For Section 7.2:

Prob. 7.2.1 In Sec. 3.7, α_i is defined such that in the conservative subsystem, Eq. 3.7.3 holds. Show that α_i satisfies Eq. 7.2.3 with $\rho \rightarrow \alpha_i$. Further, show that if a "specific" property β_i is defined such that $\beta_i \equiv \rho\alpha_i$, then by virture of conservation of mass, the convective derivative of β_i is zero.

For Section 7.6:

Prob. 7.6.1 Show that Eq. (b) of Table 7.6.2 is correct.

Prob. 7.6.2 Show that Eqs. (j) and (ℓ) from Table 7.6.2 are correct.

Prob. 7.6.3 A pair of bubbles are formed with the tube-valve system shown in the figure. Bubble 1 is blown by closing valve V_2 and opening V_1. Then, V_1 is closed and V_2 opened so that the second bubble is filled. Each bubble can be regarded as having a constant surface tension γ. With the bubbles having the same initial radius ξ_o, when t = 0, both valves are opened (with the upper inlet closed off). The object of the following steps is to describe the resulting dynamics.

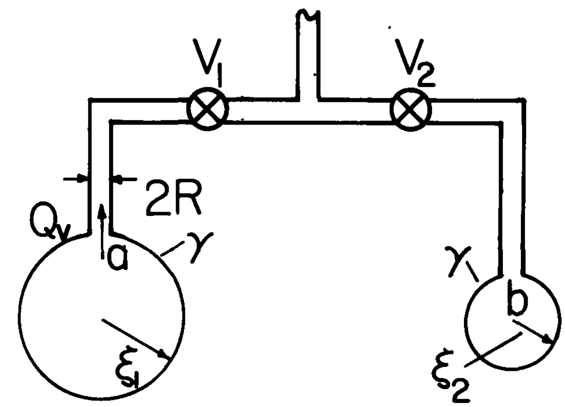

Fig. P7.6.3

(a) Flow through the tube that connects the bubbles is modeled as being fully developed and viscous dominated. Hence, for a length of tube ℓ having inner radius R and with a viscosity of the gas η, the volume rate of flow is related to the pressure difference by

$$Q_v = \frac{\pi R^4}{8\eta}\frac{(p_a - p_b)}{\ell} \qquad m^3/sec$$

The inertia of the gas and bubble is ignored, as is that of the surrounding air. Find an equation of motion for the bubble radius ξ_1.

(b) With the bubbles initially of equal radius ξ_o, there is a slight departure of the radius of one of the bubbles from equilibrium. What happens?

(c) In physical terms, explain the result of (b).

For Section 7.8:

Prob. 7.8.1 A conduit forming a closed loop consists of a pair of tubes having cross-sections with areas A_r and A_ℓ. These are arranged as shown with a fluid having density ρ_b filling the lower half and a second fluid having density ρ_a filling the upper half. The object of the following steps is to determine the dynamics of the fluid, specifically the time dependence of the interfacial positions ξ_r and ξ_ℓ.

(a) Use mass conservation to relate the displacements (ξ_r, ξ_ℓ) to each other and to the fluid velocities (v_r, v_ℓ) on the right and left respectively. Assume that the fluid is inviscid and has a uniform profile over the cross-section of a tube.

(b) Use Bernoulli's equation, Eq. 7.8.5 to relate quantities evaluated at the interfaces in the lower fluid, and in the upper fluid.

(c) Write the boundary conditions that relate quantities across the interfaces.

