CHAPTER 9 Polar Coordinates and Complex Numbers

CHAPTER 10 Infinite Series

CHAPTER 11 Vectors and Matrices

CHAPTER 12 Motion along a Curve

CHAPTER 13 Partial Derivatives

CHAPTER 10

Infinite Series

Infinite series can be a pleasure (sometimes). They throw a beautiful light on $\sin x$ and $\cos x$. They give famous numbers like π and e. Usually they produce totally unknown functions—which might be good. But on the painful side is the fact that an infinite series has infinitely many terms.

It is not easy to know the sum of those terms. More than that, it is not certain that there *is* a sum. We need tests, to decide if the series converges. We also need ideas, to discover what the series converges *to*. Here are examples of ***convergence***, ***divergence***, and ***oscillation***:

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\cdots=2 \qquad 1+1+1+\cdots=\infty \qquad 1-1+1-1\cdots=?$$

The first series converges. Its next term is 1/8, after that is 1/16—and every step brings us halfway to 2. The second series (the sum of 1's) obviously diverges to infinity. The oscillating example (with 1's and -1's) also fails to converge.

All those and more are special cases of one infinite series which is absolutely the most important of all:

$$\textbf{\textit{The geometric series is }} 1+x+x^2+x^3+\cdots=\frac{1}{1-x}.$$

This is a series of *functions*. It is a "power series." When we substitute numbers for x, the series on the left may converge to the sum on the right. We need to know when it doesn't. Choose $x=\frac{1}{2}$ and $x=1$ and $x=-1$:

$1+\frac{1}{2}+(\frac{1}{2})^2+\cdots$ is the convergent series. Its sum is $\dfrac{1}{1-\frac{1}{2}}=2$

$1+1+1+\cdots$ is divergent. Its sum is $\dfrac{1}{1-1}=\dfrac{1}{0}=\infty$

$1+(-1)+(-1)^2+\cdots$ is the oscillating series. Its sum should be $\dfrac{1}{1-(-1)}=\dfrac{1}{2}.$

The last sum bounces between one and zero, so at least its average is $\frac{1}{2}$. At $x=2$ there is no way that $1+2+4+8+\cdots$ agrees with $1/(1-2)$.

This behavior is typical of a power series—to converge in an interval of x's and to diverge when x is large. The geometric series is safe for x between -1 and 1. Outside that range it diverges.

The next example shows a ***repeating decimal*** 1.111...:

$$\text{Set } x = \frac{1}{10}. \text{ The geometric series is } 1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots$$

The decimal 1.111... is also the fraction $1/(1-\frac{1}{10})$, which is 10/9. ***Every fraction leads to a repeating decimal. Every repeating decimal adds up*** (through the geometric series) ***to a fraction.***

To get 3.333..., just multiply by 3. This is 10/3. To get 1.0101..., set $x = 1/100$. This is the fraction $1/(1-\frac{1}{100})$, which is 100/99.

Here is an unusual decimal (which eventually repeats). I don't really understand it:

$$\frac{1}{243} = .004\ 115\ 226\ 337\ 448\ \ldots$$

Most numbers are not fractions (or repeating decimals). A good example is π:

$$\pi = 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \cdots.$$

This is 3.1415..., a series that certainly converges. We happen to know the first billion terms (the billionth is given below). Nobody knows the 2 billionth term. Compare that series with this one, which also equals π:

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$$

That *alternating series* is really remarkable. It is typical of this chapter, because its pattern is clear. We know the 2 billionth term (it has a minus sign). This is not a geometric series, but in Section 10.1 it comes from a geometric series.

Question Does this series actually converge? What if all signs are $+$?
Answer The alternating series converges to π (Section 10.3). The positive series diverges to infinity (Section 10.2). The terms go to zero, but their sum is infinite.

This example begins to show what the chapter is about. Part of the subject deals with special series, adding to 10/9 or π or e^x. The other part is about series in general, adding to numbers or functions that nobody has heard of. The situation was the same for integrals—they give famous answers like $\ln x$ or unknown answers like $\int x^x\,dx$. The sum of $1 + 1/8 + 1/27 + \cdots$ is also unknown—although a lot of mathematicians have tried.

