Recitation 8

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1 Vector and Axial Symmetries

1.1 Chiralty

We work in the Weyl representation:

$$\gamma^0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Here $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4$, and satisfies

$$(\gamma_5)^\dagger = \gamma_5, \qquad \{\gamma^\mu, \gamma_5\} = 0$$

This representation is useful for the purpose of chirality, because γ^5 is diagonal. The $+1$ eigenspace corresponds to left-handed spinors ψ_L, spanned by Dirac spinors non-zero in the top 2 components. Similarly, the -1 eigenspace corresponds to right-handed spinors ψ_R, spanned by Dirac spinors non-zero in the bottom 2 components. Therefore, we a Dirac spinor can be written as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

Note that this is consistent with last recitation, where we have written the Dirac representation as the direct sum of left and right-handed Weyl representations:

$$\text{Dirac} = (\tfrac{1}{2}, 0) \oplus (0, \tfrac{1}{2}) = \text{Weyl}_L + \text{Weyl}_R$$

An arbitrary Dirac spinor can be projected onto its left/right-handed subspaces via projection operators:

$$P_L = \frac{1}{2}(1+\gamma_5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_R = \frac{1}{2}(1-\gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Because these are projectors, they satisfy:

$$P_L^2 = P_L, \qquad P_R^2 = P_R, \qquad P_L P_R = P_R P_L = 0, \qquad P_L + P_R = 1$$

Slightly abusing notation by writing $\psi_{L/R} = P_{L/R}\psi$, some useful identities are:

$$P_L\gamma^\mu = \gamma^\mu P_R, \qquad P_R\gamma^\mu = \gamma^\mu P_L, \qquad \bar{\psi}_L = \bar{\psi} P_R, \qquad \bar{\psi}_R = \bar{\psi} P_L$$

These are very important in the Standard Model because it is a chiral theory: the particles mediating forces couple differently to left and right-handed fermions.

1.2 Vector and Axial Transformations

Now consider the Lagrangian for a Dirac spinor,

$$\mathcal{L} = -i\bar{\psi}(\partial\!\!\!/ - m)\psi$$

This is invariant under the vector transformation $\psi(x) \to e^{i\alpha}\psi(x)$. The conserved quantity is the vector current, $j^\mu_V = \bar{\psi}\gamma^\mu\psi$. From the decomposition above, we see that this rotates left and right-handed spinors in the same way:

$$\psi_L \to e^{i\alpha}\psi_L, \qquad \psi_R \to e^{i\alpha}\psi_R$$

We may also consider the axial transformation $\psi(x) \to e^{i\alpha\gamma_5}\psi(x)$. This rotates left and right-handed spinors in opposite ways:

$$\psi_L \to e^{i\alpha}\psi_L, \qquad \psi_R \to e^{-i\alpha}\psi_R$$

Note that the a Dirac mass term $m\bar{\psi}\psi$ is not invariant under this transformation:

$$m\bar{\psi}\psi \to m\psi^\dagger e^{-i\alpha\gamma_5}\gamma^0 e^{i\alpha\gamma_5}\psi = m\psi^\dagger\gamma^0 e^{2i\alpha(\gamma_5)^2}\psi = e^{2i\alpha}m\bar{\psi}\psi$$

Meanwhile, the kinetic term is invariant:

$$-i\bar{\psi}\partial\!\!\!/\psi \to -i\psi^\dagger e^{-i\alpha\gamma_5}\gamma^0\partial\!\!\!/ e^{i\alpha\gamma_5}\psi = -i\bar{\psi}e^{i\alpha\gamma_5}\partial\!\!\!/ e^{i\alpha\gamma_5}\psi = -i\bar{\psi}\partial\!\!\!/ e^{-i\alpha\gamma_5}e^{i\alpha\gamma_5}\psi = -i\bar{\psi}\partial\!\!\!/\psi$$

Therefore, this is a symmetry of $\mathcal{L}$ only for a massless Dirac spinor $m = 0$. The conserved quantity is the vector current, $j^\mu_A = \bar{\psi}\gamma^\mu\gamma_5\psi$.