(d) Show that these laws combine to give an equation of motion for the right interface having the form

$$m\frac{d^2\xi_r}{dt^2} + \frac{1}{2}(\rho_b - \rho_a)\left[\left(\frac{d\xi_r}{dt}\right)^2 - \left(\frac{d\xi_\ell}{dt}\right)^2\right] + K\xi_r = 0$$

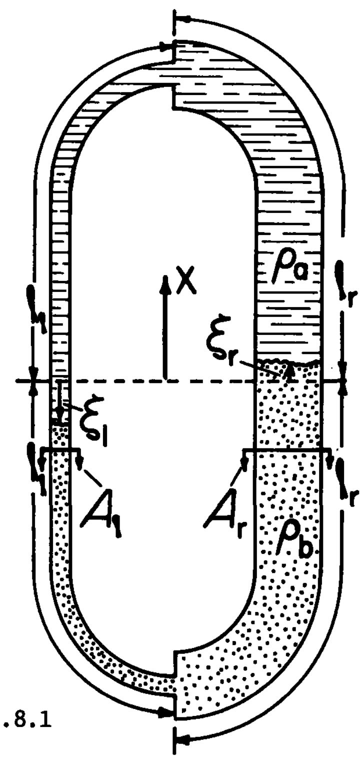

Fig. P7.8.1

Prob. 7.8.1 (continued)

What are the effective mass per unit length, m, and "spring-constant" K?

(e) Now, assume that the departures from equilibrium are small (linearize) and determine the natural frequencies of the system. Under what conditions will the system be unstable?

(f) A U tube is filled with water and open to the air. With a length of water in the tube (of uniform cross-section), ℓ, what are the natural frequencies?

Prob. 7.8.2 A hemispherical object rests on a flat plate. Fluid passes over and around the sphere with a velocity that is to be determined. The flow is uniform but a function of time far from the hemisphere.

(a) Note Eq. 7.8.11 and subsequent discussion. Find the inviscid velocity and velocity potential on the hemispherical surface.

(b) Find the pressure distribution on the hemisphere.

(c) What is the lift force on the hemisphere? (Assume that the pressure inside the sphere is the same as that at the stagnation point $r = R$, $\theta = \pi$ just outside the sphere, as would be the case if there were a small hole through the shell at this point.)

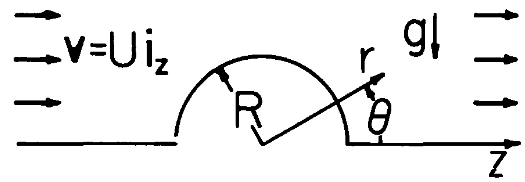

Fig. P7.8.2

Prob. 7.8.3 An electromagnetic rocket constrained by a test stand is shown in the figure. In the interior region there is a space occupied by an apparatus that produces a normal surface force density T_n on the surface S_i. A tube connects this space to the outside, and hence equalizes the pressures inside S_i and outside the rocket. The fluid inside S_i and outside the rocket has negligible mass density. There are no external forces in the fluid bulk. Thus the pressure in the surrounding homogeneous fluid is $p=T_n$. The volume is large enough that the fluid inside the rocket has negligible velocity and an essentially steady flow condition prevails. It is expelled through the throat and reaches a point where its velocity U is essentially uniform and x directed; the pressure is equal to that of the surroundings (say p=0) and the cross-sectional area is A. Gravitational effects are negligible. Use Eqs. 7.8.5, 7.3.2 and 7.4.3 to find the total force on the rocket in terms of T_n and A.

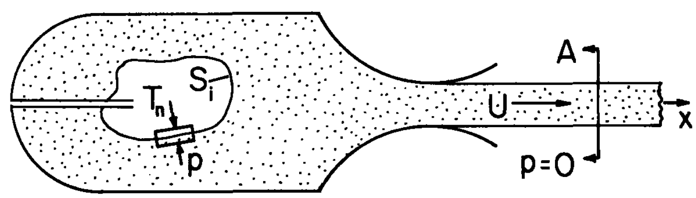

Fig. 7.8.3

For Section 7.9:

Prob. 7.9.1 In Sec. 2.17, conservation of electric energy is used to derive reciprocity conditions for the flux-potential transfer relations. The object here is a similar derivation for the transfer relations of Table 7.9.1 based on conservation of kinetic energy. Start with the assumption that for an inviscid incompressible fluid having uniform mass density, the change in kinetic energy is the result of displacements at the α and β planes.

$$\delta(W_{kin}) = -\oint_S p\delta\vec{\xi}\cdot\vec{n}da$$

Derive reciprocity conditions similar to Eq. 2.17.10.

