8.323 Problem Set 10 Solutions

April 25, 2023

Question 1: Chiral Symmetry (15 points)

Consider the Dirac action with $m = 0$. (a) Show that the action is invariant under transformations

$$\psi \to e^{i\alpha\gamma^5}\psi$$

The Dirac conjugate transforms as:

$$\bar{\psi}' = \psi^\dagger e^{-i\alpha\gamma^5}\gamma^0 = \psi^\dagger\gamma^0 e^{i\alpha\gamma_5} = \bar{\psi}e^{i\alpha\gamma_5}$$

where we use that $\{\gamma_5, \gamma^\mu\} = 0$. The second equality can be made more epxlicit by expanding the exponential, anticommuting the γ^0 through each term, and resummiong. Therefore, the massless Dirac Lagrangian transforms as

$$\mathcal{L}' = -i\bar{\psi}'\partial\!\!\!/\psi' = -i\bar{\psi}e^{i\alpha\gamma_5}\gamma^\mu e^{i\alpha\gamma^5}\psi\partial_\mu\psi = -i\bar{\psi}\gamma^\mu e^{-i\alpha\gamma_5}e^{i\alpha\gamma^5}\psi\partial_\mu\psi = -i\bar{\psi}\partial\!\!\!/\psi = \mathcal{L}$$

Therefore, the action is invariant.

(b) Construct the Noether current for the above symmetry.

Under an infinitesmial chiral rotation, $\delta\psi = i\alpha\gamma^5\psi$. Noting from (a) that the Lagrangian is invariant, the Noether current is thus:

$$j_5^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi = \bar{\psi}\gamma^\mu\gamma^5\psi$$

(c) Find how the mass term $m\bar{\psi}\psi$ transforms. Is it invariant?

The mass term transforms as:

$$m\bar{\psi}\psi \to m\bar{\psi}'\psi' = m\bar{\psi}e^{i\alpha\gamma_5}e^{i\alpha\gamma_5}\psi = e^{2i\alpha\gamma_5}m\bar{\psi}\psi$$

Since α is arbitrary, this is not left invariant under the chiral rotation.

Question 2: Quantizingf the Theory of Majorana Fermions (25 points)
Consider the theory of Majorana fermions discussed in Problem Set 9, written in terms of a 2-component complex spinor ψ_L

$$\mathcal{L}_L = i\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L - \frac{m}{2}(\psi_L^T \sigma^2 \psi_L + \psi_L^\dagger \sigma^2 \psi_L^*)$$

where ψ^T denotes the transpose of ψ, and $\sigma^\mu = (1, \vec{\sigma})$.
(a) Write down the equal time canonical quantization relations.
For the Majorana Lagrangian above, the conjugate momentum to ψ_L is

$$\pi_L = i\psi_L^\dagger$$

Therefore, the canonical anticommutation relations are

$$\{\psi_{La}(t,\mathbf{x}), \psi_{Lb}(t,\mathbf{x}')\} = \{\psi_{La}^\dagger(t,\mathbf{x}), \psi_{Lb}^\dagger(t,\mathbf{x}')\} = 0, \qquad \{\psi_{La}(t,\mathbf{x}), \psi_{Lb}^\dagger(t,\mathbf{x}')\} = \delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{x}')$$

(b) Write down the equations of motion. In momentum space a general solution can be written as

$$\psi_L(x) = u(p)e^{ip\cdot x} + v(p)e^{-ip\cdot x}$$

Using this notation, write down a complete basis of solutions in the rest frame $\mathbf{p} = 0$.
The equations of motion are given by:

$$i\sigma^\mu \partial_\mu \psi_L - m\sigma^2 \psi_L^* = 0, \qquad -i\partial_\mu \psi_L^\dagger \sigma^\mu - m\psi_L^T \sigma^2 = 0$$

Using the ansatz $\psi_L(x) = u(p)e^{ip\cdot x} + v(p)e^{-ip\cdot x}$, we obtain the momentum space equation

$$[-p_\mu \sigma^\mu u_L(p) - m\sigma^2 v_L^*(p)]e^{ip\cdot x} + [p_\mu \sigma^\mu v_L(p) - m\sigma^2 u_L^*(p)]e^{-ip\cdot x} = 0$$

