5.19 Spark ignited spherical combustion—Solution

Part (a) ¾ *See also Problem 3.7 Solution for fuller discussion of part (a)*

The density distribution is known:

$$\rho(r,t) = \begin{cases} \rho_2 & (r < V_f t) \\ \rho_1 & (r > V_f t) \end{cases} \qquad (1)$$

The velocity field is related to the density via the mass conservation law. Form A of the integral mass conservation equation is

$$\frac{d}{dt}\int_{CV(t)} \rho\, dV + \int_{CS(t)} \rho(\vec{v} - \vec{v}_c) \cdot \vec{n}\, dA = 0 \qquad (2)$$

where $\vec{v}_c$ is the local velocity of the control surface. We choose *a spherical control surface with a fixed radius* r from the origin (that is, $\vec{v}_c = 0$), and apply (2) at an instant of time t.

Consider first the case $r<V_f t$ (see Fig. 1) where r is inside the region filled by the combustion products of density ρ_2. In this case (2) reduces to

$$\frac{d}{dt}\left(\rho_2 \frac{4\pi r^3}{3}\right) + \rho_2 v 4\pi r^2 = 0 \qquad (r<V_f t) \qquad (3)$$

where v is the radial component of velocity. This gives (recall that r is the radius of the control volume and has a *fixed* value)

$$v = 0 \qquad (r < V_f t) \qquad (4)$$

For $r>V_f t$, the control volume contains two regions of constant density, one inside the flame front and the other one outside it (see Fig. 1), as expressed by (1). In this case (2) reduces to

$$\frac{d}{dt}\left\{\rho_2 \frac{4\pi}{3}\left(V_f t\right)^3 + \rho_1 \frac{4\pi}{3}\left[r^3 - \left(V_f t\right)^3\right]\right\} + \rho_1 4\pi r^2 v = 0 \qquad (5)$$

which, after differentiation with respect to time, gives

$$v = \left(1 - \frac{\rho_2}{\rho_1}\right)\frac{V_f^3 t^2}{r^2} \qquad (r>V_f t) \tag{6}$$

Based on (4) and (6), the velocity field in the gas can be expressed in dimensionless form as follows:

$$\begin{aligned} \frac{v}{V_f} &= 0 && \left(\frac{r}{V_f t} < 1\right) \\ \frac{v}{V_f} &= \left(1 - \frac{\rho_2}{\rho_1}\right)\left(\frac{r}{V_f t}\right)^{-2} && \left(\frac{r}{V_f t} > 1\right) \end{aligned} \tag{7}$$

Fig. 3 shows this dimensionless solution with the density ratio as a parameter. Note that the solution has the form

$$\frac{v}{V_f} = f\left(\frac{\rho_2}{\rho_1}, \frac{r}{V_f t}\right) \tag{8}$$

which expresses the ratio of the local velocity and the flame speed in terms of only *two* independent variables (those in the brackets), while v in the actual problem is determined by *three* independent quantities. The similarity law (or scaling law) (8) can also be derived directly by dimensional analysis.

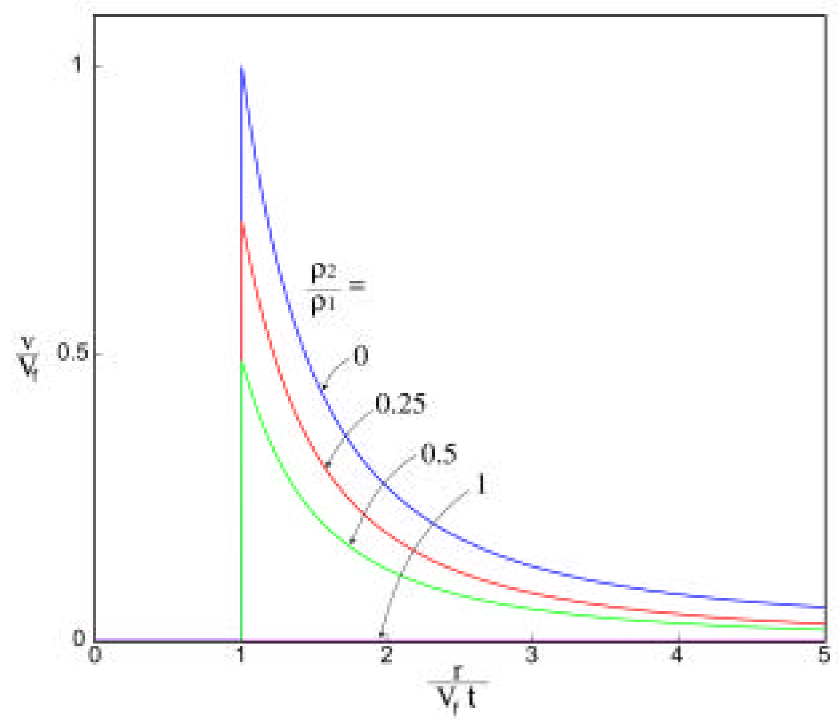


Fig. 3: Fluid velocity distribution in dimensionless coordinates.