Prob. 7.9.2 An annular region of incompressible inviscid fluid is bounded by outer and inner coaxial boundaries of radius α and β respectively, as shown in the figure. Hence, the configuration is similar to the circular cylindrical case of Table 7.9.1. However, rather than being in a state of uniform axial motion when in equilibrium, the fluid here is rotating. This equilibrium rotation is rigid body and could be established by spinning a cylinder of fluid for a long enough time that viscous shear stresses could transmit the motion to the fluid volume. Ignore gravitational effects.

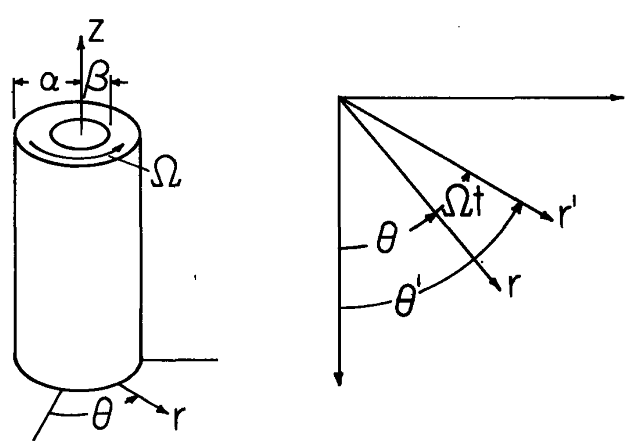

Fig. P7.9.2

Prob. 7.9.2 (continued)

(a) What is the vorticity of the equilibrium motion?

(b) Show that the equilibrium pressure distribution is

$$p_o = \frac{1}{2}\rho^2 r^2 + \Pi$$

(c) Write the perturbation continuity and force equations. Transform these expressions from the laboratory frame (primed) to a rotating frame (unprimed) where

$$r = r' \qquad v_r = v_r'$$

$$\theta = \theta' - \Omega t \qquad v_\theta = v_\theta' - \Omega r$$

$$z = z' \qquad p = p'$$

$$t = t'$$

and show that the perturbation equations are

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

$$\rho\left(\frac{\partial v_r}{\partial t} - 2\Omega v_\theta\right) + \frac{\partial p}{\partial r} = 0$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + 2\Omega v_r\right) + \frac{1}{r}\frac{\partial p}{\partial \theta} = 0$$

$$\rho\left(\frac{\partial v_z}{\partial t}\right) + \frac{\partial p}{\partial z} = 0$$

(d) Show that the pressure complex amplitude satisfies the equation

$$r^2\frac{d^2\hat{p}}{dr^2} + r\frac{d\hat{p}}{dr} - \hat{p}\left[m^2 + r^2k^2\left(1 - \frac{4\Omega^2}{\omega^2}\right)\right] = 0$$

where ω is the frequency in the rotating frame of reference.

(e) Show that transfer relations are

$$\begin{bmatrix}\hat{p}^\alpha \\ \\ \hat{p}^\beta\end{bmatrix} = \frac{\rho(4\Omega^2-\omega^2)}{j\omega D}\begin{bmatrix}\left[f_m(\alpha,\beta,\gamma) + \frac{2\Omega m}{\omega\beta}\right] & -g_m(\alpha,\beta,\gamma) \\ \\ -g_m(\beta,\alpha,\gamma) & \left[f_m(\beta,\alpha,\gamma) + \frac{2\Omega m}{\omega\alpha}\right]\end{bmatrix}\begin{bmatrix}\hat{v}_r^\alpha \\ \\ \hat{v}_r^\beta\end{bmatrix}$$

where

$$\gamma^2 \equiv k^2\left(1 - \frac{4\Omega^2}{\omega^2}\right);$$

$$D \equiv \left[f_m(\beta,\alpha,\gamma) - \frac{m\omega}{2\Omega\alpha}\right]\left[f_m(\alpha,\beta,\gamma) - \frac{m\omega}{2\Omega\beta}\right] - g_m(\beta,\alpha,\gamma)\, g_m(\alpha,\beta,\gamma)$$

For Section 7.11:

Prob. 7.11.1 Determine the transfer relations for the spherical shell of Table 7.9.1. Define functions F_n and G_n such that these relations take the form of Eq. (i) of that table. Describe the temporal modes of a gas surrounded at a radius r = R by a rigid boundary.