The chapter is not long, but it is full. The last half studies *power series.* We begin with a linear approximation like $1 + x$. Next is a quadratic approximation like $1 + x + x^2$. In the end we match *all* the derivatives of $f(x)$. This is the "***Taylor series,***" a new way to create functions—not by formulas or integrals but by infinite series.

No example can be better than $1/(1-x)$, which dominates Section 10.1. Then we define convergence and test for it. (Most tests are really comparisons with a geometric series.) The second most important series in mathematics is the *exponential series* $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$. It includes the series for $\sin x$ and $\cos x$, because of the formula $e^{ix} = \cos x + i \sin x$. Finally a whole range of new and old functions will come from Taylor series.

In the end, all the key functions of calculus appear as "*infinite polynomials*" (except the step function). This is the ultimate voyage from the linear function $y = mx + b$.

10.1 The Geometric Series

We begin by looking at both sides of the geometric series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}. \tag{1}$$

How does the series on the left produce the function on the right? How does $1/(1-x)$ produce the series? Add up two terms of the series, then three terms, then n terms:

$$1 + x = \frac{1-x^2}{1-x} \qquad 1 + x + x^2 = \frac{1-x^3}{1-x} \qquad 1 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}. \tag{2}$$

For the first, $1 + x$ times $1 - x$ equals $1 - x^2$ by ordinary algebra. The second begins to make the point: $1 + x + x^2$ times $1 - x$ gives $1 - x + x - x^2 + x^2 - x^3$. Between 1 at the start and $-x^3$ at the end, everything cancels. The same happens in all cases: $1 + \cdots + x^{n-1}$ times $1 - x$ leaves 1 at the start and $-x^n$ at the end. This proves equation (2)—the sum of n terms of the series.

For the whole series we will push n towards infinity. On a graph you can see what is happening. Figure 10.1 shows $n = 1$ and $n = 2$ and $n = 3$ and $n = \infty$.

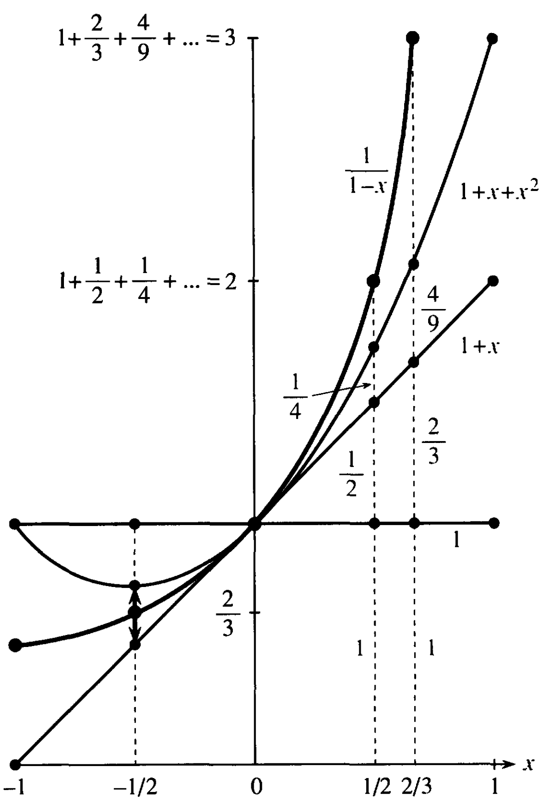

Fig. 10.1 Two terms, then three terms, then full series:

$$1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

$$\begin{array}{r l}
 & 1 + x + x^2 + \cdots \\
1 - x & \overline{)\,1 \qquad\qquad\quad} \\
 & \underline{1 - x} \\
 & \quad x \\
 & \quad \underline{x - x^2} \\
 & \qquad x^2 \\
 & \qquad \underline{x^2 - x^3} \\
 & \qquad\quad \cdots
\end{array}$$