To shed more light on this, it is instructive to write the $\mathcal{L}$ in terms of ψ_L and ψ_R:

$$\begin{aligned}
\bar{\psi}\partial\!\!\!/\psi &= \bar{\psi}\partial\!\!\!/(P_L + P_R)\psi = \bar{\psi}\partial\!\!\!/ P_L P_L\psi + \bar{\psi}\partial\!\!\!/ P_R P_R\psi = \bar{\psi}P_R\partial\!\!\!/ P_L\psi + \bar{\psi}P_L\partial\!\!\!/ P_R\psi = \bar{\psi}_L\partial\!\!\!/\psi_L + \bar{\psi}_R\partial\!\!\!/\psi_R \\
m\bar{\psi}\psi &= \bar{\psi}(P_L + P_R)\psi = \bar{\psi}P_L P_L\psi + \bar{\psi}P_R P_R\psi = \bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R
\end{aligned}$$

The kinetic term decomposes into a kinetic term for ψ_L and a kinetic term for ψ_R. The mass term couples the left and right-handed Weyl spinors. When $m = 0$ the theory becomes decoupled, and the Dirac Lagrangian reduces to a free ψ_L and free ψ_R.

How now, do the vector and axial symmeties manifest? The massless Lagrangian $\mathcal{L} = -i(\bar{\psi}_L\partial\!\!\!/\psi_L + \bar{\psi}_R\partial\!\!\!/\psi_R)$ has the symmetries $\psi_L \to e^{i\alpha_L}\psi_L$, and $\psi_R \to e^{i\alpha_L}\psi_R$, where we rotate the left spinor by α_L, and the right by α_R independently. Equivalently, we may rotate left/right by the same $e^{i\alpha_V}$, or by opposites $e^{\pm\alpha_A}$. These are the vector and axial symmetries. The coupling term $m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$ is not invariant if we rotate $\psi_{L,R}$ differently.

1.3 The Chiral Anomaly

The moral of the previous section is that the Lagrangian for a massless Dirac fermion enjoys the vector and axial symmetries. Noethers theorem tells us that the corresponding currents are conserved:

$$\partial_\mu j^\mu_V = \partial_\mu j^\mu_A = 0$$

However, this symmetry is broken when we quantize the theory: if we couple a massless Dirac fermion to an EM field, we find that the Noether current is not conserved:

$$\partial_\mu j^0_A = \frac{\alpha}{4\pi}e^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \frac{\alpha}{4\pi}F \wedge F$$

Here $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength, which may be familiar to you from your undergraduate EM class.

Generally, an anomaly is a symmetry of a classical theory, which is not a symmetry of the corresponding quantum theory. The one we have just seen is called the chiral or Adler-Bell-Jackiw anomaly. It is responsible for much physics, such as the very short lifetime of the neutral pion, by mediating the would-be-forbidden process $\pi^0 \to \gamma\gamma$.

Where does Noether's theorem break down? The anomaly is easiest to see using the path-integral, which is the central object in a quantum theory. We can write

$$Z = \int D\psi D\bar{\psi} e^{iS[\bar{\psi},\psi]} = \int D\psi' D\bar{\psi}' e^{iS[\bar{\psi}',\psi']}$$

Given a symmetry of the action $(\psi, \bar{\psi}) \to (\psi', \bar{\psi}')$, the Lagrangian density must change by only a total derivative (i.e. a surface term):

$$iS[\psi', \bar{\psi}'] = iS[\psi, \bar{\psi}] + \int d^4x \partial_\mu j^\mu_A(x)$$

Noether's procedure shows how to construct j^μ_A such that $\partial_\mu j^\mu_A(x) = 0$. However, this is predicated on the assumption that the path-integral measure is invariant under our symmetry, $D\psi' D\bar{\psi}' = D\psi D\bar{\psi}$. In general this is not true. Instead,

$$D\psi' D\bar{\psi}' = D\psi D\bar{\psi} \det \Delta^{-1} = D\psi D\bar{\psi} e^{\ln \det \Delta^{-1}} = D\psi D\bar{\psi} e^{\mathrm{Tr} \ln \Delta^{-1}} = D\psi D\bar{\psi} e^{-\int d^4x \ln \Delta}$$

Putting everything together, we have

$$\begin{aligned}
\int D\psi D\bar{\psi} e^{iS[\bar{\psi},\psi]} &= \int D\psi' D\bar{\psi}' e^{iS[\bar{\psi}',\psi']} \\
&= \int D\psi D\bar{\psi} e^{-\int d^4x \ln \Delta} e^{iS[\psi,\bar{\psi}] + \int d^4x \partial_\mu j^\mu_A(x)} \\
&= \int D\psi D\bar{\psi} e^{iS[\psi,\bar{\psi}]} e^{\int d^4x (\partial_\mu j^\mu_A(x) - \ln \Delta)}
\end{aligned}$$

For these to be equal, we must have that

$$\partial_\mu j^\mu_A = \ln \Delta$$

That is, the current j^μ_A is no longer conserved.

2 Weyl, Dirac, Majorana

Moved to next time.

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