For Section 7.12:

Prob. 7.12.1 Gas flows with a uniform velocity U in the z direction through a rigid tube having inner radius R.

(a) Determine the dispersion equation for acoustic waves propagating in the z direction with pressure dependence of the form $p = \text{Re}\hat{p}(r)\exp j(\omega t - m\theta - kz)$.

(b) What are the wavenumbers of the spatial modes? Sketch the dispersion equation for a < U and for a > U.

Prob. 7.12.2 An acoustic waveguide consists of rigid plane parallel walls in y-z planes having the spacing (a+b) bounding planar layers of fluid respectively having the thicknesses (a) and (b), mass densities ρ_a and ρ_b and acoustic velocities a_a and a_b. Ignore gravity and surface tension effects at the interface. Determine the dispersion equation for waves propagating in the z direction between the walls. This expression is transcendental, and hence requires numerical solution. Consider two limits in which explicit expressions can be derived.

(a) The waves are very long, so that $\gamma_a a \ll 1$ and $\gamma_b b \ll 1$. What is the wave velocity for the resulting "principal" mode?

(b) Here, $a_a \ll a_b$ (for example, air and water) and $k^2 \gg \omega^2/a_b^2$. Use a graphical solution to find the wavenumbers of the approximate spatial modes.

For Section 7.13:

Prob. 7.13.1 The equations describing the incremental motions of a perfectly elastic isotropic solid can be developed in steps that follow those for a Newtonian fluid. The first problem following each of Secs. 7.13.1 through 7.16.1 is a step in developing these equations, which are summarized for reference in the table of Prob. 7.16.1.

(a) It is natural to use the displacement $\vec{\xi}(\vec{r}_o,t)$, rather than the velocity $\vec{v}(\vec{r},t)$, as a variable. By contrast with the Eulerian variable, the displacement is a Lagrangian variable in that a material particle originally at $\vec{r}_o$ is found at the position $r_o + \vec{\xi}(\vec{r}_o,t)$ (see Sec. 2.4). Show that to linear terms $\vec{\xi}(\vec{r}_o,t) \approx \vec{\xi}(\vec{r}_o+\vec{\xi},t)$, so that for incremental displacements $\vec{\xi}$ can be regarded as either an Eulerian or Lagrangian variable.

(b) The annulus of Fig. 7.13.1 is filled with an elastic solid rigidly attached to the walls. Instead of being given a steady velocity U, the boundary is given a steady displacement Ξ_z. It is found that $T_z = G_s(\Xi_z/d)$, where the coefficient G_s is the shear modulus. What is the elasticity version of Eq. 7.13.3?

(c) A thin rod of initial length ℓ is fixed at x=0 and subjected to a surface force density T_x at its end (where x=ℓ originally). Because the rod is "thin," the transverse stresses are negligible compared to T_{xx}. It is found that $T_x = E_s \Xi_x/\ell$, where the coefficient E_s is the elastic modulus. Write an equation expressing force equilibrium for an incremental length Δx of the rod, and obtain the analogue of Eq. 7.13.3 for dilatational deformations.

Typical values of G_s and E_s are given in Table P7.13.1.

Table P7.13.1. Elastic properties of various materials.