The infinite sum gives a finite answer, provided x ***is between*** -1 ***and*** 1. ***Then*** x^n ***goes to zero***:

$$\frac{1-x^n}{1-x} \to \frac{1}{1-x}.$$

Now start with the function $1/(1-x)$. How does it produce the series? One way is elementary but brutal, to do "long division" of $1 - x$ into 1 (next to the figure). Another way is to look up the binomial formula for $(1-x)^{-1}$. That is cheating—we want to discover the series, not just memorize it. The successful approach uses calculus. ***Compute the derivatives of*** $f(x) = 1/(1-x)$:

$$f' = (1-x)^{-2} \qquad f'' = 2(1-x)^{-3} \qquad f''' = 6(1-x)^{-4} \quad \cdots \tag{3}$$

At $x = 0$ these derivatives are 1, 2, 6, 24, ... Notice how -1 from the chain rule keeps them positive. *The nth derivative at* $x = 0$ *is n factorial*:

$$f(0) = 1 \qquad f'(0) = 1 \qquad f''(0) = 2 \qquad f'''(0) = 6 \qquad \cdots \qquad f^{(n)}(0) = n!.$$

Now comes the idea. ***To match the series with*** $1/(1-x)$, ***match all those derivatives at*** $x = 0$. Each power x^n gets one derivative right. Its derivatives at $x = 0$ are zero, except the nth derivative, which is $n!$ By adding all powers we get every derivative right—so the geometric series matches the function:

$$1 + x + x^2 + x^3 + \cdots \textbf{ has the same derivatives at } x = 0 \textbf{ as } 1/(1-x).$$

The linear approximation is $1 + x$. Then comes $\frac{1}{2}f''(0)x^2 = x^2$. The third derivative is supposed to be 6, and x^3 is just what we need. *Through its derivatives, the function produces the series.*

With that example, you have seen a part of this subject. The geometric series diverges if $|x| \geqslant 1$. Otherwise it adds up to the function it comes from (when $-1 < x < 1$). To get familiar with other series, we now apply algebra or calculus—to reach the square of $1/(1-x)$ or its derivative or its integral. The point is that these operations are applied *to the series.*

The best I know is to show you eight operations that produce something useful. At the end we discover series for ln 2 and π.

1. ***Multiply the geometric series by a or ax***:

$$a + ax + ax^2 + \cdots = \frac{a}{1-x} \qquad ax + ax^2 + ax^3 + \cdots = \frac{ax}{1-x}. \tag{4}$$

The first series fits the decimal 3.333.... In that case $a = 3$. The geometric series for $x = \frac{1}{10}$ gave $1.111\ldots = 10/9$, and this series is just three times larger. Its sum is 10/3.

The second series fits other decimals that are fractions in disguise. To get 12/99, choose $a = 12$ and $x = 1/100$:

$$.121212\ldots = \frac{12}{100} + \frac{12}{100^2} + \frac{12}{100^3} + \cdots = \frac{12/100}{1 - 1/100} = \frac{12}{99}.$$

Problem 13 asks about .8787... and .123123.... It is usual in precalculus to write $a + ar + ar^2 + \cdots = a/(1-r)$. But we use x instead of r to emphasize that *this is a function*—which we can now differentiate.

2. ***The derivative of the geometric series*** $1 + x + x^2 + \cdots$ ***is*** $1/(1-x)^2$:

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}. \tag{5}$$

At $x = \frac{1}{10}$ the left side starts with 1.23456789. The right side is $1/(1-\frac{1}{10})^2 = 1/(9/10)^2$, which is 100/81. If you have a calculator, divide 100 by 81.

The answer should also be near $(1.11111111)^2$, which is 1.2345678987654321.

3. ***Subtract*** $1 + x + x^2 + \cdots$ ***from*** $1 + 2x + 3x^2 + \cdots$ ***as you subtract functions***:

$$x + 2x^2 + 3x^3 + \cdots = \frac{1}{(1-x)^2} - \frac{1}{(1-x)} = \frac{x}{(1-x)^2}. \tag{6}$$

Curiously, the same series comes from multiplying (5) by x. It answers a question left open in Section 8.4—the average number of coin tosses until the result is heads. This is the sum $1(p_1) + 2(p_2) + \cdots$ from probability, with $x = \frac{1}{2}$:

$$1(\tfrac{1}{2}) + 2(\tfrac{1}{2})^2 + 3(\tfrac{1}{2})^3 + \cdots = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2. \tag{7}$$

The probability of waiting until the nth toss is $p_n = (\frac{1}{2})^n$. The expected value is *two tosses*. I suggested experiments, but now this mean value is exact.