Both positive and negative frequency parts must vanish independently, which gives

$$p_\mu \sigma^\mu u_L(p) + m\sigma^2 v_L^*(p) = 0, \qquad p_\mu \sigma^\mu v_L(p) - m\sigma^2 u_L^*(p) = 0$$

Note that these are not independent: the first equation implies the second, as

$$p_\mu \sigma^\mu v_L = -\frac{1}{m}(p\cdot\sigma)\sigma_2(p\cdot\sigma)^* u_L^* = -\frac{1}{m}(p\cdot\sigma)(p\cdot\bar{\sigma})\sigma_2 u_L^* = -m\sigma_2 u_L^*$$

where in the first equality we use the complex conjugate form of $p_\mu \sigma^\mu u_L(p) + m\sigma^2 v_L^*(p) = 0$, in the second we use from Problem Set 9 that $\sigma_2 \sigma^* \sigma_2 = -\sigma_2$, and in the last equality we use that $(p\cdot\sigma)(p\cdot\bar{\sigma}) = m^2$.

Thus, without loss of generality, we look for solutions to $p_\mu \sigma^\mu u_L(p) + m\sigma^2 v_L^*(p) = 0$. In the rest frame, this becomes

$$u_L(\mathbf{0}) = \sigma_2 v_L(\mathbf{0})^* \quad \implies \quad v_L(\mathbf{0}) = -\sigma_2 u_L^*(\mathbf{0})$$

This relates v_L to u_L. We still have the freedom to specify the polarization $u_L(\mathbf{0})$. First, consider the normalized eigenvectors $\xi_{\{s\in\pm\}}$ of σ^3. That is, $\sigma^3 \xi_\pm = \pm 1$, and $\xi_s^\dagger \xi_r = \delta_{rs}$. We let this be our solution space of $u_L(\mathbf{0})$, and set:

$$u_{Ls}(\mathbf{0}) = \sqrt{m}\xi_s, \quad v_{Ls} = -\sqrt{m}\sigma^2 \xi_s, \qquad \xi_+ = \begin{pmatrix}1\\0\end{pmatrix}, \quad \xi_- = \begin{pmatrix}0\\1\end{pmatrix}$$

These are normalized so that $u_{Lr}^\dagger(\mathbf{0}) u_{Lr}(\mathbf{0}) = m\delta_{rs}$.

(c) Verify the following expresssions give a complete basis of solutions for general p:

$$u_s(p) = \sqrt{-p\cdot\bar{\sigma}}\xi_s, \qquad v_s(p) = -\sqrt{-p\cdot\bar{\sigma}}\sigma^2\xi_s$$

where ξ_s, $s=\pm$ are respectively eigenvectors of σ^3 with eigenvalues ± 1.

By the discussion in (b), to check these spinors are solutions of the equations of motion, it is sufficient to check that $p\cdot\sigma u_L(p) + m\sigma^2 v_L^*(p) = 0$. We compute:

$$\begin{aligned}
p\cdot\sigma u_L(p) + m\sigma^2 v_L^*(p) &= (p\cdot\sigma)(-p\cdot\bar{\sigma})^{1/2}\xi_s + m\sigma^2(-\sqrt{-p\cdot\bar{\sigma}}\sigma^2\xi_s)^* \\
&= -(-p\cdot\sigma)^{1/2}(-p\cdot\sigma)^{1/2}(-p\cdot\bar{\sigma})^{1/2}\xi_s + m\sigma^2(\sqrt{-p\cdot\bar{\sigma}})^*\sigma^2\xi_s \\
&= -m\sqrt{-p\cdot\sigma}\xi_s + m(\sqrt{-p\cdot\sigma})\xi_s = 0
\end{aligned}$$

In the second line we use $p\cdot\sigma = -(-p\cdot\sigma)^{1/2}(-p\cdot\sigma)^{1/2}$. In the third line, we use that $\sqrt{-p\cdot\sigma}\sqrt{-p\cdot\bar{\sigma}} = m$. We also use that $\sigma^2(\sqrt{-p\cdot\bar{\sigma}})^*\sigma^2 = \sqrt{-p\cdot\sigma}$: this can be showed by expanding the square root as a Taylor series, and repeatedly applying the identity $\sigma_2\sigma^*\sigma_2 = -\sigma_2$ from Problem Set 9.