Material	Shear Modulus G_s (N/m^2)	Elastic Modulus E_s (N/m^2)	Poisson's Ratio ν_s	Mass Density kg/m^3
Aluminum	$2.6\text{x}10^{10}$	$7.3\text{x}10^{10}$	0.33	$2.8\text{x}10^3$
Steel	$7.8\text{x}10^{10}$	$2.1\text{x}10^{11}$	0.27	$7.8\text{x}10^3$
Rubber	$\sim.6\text{x}10^6$	$\sim 2\text{x}10^6$	~ 0.50	$1.1\text{x}10^3$

For Section 7.14:

Prob. 7.14.1 Define the strain tensor e_{ij} for incremental deformations, using arguments paralleling those from Eq. 7.14.1 to 7.14.3. The result should be Eq. (a) of Table P7.16.1. Geometrically interpret the shear and normal components of e_{ij}.

For Section 7.15:

Prob. 7.15.1 Starting with the assumption that the stress-strain constitutive laws for an isotropic perfectly elastic solid take the form

$$T_{ij} = c_{ijk\ell}e_{k\ell}$$

show that Eq. (b) of Table P7.16.1 is the desired relation with G_s as defined in Prob. 7.13.1 and λ_s, a second property of the material. Remember that in the thin-rod experiment of Prob. 7.13.1, the transverse stress components were essentially zero. Use this fact to show that G_s and E_s are related to λ_s by Eq. f of Table P7.16.1. In terms of the thin-rod experiment, Poisson's ratio ν_s is defined as the negative of the strain ratio e_{yy}/e_{xx} or e_{zz}/e_{xx}. Show that this property is related to G_s and E_s by Eq. (g) of the table.

Prob. 7.15.2 Following Eq. 7.15.15, it is argued that $\overset{o}{e}_{nn}$ is invariant under a transformation between coordinate systems. Confirm this by using the transformation properties of $\overset{o}{e}_{ij}$ (note Eq. 3.9.14).

Prob. 7.15.3 For the velocity distribution of Eq. 7.14.4

(a) What is S_{ij}?

(b) Use Eq. 7.15.5 to find the principal axes and the associated normal stresses.

For Section 7.16:

Prob. 7.16.1 The relations required to write the force equation representing an isotropic perfectly elastic solid, as derived in Probs. 7.13.1, 7.14.1 and 7.15.1 are summarized in Table P7.16.1. Show that the force equation can be written as Eq. (d) and, hence, as Eq. (e). (Note the discussion in Sec. 2.4.)

Table P7.16.1. Definitions, relations and equations of motion for an isotropic perfectly elastic solid.

Strain-displacement	$e_{ij} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right)$	(a)
Stress-strain	$T_{ij} = 2G_s e_{ij} + \lambda_s \delta_{ij} e_{kk}$	(b)
	$e_{ij} = \frac{1}{2G_s} T_{ij} - \frac{\nu_s}{E_s} \delta_{ij} T_{kk}$	(c)
Force equations	$\rho \frac{\partial^2 \xi_i}{\partial t^2} = \frac{\partial T_{ij}}{\partial x_j} + (F_{ex})_i$	(d)
	$\rho \frac{\partial^2 \bar{\xi}}{\partial t^2} = (2G_s+\lambda_s)\nabla(\nabla\cdot \bar{\xi}) - G_s \nabla \times (\nabla \times \bar{\xi}) + \bar{F}_{ex}$	(e)
Constitutive relations	$\lambda_s = (E_s - 2G_s)/[3-(E_s/G_s)]$	(f)
	$\nu_s = (E_s/2G_s) - 1$	(g)

For Section 7.18:

Prob. 7.18.1 The equation of motion for a perfectly elastic isotropic solid is Eq. (e) of Table P7.16.1. The external force density is represented by Eq. 7.18.2, while the displacement is represented in terms of vector and scaler potentials

$$\vec{\xi} = \nabla\times\vec{A}_s - \nabla\psi_s \;;\; \nabla\cdot\vec{A}_s = 0$$

Show that $\vec{A}_s$ and ψ_s respectively represent rotational and dilational deformations. Show that A_s and ψ_s respectively satisfy wave equations with the wave velocities $v_s = \sqrt{G_s/\rho}$ and $v_c = \sqrt{(2G_s+\lambda_s)/\rho}$.