4. ***Multiply series: the geometric series times itself is*** $1/(1-x)$ ***squared***:

$$(1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots) = 1 + 2x + 3x^2 + \cdots. \tag{8}$$

The series on the right is not new! In equation (5) it was the ***derivative*** of $y = 1/(1-x)$. Now it is the ***square*** of the same y. The geometric series satisfies $dy/dx = y^2$, so the function does too. We have stumbled onto a differential equation.

Notice how the series was squared. A typical term in equation (8) is $3x^2$, coming from 1 times x^2 and x times x and x^2 times 1 on the left side. It is a lot quicker to square $1/(1-x)$—but other series can be multiplied when we don't know what functions they add up to.

5. ***Solve*** $dy/dx = y^2$ ***from any starting value—a new application of series***:

Suppose the starting value is $y = 1$ at $x = 0$. The equation $y' = y^2$ gives 1^2 for the derivative. Now a key step: ***The derivative of the equation gives*** $y'' = 2yy'$. At $x = 0$ that is $2 \cdot 1 \cdot 1$. Continuing upwards, the derivative of $2yy'$ is $2yy'' + 2(y')^2$. At $x = 0$ that is $y''' = 4 + 2 = 6$.

All derivatives are factorials: 1, 2, 6, 24, ... We are matching the derivatives of the geometric series $1 + x + x^2 + x^3 + \ldots$. Term by term, we rediscover the solution to $y' = y^2$. The solution starting from $y(0) = 1$ is $y = 1/(1-x)$.

A different starting value is -1. Then $y' = (-1)^2 = 1$ as before. The chain rule gives $y'' = 2yy' = -2$ and then $y''' = 6$. With alternating signs to match these derivatives, the solution starting from -1 is

$$y = -1 + x - x^2 + x^3 \cdots = -1/(1+x). \tag{9}$$

It is a small challenge to recognize the function on the right from the series on the left. The series has $-x$ in place of x; then multiply by -1. The sum $y = -1/(1+x)$ also satisfies $y' = y^2$. *We can solve differential equations from all starting values by infinite series.* Essentially we substitute an unknown series into the equation, and calculate one term at a time.

6. ***The integrals of*** $1 + x + x^2 + \cdots$ ***and*** $1 - x + x^2 - \cdots$ ***are logarithms***:

$$x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots = \int_0^x \frac{dx}{1-x} = -\ln(1-x) \tag{10a}$$

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots = \int_0^x \frac{dx}{1+x} = +\ln(1+x) \tag{10b}$$

The derivative of (10a) brings back the geometric series. For logarithms we find $1/n$ not $1/n!$ The first term x and second term $\frac{1}{2}x^2$ give linear and quadratic approximations. Now we have the whole series. I cannot fail to substitute 1 and $\frac{1}{2}$, to find $\ln(1-1)$ and $\ln(1+1)$ and $\ln(1-\frac{1}{2})$:

$$x = 1: \quad 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots = -\ln 0 = +\infty \tag{11a}$$

$$x = 1: \quad 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots = \ln 2 = .693 \tag{11b}$$

$$x = \tfrac{1}{2}: \quad \tfrac{1}{2} + \tfrac{1}{8} + \tfrac{1}{24} + \tfrac{1}{64} + \cdots = -\ln \tfrac{1}{2} = \ln 2. \tag{12}$$

The first series diverges to infinity. This ***harmonic series*** $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ came into the earliest discussion of limits (Section 2.6). The second series has alternating signs and converges to ln 2. The third has plus signs and also converges to ln 2. These will be examples for a major topic in infinite series—tests for convergence.

For the first time in this book we are able to compute a logarithm! Something remarkable is involved. *The sums of numbers in* (11) and (12) *were discovered from the sums of functions in* (10). You might think it would be easier to deal only with numbers, to compute ln 2. But then we would never have integrated the series for $1/(1-x)$ and detected (10). It is better to work with x, and substitute special values like $\frac{1}{2}$ at the end.

