Class Exam 1 Review Problems
18.05, Spring 2022

There are certainly too many problems here to do in class. Pick and choose the ones that will be most helpful to you. The actual test will be much much shorter.

1 Normal probability table

Problem 1. (Table of normal probabilities)
Use the table of standard normal probabilities to compute the following. (Z is the standard normal.)

(a) (i) $P(Z \le 1.5)$ (ii) $P(-1.5 < Z < 1.5)$ $P(Z > -0.75)$.

(b) Suppose $X \sim \mathrm{N}(2, (0.5)^2)$. Find (i) $P(X \le 2)$ (ii) $P(1 < X \le 1.75)$.

2 Counting and Probability Problems

Problem 2. (a) How many ways can you arrange the letters in the word STATISTICS? (e.g. SSSTTTIIAC counts a one arrangement.)

(b) If all arrangements are equally likely, what is the probabilitiy the two 'i's are next to each other.

3 Conditional Probability and Bayes' Theorem Problems

Problem 3. Corrupted by their power, the judges running the popular game show *America's Next Top Mathematician* have been taking bribes from many of the contestants. Each episode, a given contestant is either allowed to stay on the show or is kicked off.

If the contestant has been bribing the judges they will be allowed to stay with probability 1. If the contestant has not been bribing the judges, they will be allowed to stay with probability 1/3.

Suppose that 1/4 of the contestants have been bribing the judges. The same contestants bribe the judges in both rounds, i.e., if a contestant bribes them in the first round, they bribe them in the second round too (and vice versa).

(a) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that they were bribing the judges?

(b) If you pick a random contestant, what is the probability that they are allowed to stay during both of the first two episodes?

(c) If you pick random contestant who was allowed to stay during the first episode, what is the probability that they get kicked off during the second episode?

4 Independence Problems

Problem 4. You roll a twenty-sided die. Determine whether the following pairs of events are independent.

(a) 'You roll an even number' and 'You roll a number less than or equal to 10'.

(b) 'You roll an even number' and 'You roll a prime number'.

5 Expectation and Variance Problems

Problem 5. The random variable X takes values -1, 0, 1 with probabilities 1/8, 2/8, 5/8 respectively.

(a) Compute $E[X]$.

(b) Give the pmf of $Y = X^2$ and use it to compute $E[Y]$.

(c) Instead, compute $E[X^2]$ directly from an extended table.

(d) Compute $\text{Var}(X)$.

Problem 6. Suppose 100 people all toss a hat into a box and then proceed to randomly pick out of a hat. What is the expected number of people to get their own hat back.

Hint: express the number of people who get their own hat as a sum of random variables whose expected value is easy to compute.

6 Probability Mass Functions, Probability Density Functions and Cumulative Distribution Functions Problems

Problem 7. **(a)** Suppose that X has probability density function $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Compute the cdf, $F_X(x)$.

(b) If $Y = X^2$, compute the pdf and cdf of Y.

Problem 8. Suppose you roll a fair 6-sided die 100 times (independently), and you get \$3 every time you roll a 6.

Let X_1 be the number of dollars you win on rolls 1 through 25.
Let X_2 be the number of dollars you win on rolls 26 through 50.
Let X_3 be the number of dollars you win on rolls 51 through 75.
Let X_4 be the number of dollars you win on rolls 76 throught 100.

Let $X = X_1 + X_2 + X_3 + X_4$ be the total number of dollars you win over all 100 rolls.

(a) What is the probability mass function of X?

(b) What is the expectation and variance of X?

(c) Let $Y = 4X_1$. (So instead of rolling 100 times, you just roll 25 times and multiply your winnings by 4.)

(i) What are the expectation and variance of Y?

(ii) How do the expectation and variance of Y compare to those of X? (That is, are they bigger, smaller, or equal?) Explain (briefly) why this makes sense.

7 Joint Probability, Covariance, Correlation Problems

Problem 9. Covariance and Independence
Let X be a random variable that takes values -2, -1, 0, 1, 2; each with probability 1/5. Let $Y = X^2$.

(a) Fill out the following table giving the joint frequency function for X and Y. Be sure to include the marginal probabilities.

X / Y	-2	-1	0	1	2	total
0						
1						
4						
total						

(b) Find $E[X]$ and $E[Y]$.

(c) Show X and Y are not independent.

(d) Show $\text{Cov}(X, Y) = 0$.
This is an example of uncorrelated but non-independent random variables. The reason this can happen is that correlation only measures the linear dependence between the two variables. In this case, X and Y are not at all linearly related.

Problem 10. Continuous Joint Distributions
Suppose X and Y are continuous random variables with joint density function $f(x, y) = c(x + 2y)$ on the rectangle $[0, 1] \times [0, 2]$.

When doing this problem in class: Only compute the integrals in parts (a) and (b). For the others, just give the integrals in a form like $\int_0^1 \int_0^2 xf(x, y)\, dy\, dx$, but don't compute them.

(a) Find the value of c.

(b) Let $F(x, y)$ be the joint CDF. Compute $F(x, y)$. Compute $F(1, 2)$.

(c) Compute the marginal densities for X and Y.

(d) Are X and Y independent?

(e) Compute $E[X]$, $E[Y]$, $E[X^2 + Y^2]$, $\text{Cov}(X, Y)$, $\text{Cor}(X, Y)$.

8 Law of Large Numbers, Central Limit Theorem Problems

Problem 11. Suppose $X_1, \ldots, X_{100}$ are i.i.d. with mean 1/5 and variance 1/9. Use the central limit theorem to estimate $P(\sum X_i < 30)$.

Problem 12. (**More Central Limit Theorem**)

The average IQ in a population is 100 with standard deviation 15 (by definition, IQ is normalized so this is the case). What is the probability that a randomly selected group of 100 people has an average IQ above 115?

9 More problems

Problem 13. (**Arithmetic Puzzle**)
The joint and marginal pmf's of X and Y are partly given in the following table.

$X \backslash Y$	1	2	3	
1	1/6	0	...	1/3
2	...	1/4	...	1/3
3	...	...	1/4	...
	1/6	1/3	...	1

(a) Complete the table.

(b) Are X and Y independent?

Problem 14. Compute the expectation and variance of a Bernoulli(p) random variable.