Prob. 7.18.2 In Sec. 7.21, it is shown that the viscous drag force on a rigid sphere having radius R moving through a fluid with velocity U is $6\pi\eta RU$. A spherical particle has mass density ρ_p much greater than that of the surrounding fluid so that the mass of the fluid can be ignored (the Reynolds number is low, as it must be for the Stokes drag force to be correct). Write the force equation for the slowing of the particle from some initial velocity. Show that the velocity decreases exponentially, with a time constant 2/9 of the viscous "diffusion" time based on the particle radius and density and the fluid viscosity.

For Section 7.19:

Prob. 7.19.1 An incompressible elastic solid is one in which deformations are solenoidal. It can be pictured as having $\nabla\cdot\vec{\xi}\to 0$ and $2G_s+\lambda_s\to\infty$ in such a way that the product $(2G_s+\lambda_s)\nabla\cdot\vec{\xi}\to -p$, where the pressure p is finite. It is an appropriate model if the transit time of compressional waves having velocity v_c as found in Prob. 7.18.1 is very short compared to times of interest but that of shear waves is arbitrary. Equations (f) and (g) of Table P7.16.1 combine to show that $(2G_s+\lambda_s)\to\infty$ as $\nu_s\to 0.5$, so the incompressible model is especially appropriate in working with materials such as Jello or rubber. The force equation, Eq. e of Table 7.16.1, becomes

$$\rho\frac{\partial^2\vec{\xi}}{\partial t^2} = -\nabla p + G_s\nabla^2\vec{\xi} + \vec{F}_{ex}$$

(a) Show that the associated stress tensor is

$$S_{ij} = -p + G_s\left(\frac{\partial\xi_i}{\partial x_j} + \frac{\partial\xi_j}{\partial x_i}\right)$$

(b) Show that the transfer relations derived in this section, Eqs. 7.19.13 and 7.19.19, can be adapted to an incompressible solid by making the identification of variables

$$\hat{\vec{v}} \to j\omega\hat{\vec{\xi}} \;;\; j\omega\eta \to G_s \;;\; \hat{p}\to\hat{p},\; \hat{S}_{ij}\to\hat{S}_{ij}$$

Prob. 7.19.2 Show that an infinite half-space of elastic material is described by the transfer relations

$$\begin{bmatrix}\hat{S}^{\alpha}_{xx}{}^{\beta}\\ \hat{S}^{\alpha}_{yx}{}^{\beta}\end{bmatrix} = \frac{\rho}{\gamma_c\gamma_s-k^2}\begin{bmatrix}\pm\gamma_s v_c^2(\gamma_c^2-k^2) & jk[k^2(v_c^2-2v_s^2) - v_c^2\gamma_c^2+2\gamma_c\gamma_s v_s{}^2] \\ jkv_s^2(\gamma_s^2+k^2-2\gamma_c\gamma_s) & \pm v_s^2\gamma_c(\gamma_s^2-k^2)\end{bmatrix}\begin{bmatrix}\hat{\xi}^{\alpha}_x{}^{\beta}\\ \hat{\xi}^{\alpha}_y{}^{\beta}\end{bmatrix}$$

where $\gamma_c^2 \equiv k^2-\omega^2/v_c^2$ and $\gamma_s^2 \equiv k^2-\omega^2/v_s^2$ and v_s and v_c are the velocities of shear and compressional waves as defined in Prob. 7.18.1.

Prob. 7.19.3 Rayleigh waves propagate on the free surface of an elastic material without dispersion. Because the associated deformations are readily accessible from the adjacent free space, these waves have been made the basis for surface acoustic wave (SAW) devices. The Rayleigh wave is neither a shear wave nor a dilatational wave, but rather a combination of these.

(a) Using the transfer relations for the lower half space derived in Prob. 7.19.2, show that the dispersion equation for this wave is

$$(\gamma_s^2+k^2)^2 = 4\gamma_s\gamma_c k^2$$

(b) Use the definition of γ_s and γ_c to show that the dispersion equation can be expressed as

$$\underline{\omega}^6 - 8\underline{\omega}^4 + 8\underline{\omega}^2\left(3 - \frac{2v_s^2}{v_c^2}\right) - 16\left(1 - \frac{v_s^2}{v_c^2}\right) = 0$$

where $\underline{\omega} \equiv \omega/kv_s$.