5.19 Part (b)

The flame speed U_f relative to the gas just ahead of the flame front is by definition the difference between the flame speed in fixed reference frame and the gas velocity just ahead of the flame front (also measured in the fixed reference frame), that is,

$$U_f \equiv V_f - (v)_{r=V_f t+\varepsilon} \tag{9}$$

From our solution (7), we see that

$$(v)_{r=V_f t+\varepsilon} = \left(1 - \frac{\rho_2}{\rho_1}\right) V_f \tag{10}$$

From (9) and (10), we get

$$U_f = \frac{\rho_2}{\rho_1} V_f = \text{constant} \tag{11}$$

5.19 Part (c)

In the region $r > V_f t$, the pressure gradient is given by Euler's equation,

$$-\frac{\partial p}{\partial r} = \rho_1 \left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}\right) \tag{12}$$

Substituting for v from (6), (12) becomes

$$-\frac{\partial p}{\partial r} = 2(\rho_1 - \rho_2) V_f^3 t \left[\frac{1}{r^2} + \left(\frac{\rho_1 - \rho_2}{\rho_1}\right)\frac{V_f^3 t^3}{r^5}\right] \tag{13}$$

Integrating this equation from some point at $r > V_f t$, where the pressure is $p(r,t)$, to infinity, where the pressure is p_∞, we find that

$$P = \frac{2}{s}\left(1 - \frac{\gamma}{4s^3}\right) \qquad (r > V_f t) \tag{14}$$

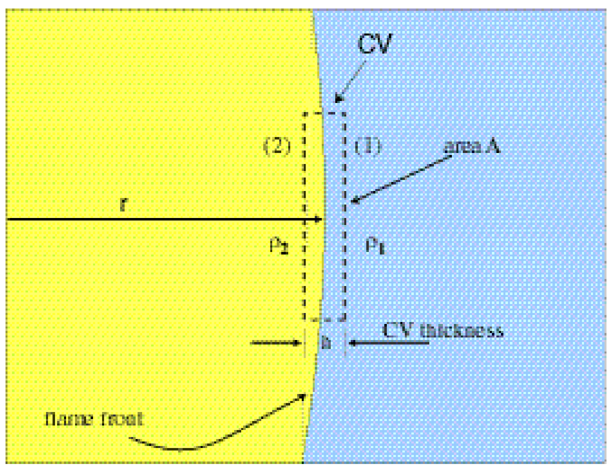
where

$$P = \frac{p - p_\infty}{(\rho_1 - \rho_2) V_f^2}\ , \ s = \frac{r}{V_f t}, \text{ and } \gamma = \frac{\rho_1 - \rho_2}{\rho_1} \tag{15}$$

are dimensionless variables.

5.19 Part (d)

The pressure in the static gas at $r<V_f t$ must be uniform. There is no reason to think that this pressure is continuous across the flame front, however, since the velocity is discontinuous, and so is the momentum flux. The pressure in the product gas can be determined by applying the momentum theorem to a small "pill-box" of a control volume that straddles the frame front, with one side being in the region (1) just ahead the flame and the other in the region (2) just behind it and the flame halfway between (see Fig. 2 in the problem statement, reproduced below). Let the pillbox have a small area A in the plane of the flame front with a small thickness h that approaches zero. Furthermore, let the pillbox *move with the flame front*.



Now apply Form A of the momentum theorem,

$$\frac{d}{dt}\int_{CV} \rho v_x dV + \int_{CS} \rho v_x (v_x - v_{CS}) dA = \left(F_x\right)_{CV} \qquad (16)$$

Choosing an inertial *reference frame fixed in the origin* (relative to which our CV is moving at speed V_f), this becomes

$$\frac{d}{dt}\left[\left(\frac{\rho_1 v_1 + \rho_2 v_2}{2}\right) hA\right] + \rho_1 v_1 (v_1 - V_f)A - \rho_2 v_2 (v_2 - V_f)A = (p_2 - p_1)A \qquad (17)$$

From part (a) we have that

$$v_1 = \frac{(\rho_1 - \rho_2)V_f}{\rho_1}$$

$$v_2 = 0 \qquad (18)$$

Substituting these into (17) and taking the limit $h \to 0$, we obtain

$$p_2 - p_1 = -\frac{(\rho_1 - \rho_2)}{\rho_1} \rho_1 V_f^2 \qquad (19)$$

or, in dimensionless terms [see (15)],

$$P_2 = P_1 - (1 - \gamma) \quad , \qquad (20)$$

which indicates, surprisingly, that the pressure is *lower* in the products region just behind the flame front than it is just ahead of it, γ being less than unity. This begs the question, how can this be, given that the gas outside is being *pushed* outward?