We show that $u_s(p)$ give a complete basis of solutions for any p. Solutions only exist on the mass-shell, so without loss of generality consider any fixed $p = (\omega_{\mathbf{p}}, \mathbf{p})$. It follows from the discussion in (b) that the solution space is 2-dimensional. Furthermore,

$$u_+^\dagger(\mathbf{p})u_-(-\mathbf{p}) = \xi_+\sqrt{-p\cdot\bar{\sigma}}^\dagger\sqrt{-p\cdot\sigma}\xi_- = \xi_+\sqrt{-p\cdot\bar{\sigma}}\sqrt{-p\cdot\sigma}\xi_- = m\xi_+\xi_- = 0$$

Therefore, the $u_\pm(\mathbf{p})$ are always orthogonal. Since neither vanish for any $\mathbf{p}$, they are always linearly independent, and thus span the solution space for given momentum.

(d) Write down the mode expansion for the quantum operator ψ_L

The mode expansion is given by

$$\begin{aligned}
\psi_L(x) &= \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\left(a_{\mathbf{k}}^{(s)}u_{Ls}(\mathbf{k})e^{ik\cdot x} + a_{\mathbf{k}}^{\dagger(s)}v_{Ls}(\mathbf{k})e^{-ik\cdot x}\right) \\
\psi_L^\dagger(x) &= \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\left(a_{\mathbf{k}}^{\dagger(s)}u_{Ls}^\dagger(\mathbf{k})e^{-ik\cdot x} + a_{\mathbf{k}}^{(s)}v_{Ls}^\dagger(\mathbf{k})e^{ik\cdot x}\right)
\end{aligned}$$

where the relation between $u_L(\mathbf{p})$ and $v_L(\mathbf{p})$ is given in part (b).

(e) Define the vacuum and construct the single particle states, with proper normalization. Discuss the differences between the particles in this theory and those of the Dirac theory.

The vacuum is defined by the state annihilated by all the $a_{\mathbf{k}}^{(s)}$'s, i.e. $a_{\mathbf{k}}^{(s)}|0\rangle = 0$.

The single particle states are defined as usual (with Lorentz invariant inner product):

$$|k,s\rangle = \sqrt{2\omega_{\mathbf{k}}}a_{\mathbf{k}}^{\dagger(s)}|0\rangle$$

Note that the theory for Majoranas (as constrasted with the Dirac theory) has only particles with no antiparticles. Equivalently, each particle is its own antiparticle.

Question 3: Gaussian Integrals for Grassmann Variables (16 points)
Show the following identities:

$$\int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) e^{-\theta_i^* A_{ij} \theta_j} = \det A$$

$$\int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) \theta_k \theta_l^* e^{-\theta_i^* A_{ij} \theta_j} = (A^{-1})_{kl} \det A$$

We first expand out the exponential as a Taylor series:

$$e^{-\theta_i^* A_{ij} \theta_j} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \prod_{k=1}^{n} A_{i_k j_k} \theta_{i_k}^* \theta_{j_k} = \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{k=1}^{n} A_{i_k j_k} \theta_{j_k} \theta_{i_k}^*$$

Note that any term of order $k > N$ will vanish: there must be at least 2 of some θ_k, thus the term is zero because $\theta_k^2 = 0$. Furthermore, any term with $k < N$ vanishes when taking the $2N$ integrals, because $\int d\theta \, 1 = 0$. Therefore only the Nth order term survives:

