, if I take 75 degrees 6 times, notice that I come back, what, 75 times 6 is 450. It means I go all the way around and come back to i when I raise this to the sixth power.

To help you see this geometrically, I'll pick-- of these six, one of these happens to be very easy, at least, to me. 9 pi over 12 is 3 pi over 4, which happens to be 135 degrees. And i-- 1 comma 3 pi over 4 is cosine 3 pi over 4 plus i sine 3 pi over 4. The cosine of 3 pi over 4 is minus 1 over square root of 2. The sine is plus 1 over square root of 2.

So in typical x plus iy form, one of the roots is 1 over the square root of 2 times minus 1 plus i. In other words, the real part of this complex number is minus 1 over the square root of 2. The imaginary part is 1 over the square root of 2. And I leave it, again, as a voluntary exercise for you to do to actually raise this to the sixth power and find, amazingly enough, that you do get 1 for an answer-- i for an answer.

Now you see, I just picked one particular example. But this would have worked for any roots that I wanted to extract. And this is very important from a mathematical point of view. The complex numbers are closed, with respect to extracting roots. And let me summarize that for a part of today's lesson over here. The idea is that one of the reasons that we had to invent the fractions after we knew the integers was the fact that the integers were not closed with respect to division, that the quotient of two integers didn't have to be an integer.

One of the reasons that we had to invent the complex numbers after we had the real numbers was that the real numbers were not closed with respect to extracting roots. What I've just shown you in terms of a particular example is that the complex numbers are closed with respect to extracting roots. This means that, in particular, the basic operations that are involved in solving polynomial equations-- in other words, what you have to do is solve a polynomial equation, nothing more than the basic operations of adding, subtracting, multiplying, dividing, raising the powers, and extracting roots. All of these operations are closed with respect to the complex numbers.

And what this means is if you wanted to write a polynomial equation which had complex numbers as coefficients, you would not have to invent any more complex numbers. You would not have to invent a new number system to solve this equation, namely, any polynomial with complex coefficients has complex roots. And I think that's enough for today. I want you to drill now on the exercises.

Next time, we will talk about functions using complex numbers. At any rate, until next time, goodbye.

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