(c) Note that the coefficients of this expression do not depend on k. Extraneous roots have been generated in deriving this expression. One root represents the Rayleigh wave. Argue that the surface wave propagates without dispersion.

(d) Show that $v_s^2/v_c^2 = (1-2\nu_s)/2(1-\nu_s)$, so that $\underline{\omega}$ is determined by ν_s.

Prob. 7.19.4 The transfer relations, Eq. 7.19.3, are to be extended to describe the fluid response not only because of external interactions which have their effect on the layer through the surfaces, but also because of an imposed force density

$$\vec{F}_{ex} = \nabla \times \vec{G}; \quad \vec{G} = \text{Re}\, \hat{G}(x) \exp j(\omega t - ky)\vec{i}_z \tag{1}$$

The extension follows lines similar to those taken in Sec. 4.5 for the flux-potential relations.

(a) Write Eq. 7.19.1 including the effect of the force density.

(b) Given any particular solution to this equation, $\hat{A}_P(x)$ with associated velocity and stress functions denoted by subscripts P, show that the transfer functions are:

$$\begin{bmatrix} \hat{S}_{xx}^{\alpha} \\ \hat{S}_{xx}^{\beta} \\ \hat{S}_{yx}^{\alpha} \\ \hat{S}_{yx}^{\beta} \end{bmatrix} = \eta[P_{ij}] \begin{bmatrix} \hat{v}_x^{\alpha} \\ \hat{v}_x^{\beta} \\ \hat{v}_y^{\alpha} \\ \hat{v}_y^{\beta} \end{bmatrix} + \begin{bmatrix} (\hat{S}_{xx}^{\alpha})_P \\ (\hat{S}_{xx}^{\beta})_P \\ (\hat{S}_{yx}^{\alpha})_P \\ (\hat{S}_{yx}^{\beta})_P \end{bmatrix} - \eta[P_{ij}] \begin{bmatrix} (\hat{v}_x^{\alpha})_P \\ (\hat{v}_x^{\beta})_P \\ (\hat{v}_y^{\alpha})_P \\ (\hat{v}_y^{\beta})_P \end{bmatrix} \tag{2}$$

(c) For a y directed force density F_o that is independent of x, $\hat{G}(x) = F_o x$. Evaluate Eq. 2 in this case.

Prob. 7.19.5 The fluid layer shown in Fig. 7.19.1 is bounded in the x=0 and x=d planes by rigid walls. Find the frequencies of the temporal modes. To do this use γ/k as a parameter representing the frequency ω, and write a transcendental equation of the form $D(\gamma/k, kd)=0$ which (given kd) can be solved for γ/k and hence ω. Illustrate how a graphical construction can be used to find roots of this expression, wherein γ/k is imaginary.

For Section 7.20:

Prob. 7.20.1 The equations of motion for an elastic solid are summarized in Prob. 7.19.1.

(a) Show that the transfer relations developed in this section can be used to describe an incompressible "inertia-less" elastic material by making the substitution

$$\hat{\vec{v}} \rightarrow \hat{\vec{\xi}},\ \hat{S}_{ij} \rightarrow \hat{S}_{ij},\ \eta \rightarrow G_s.$$

(b) Argue that the relations hold for deformation that are quasistatic with respect to the transit times of both the compressional and shear waves.

Prob. 7.21.1 Use Eqs. 7.21.17 and 7.21.13 to show that the Stoke's flow around a sphere is represented by Eq. 5.5.5.

Prob. 7.21.2 A rigid sphere having radius R is subject to an externally applied slowly varying z directed force f_z. Show that the resulting displacement Ξ in the z direction is related to this force by $f_z = 6\pi G_s R\Xi$. (See Prob. 7.20.1.)