$$\begin{aligned}
\int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) e^{-\theta_i^* A_{ij} \theta_j} &= \frac{1}{N!} A_{i_1 j_1} \cdots A_{i_N j_N} \int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) (\theta_{j_1} \theta_{i_1}^*) \cdots (\theta_{j_N} \theta_{i_N}^*) \\
&= \frac{(-1)^{N(N-1)/2}}{N!} A_{i_1 j_1} \cdots A_{i_N j_N} \int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) (\theta_{j_1} \cdots \theta_{j_N}) (\theta_{i_1}^* \cdots \theta_{i_N}^*) \\
&= \frac{(-1)^{N(N-1)/2}}{N!} A_{i_1 j_1} \cdots A_{i_N j_N} \int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) \epsilon_{j_1 \cdots j_N} \epsilon_{i_1 \cdots i_N} (\theta_1 \cdots \theta_N) (\theta_1^* \cdots \theta_N^*) \\
&= \frac{1}{N!} A_{i_1 j_1} \cdots A_{i_N j_N} \epsilon_{j_1 \cdots j_N} \epsilon_{i_1 \cdots i_N} \int d\theta_N^* d\theta_N \cdots d\theta_1^* d\theta_1 (\theta_1 \theta_1^*) \cdots (\theta_N \theta_N^*) \\
&= \frac{1}{N!} \epsilon_{i_1 \cdots i_N} \epsilon_{j_1 \cdots j_N} A_{i_1 j_1} \cdots A_{i_N j_N} = \det A
\end{aligned}$$

In line 2, we have moved all of the θ's to the left and θ^*'s to the right, picking up a total factor of $(-1)^{(N-1)+(N-2)+\cdots+1} = (-1)^{N(N-1)/2}$. In line 3 we use that $\theta_{i_1} \cdots \theta_{i_N} = \epsilon_{i_1 \cdots i_N} \theta_1 \cdots \theta_N$, and likewise for the θ^*'s. In line 4 we change the order of the Grassmann variables again, picking up another factor of $(-1)^{N(N-1)/2}$ cancelling the factor from before. Note also that it doesn't matter what order we place the $d\theta_i^* d\theta_i$'s, since pairs of Grassmann variables always commute with each other. In line 5 we do the Grassmann integrals successively. Finally we recognize the Leibniz formula for the determinant.

To obtain the second equation, we differentiate both sides of the first equation by A_{lk}. On the left hand side we get

$$\int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) e^{-\theta_i^* A_{ij} \theta_j} (-\theta_l^* \theta_k) = \int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) e^{-\theta_i^* A_{ij} \theta_j} \theta_k \theta_l^*$$

On the right-hand side we can Jacobi's formula, $\delta \det A = \det A \; \mathrm{Tr}(A^{-1} \delta A) = (A^{-1})_{kl} \det A$, for δA zero everywhere except the kl-th entry. Therefore,

$$\int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) e^{-\theta_i^* A_{ij} \theta_j} \theta_k \theta_l^* = (A^{-1})_{kl} \det A$$

Question 4: Yukawa Theory (24 points)

Consider the Yukawa theory discussed in lecture,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 = i\bar{\psi}(\partial\!\!\!/ - m)\psi = g\phi\bar{\psi}\psi$$

Denote the propagator of a ϕ particle by a dashed line, and that of ψ by a solid line (with arrow). We will call p the particle excitation of ψ, and $\bar{p}$ the antiparticle excitation of ψ.

(a) Consider the process

$$\bar{p} + \bar{p} \to \bar{p} + \bar{p}$$

Draw the lowest order Feynman diagrams, and write down the corresponding scattering amplitude. Take the initial and final states to have momenta and polarizations (p_1, s_1) (p_2, s_2) and (p_1', s_1') (p_2', s_2') respectively.

We identify the Mandelstam variables $s = -(p_1 + p_2)^2$, $t = -(p_1 - p_1')^2$, and $u = -(p_1 - p_2')^2$. There is an t-channel and a u-channel diagram. Using the Feynman rules, the amplitude is:

$$\begin{aligned}
\mathcal{M} &= ig^2 \left[\frac{\bar{v}_{s_1}(p_1)v_{s_1'}(p_1')\bar{v}_{s_2}(p_2)v_{s_2'}(p_2')}{(p_1 - p_1')^2 + m_\phi^2 - i\epsilon} - \frac{\bar{v}_{s_1}(p_1)v_{s_2'}(p_2')\bar{v}_{s_2}(p_2)v_{s_1'}(p_1')}{(p_1 - p_2')^2 + m_\phi^2 - i\epsilon} \right] \\
&= -ig^2 \left[\frac{\bar{v}_{s_1}(p_1)v_{s_1'}(p_1')\bar{v}_{s_2}(p_2)v_{s_2'}(p_2')}{t - m_\phi^2 + i\epsilon} - \frac{\bar{v}_{s_1}(p_1)v_{s_2'}(p_2')\bar{v}_{s_2}(p_2)v_{s_1'}(p_1')}{u - m_\phi^2 + i\epsilon} \right]
\end{aligned}$$

