Quantum Field Theory I (8.323) Spring 2023
Assignment 9

Apr. 11, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 3

Notes: conventions and some useful formulae

1. Conventions of γ matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{1}$$

and

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \ . \tag{2}$$

2. The Dirac equation has the form

$$(\gamma^\mu\partial_\mu - m)\psi = 0 \tag{3}$$

and the action is given by

$$S = -i\int d^4x\, \bar{\psi}(\partial\!\!\!/ - m)\psi \ . \tag{4}$$

3. A spinor ψ transforms under a Lorentz transformation Λ as

$$\psi'_\alpha(x') = S_\alpha{}^\beta(\Lambda)\psi_\beta(x), \qquad x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{5}$$

with

$$\Lambda^\mu{}_\nu = \left(e^{-\frac{i}{2}\omega_{\lambda\rho}\mathcal{J}^{\lambda\rho}}\right)^\mu{}_\nu, \qquad S(\Lambda) = e^{-\frac{i}{2}\omega_{\lambda\rho}\Sigma^{\lambda\rho}}, \tag{6}$$

and

$$\left(\mathcal{J}^{\lambda\rho}\right)^\mu{}_\nu = i\left(\eta^{\lambda\mu}\delta^\rho{}_\nu - \eta^{\rho\mu}\delta^\lambda{}_\nu\right), \qquad \Sigma^{\lambda\rho} = \frac{i}{4}[\gamma^\lambda, \gamma^\rho] \ . \tag{7}$$

Also note

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu\gamma^\nu \ . \tag{8}$$

4. $u_s(\vec{k})e^{ik\cdot x}$ and $v_s(\vec{k})e^{-ik\cdot x}$, $s = 1, 2$ denote respectively a basis of positive and negative energy solutions to the Dirac equation, with $k^2 = -m^2$.

5. We normalize $u_s(\vec{k})$ and $v_s(\vec{k})$ as

$$\bar{u}_r(\vec{k})u_s(\vec{k}) = 2mi\delta_{rs}, \qquad \bar{v}_r(\vec{k})v_s(\vec{k}) = -2mi\delta_{rs} \ . \tag{9}$$

$u_s(\vec{k})$ and $v_s(\vec{k})$ are orthogonal

$$\bar{u}_r(\vec{k})v_s(\vec{k}) = 0, \qquad \bar{v}_r(\vec{k})u_s(\vec{k}) = 0 \ . \tag{10}$$

6. With normalization (9), we have

$$u_r^\dagger(\vec{k})u_s(\vec{k}) = 2E\delta_{rs}, \qquad v_r^\dagger(\vec{k})v_s(\vec{k}) = 2E\delta_{rs} \ , \tag{11}$$

and the orthogonal relations (10) can also be written as

$$u_r^\dagger(\vec{k})v_s(-\vec{k}) = 0, \qquad v_r^\dagger(\vec{k})u_s(-\vec{k}) = 0 \ . \tag{12}$$

These relations are valid for any choices of basis and any representation of gamma matrices once the normalizations are fixed as in (9).

7. With normalization (9), one can also show that

$$\Lambda_+(\vec{k}) = \sum_{s=1,2} u_s(\vec{k}) \otimes \bar{u}_s(\vec{k}) = i(i\not{k} + m), \tag{13}$$

$$\Lambda_-(\vec{k}) = \sum_{s=1,2} v_s(\vec{k}) \otimes \bar{v}_s(\vec{k}) = -i(-i\not{k} + m) \ . \tag{14}$$

8. An operator solution $\psi(x)$ to the Dirac equation can be expanded as

$$\psi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left[a_{\vec{k}}^{(s)} u_s(\vec{k}) e^{ik\cdot x} + \left(c_{\vec{k}}^{(s)}\right)^\dagger v_s(\vec{k}) e^{-ik\cdot x} \right] \ . \tag{15}$$

where the operators $a_{\vec{k}}^{(s)}, (a_{\vec{k}}^{(s)})^\dagger$ and $c_{\vec{k}}^{(s)}, (c_{\vec{k}}^{(s)})^\dagger$ satisfy the relations

$$\{a_{\vec{k}}^{(r)}, (a_{\vec{k}'}^{(s)})^\dagger\} = \{c_{\vec{k}}^{(r)}, (c_{\vec{k}'}^{(s)})^\dagger\} = \delta_{rs}(2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}'), \tag{16}$$

$$\{a_{\vec{k}}^{(r)}, a_{\vec{k}'}^{(s)}\} = \{c_{\vec{k}}^{(r)}, c_{\vec{k}'}^{(s)}\} = 0 \ . \tag{17}$$

9. In the chiral representation, the gamma matrices are given by

$$\gamma^0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix} \ . \tag{18}$$

10. In this course we will not be able to talk about applications of Majorana fermions, which have played important roles in models for neutrinos. During the last decade, it has also received wide interests in condensed matter physics and quantum computing as a possible avenue for topological quantum computing. Ettore Majorana came up with the idea of Majorana fermions in 1937, at the age of 31. Less than one year later he mysteriously disappeared; he boarded a ship from Palermo to Naples, but never got off it.