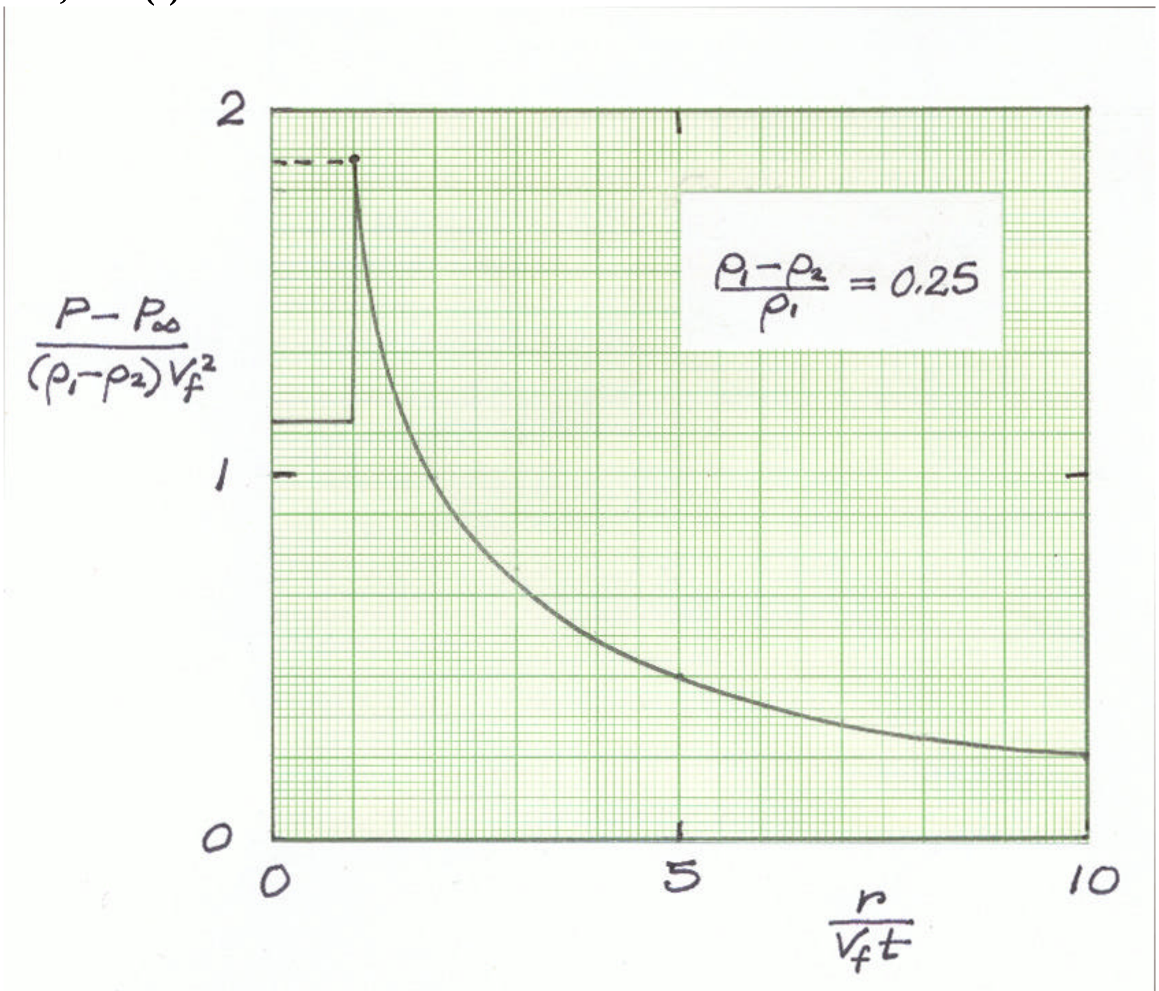
The answer is found by thinking of what happens to a fluid particle when the flame front overtakes it. A millisecond before the flame front arrives, the particle is moving outward at the speed v_1 given by (18). Two milliseconds later, it is standing still (and its density has dropped from ρ_1 to ρ_2). What force decelerated it as it traversed the flame front? Clearly, the radial pressure gradient must be *negative* inside the flame front; that is, the pressure ahead if it must be higher than behind it, consistent with our result.

What, then, pushes the fluid outside away from the product gases? It is the momentum flux, not the pressure. Relative to the inertial frame fixed in the center, the combustion process inside the flame front generates a net radial momentum flux from the flame toward the outside. This gives raises the pressure just ahead of it and drives the fluid outward.

Part (e)

See next page.

5.19, Part (e)



2
1
0
0
5
10
$\frac{P - P_\infty}{(\rho_1 - \rho_2) V_f^2}$
$\frac{r}{V_f t}$
$\frac{\rho_1 - \rho_2}{\rho_1} = 0.25$