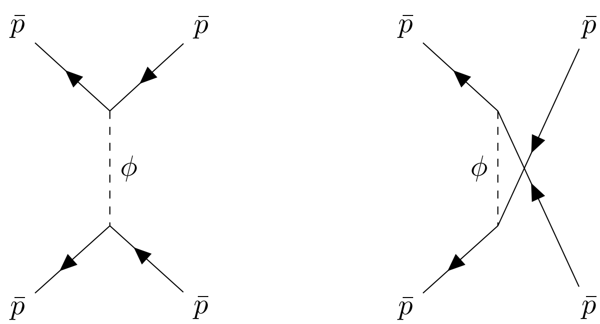

Figure 1: $\bar{p}\bar{p} \to \bar{p}\bar{p}$ scattering

(b) Consider the process

$$p + \bar{p} \to \phi + \phi$$

Draw the lowest order Feynman diagrams, and write down the corresponding scattering amplitude. Take the initial and final states to have momenta and polarizations (p_1, s_1) (p_2, s_2) and p_1', p_2'.

There is an t-channel and a u-channel diagram. Using the Feynman rules, the amplitude is:

$$\mathcal{M} = g^2 \left[\frac{\bar{v}_{s_2}(p_2)(m_p + i(p\!\!\!/_1 - p\!\!\!/_1'))u_{s_1}(p_1)}{t - m_p^2 + i\epsilon} + \frac{\bar{v}_{s_2}(p_2)(m_p + i(p\!\!\!/_1 - p\!\!\!/_2'))u_{s_1}(p_1)}{u - m_p^2 + i\epsilon} \right]$$

We have made use of the expression for the fermion propagator, $-\dfrac{1}{ik\!\!\!/ - m + i\epsilon} = \dfrac{m + ik\!\!\!/}{k^2 + m^2 - i\epsilon}$.

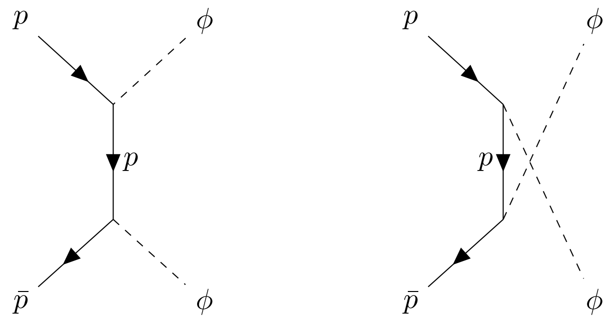

Figure 2: $p\bar{p} \to \phi\phi$ scattering

(c) Consider the process

$$p + \phi \to p + \phi$$

Draw the lowest order Feynman diagrams, and write down the corresponding scattering amplitude. Take the initial and final states to have momenta and polarizations (p_1, s_1), p_2 and (p_1', s_1'), p_2'.

There is an s-channel and a u-channel diagram. Using the Feynman rules, the amplitude is:

$$\mathcal{M} = g^2 \left[\frac{\bar{u}_{s_1'}(p_1')(m_p + i(\not{p}_1 + \not{p}_2))u_{s_1}(p_1)}{s - m_p^2 + i\epsilon} + \frac{\bar{u}_{s_1'}(p_1')(m_p + i(\not{p}_1 - \not{p}_2'))u_{s_1}(p_1)}{u - m_p^2 + i\epsilon} \right]$$

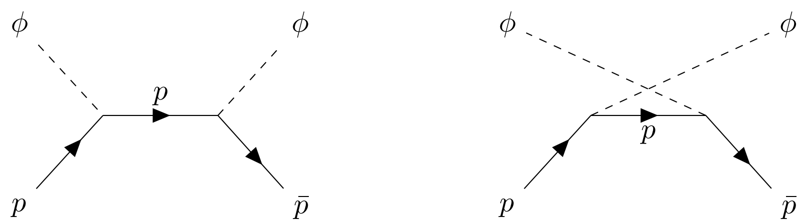

Figure 3: $p\phi \to p\phi$ scattering

8.323 Relativistic Quantum Field Theory I
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