    Fermi once said:

    "There are several categories of scientists in the world; those of second or third rank do their best but never get very far. Then there is the first rank, those who make important discoveries, fundamental to scientific progress. But then there are the geniuses, like Galilei and Newton. Majorana was one of these."

    You can read more about Majorana at:

    http://www.ccsem.infn.it/em/EM_genius_and_mystery.pdf.

Problem Set 9

1. **Some identities (10 points)**

   Define γ^5 as
   $$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \ . \tag{19}$$
   Show that it has the following properties:

   (a) $(\gamma^5)^\dagger = \gamma^5$ and $(\gamma^5)^2 = 1$.

   (b) $\{\gamma^5, \gamma^\mu\} = 0$ and $\mathrm{Tr}\,\gamma^5 = 0$.

2. **Feynman propagator for Dirac spinors (10 points)**

   Show that the Feynman Green function
   $$D_F^{\alpha\beta}(x-y) \equiv \left\langle 0 | T\psi_\alpha(x)\bar{\psi}_\beta(y) | 0 \right\rangle = i(\partial\!\!\!/ + m)_{\alpha\beta} G_F(x-y) \tag{20}$$
   where G_F is the Feynman propagator for a free complex scalar of the same mass m.

3. **Chiral and Majorana fermions (50 points)**

   In this problem we consider the chiral representation (18), and write a Dirac spinor ψ in terms of two chiral spinors ψ_L and ψ_R as
   $$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \ . \tag{21}$$

(a) Show that under a rotation with parameters $\omega_{ij} = \epsilon_{ijk}\theta_k$, $\psi_{L,R}$ transform as

$$\psi'_L(x') = e^{\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}}\psi_L(x), \qquad \psi'_R(x') = e^{\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}}\psi_R(x) \ . \tag{22}$$

(b) Show that under a boost with parameters $\omega_{0i} = \beta_i$, $\psi_{L,R}$ transform as

$$\psi'_L(x') = e^{-\frac{1}{2}\vec{\beta}\cdot\vec{\sigma}}\psi_L(x), \qquad \psi'_R(x') = e^{\frac{1}{2}\vec{\beta}\cdot\vec{\sigma}}\psi_R(x) \ . \tag{23}$$

(c) The Lagrangian density for the Dirac theory contains a mass term of the form

$$\mathcal{L} = \cdots + im\bar{\psi}\psi = \cdots + im(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L) \ . \tag{24}$$

Using the transformations of parts (a) and (b) show that the above mass term is Lorentz invariant, while a term of the form

$$m\psi_L^\dagger\psi_L \quad \text{or} \quad m\psi_R^\dagger\psi_R \tag{25}$$

is not.

(d) The discussion in part (c) might give the impression that it is not possible to write down a mass term with ψ_L or ψ_R alone. In fact, it is possible, with a bit more sophistication. For this purpose, first show that

$$\sigma^2\vec{\sigma}\sigma^2 = -\vec{\sigma}^* \ . \tag{26}$$

From the above equation show that $\sigma^2\psi_L^*$ transforms under Lorentz transformation in the same way as ψ_R.

(e) From the observation of part (d), construct a mass term using ψ_L alone, which is both Lorentz invariant and real. (This mass term is called the Majorana mass term for reasons which will be clear in problem 4.)

Show that the mass term is *identically* zero if ψ_L consists of ordinary functions, while it is nonzero if ψ_L are anti-commuting variables.

(f) Now write down a Lorentz invariant full action using ψ_L alone which includes both kinetic and mass terms. Write down equations of motion.

(g) Does the theory of part (f) has a conserved charge? Argue such a theory can only describe neutral particles (thus cannot be a theory of electron).

4. **Majorana fermions (10 points)**

In the Majorana representation, γ^μ are real and thus ψ can be chosen to be real. Such a spinor is call a Majorana spinor. This has important physical consequences. Upon quantization, being real, a Majorana particle should not have an anti-particle (or equivalently it is its own anti-particle).

We discussed in lecture that the concept of a Majorana spinor can be generalized to any representation of γ, if we could find a matrix B satisfying

$$B\gamma^{\mu}B^{-1} = \gamma^{\mu *}, \tag{27}$$

and the Majorana condition is

$$\psi^{*} = B\psi \ . \tag{28}$$

(a) In lecture we showed that in the chiral representation (18) we can choose $B = \gamma^2$. Solve the Majorana condition (28) in the chiral representation. Show that in this representation a Majorana spinor ψ can be expressed in terms of ψ_L alone.

(b) Plug in the expression (in terms of ψ_L) for Majorana ψ you obtained in part (a) into the Dirac action. Show that it reduces to the action you found in part 3(f)! (That is why the mass term you found in part 3(e) is called the Majorana mass term.)

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