There are two practical problems with these series. For ln 2 they converge slowly. For ln e they blow up. The correct answer is $\ln e = 1$, but the series can't find it. Both problems are solved by adding (10a) to (10b), which cancels the even powers:

$$2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) = \ln(1+x) - \ln(1-x) = \ln\frac{1+x}{1-x}. \tag{13}$$

At $x = \frac{1}{3}$, the right side is $\ln\frac{4}{3} - \ln\frac{2}{3} = \ln 2$. Powers of $\frac{1}{3}$ are much smaller than powers of 1 or $\frac{1}{2}$, so ln 2 is quickly computed. All logarithms can be found from the improved series (13).

7. ***Change variables in the geometric series (replace*** x ***by*** x^2 ***or*** $-x^2$***):***

$$1 + x^2 + x^4 + x^6 + \cdots = 1/(1-x^2) \tag{14}$$

$$1 - x^2 + x^4 - x^6 + \cdots = 1/(1+x^2). \tag{15}$$

This produces new functions (always our goal). They involve even powers of x. The second series will soon be used to calculate π. Other changes are valuable:

$$\frac{x}{2} \text{ in place of } x\text{:}\quad 1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \cdots = \frac{1}{1-(x/2)} = \frac{2}{2-x} \tag{16}$$

$$\frac{1}{x} \text{ in place of } x\text{:}\quad 1 + \frac{1}{x} + \frac{1}{x^2} + \cdots = \frac{1}{1-(1/x)} = \frac{x}{x-1}. \tag{17}$$

Equation (17) is a series of *negative powers* x^{-n}. It converges when $|x|$ is *greater* than 1. Convergence in (17) is for large x. Convergence in (16) is for $|x| < 2$.

8. ***The integral of*** $1 - x^2 + x^4 - x^6 + \cdots$ ***yields the inverse tangent of*** x:

$$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots = \int\frac{dx}{1+x^2} = \tan^{-1} x. \tag{18}$$

We integrated (15) and got odd powers. The magical formula for π (discovered by Leibniz) comes when $x = 1$. The angle with tangent 1 is $\pi/4$:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}. \tag{19}$$

The first three terms give $\pi \approx 3.47$ (not very close). The 5000th term is still of size .0001, so the fourth decimal is still not settled. By changing to $x = 1/\sqrt{3}$, the astronomer Halley and his assistant found 71 correct digits of $\pi/6$ (while waiting for the comet). That is one step in the long and amazing story of calculating π. The Chudnovsky brothers recently took the latest step with a supercomputer—*they have found more than* ***one billion*** *decimal places of* π (see *Science*, June 1989). The digits look completely random, as everyone expected. But so far we have no proof that all ten digits occur $\frac{1}{10}$ of the time.

Historical note Archimedes located π above 3.14 and below $3\frac{1}{7}$. Variations of his method (polygons in circles) reached as far as 34 digits—but not for 1800 years. Then Halley found 71 digits of $\pi/6$ with equation (18). For faster convergence that series was replaced by other inverse tangents, using smaller values of x:

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}. \tag{20}$$

A prodigy named Dase, who could multiply 100-digit numbers in his head in 8 hours, finally passed 200 digits of π. The climax of hand calculation came when Shanks published 607 digits. I am sorry to say that only 527 were correct. (With years of calculation he went on to 707 digits, but still only 527 were correct.) The mistake was not noticed until 1945! Then Ferguson reached 808 digits with a desk calculator.

Now comes the computer. Three days on an ENIAC (1949) gave 2000 digits. A hundred minutes on an IBM 704 (1958) gave 10,000 digits. Shanks (no relation) reached 100,000 digits. Finally a million digits were found in a day in 1973, with a CDC 7600. All these calculations used variations of equation (20).

The record after that went between Cray and Hitachi and now IBM. But the method changed. The calculations rely on an incredibly accurate algorithm, based on the "arithmetic-geometric mean iteration" of Gauss. It is also incredibly simple, all things considered:

$$a_{n+1} = \frac{a_n + b_n}{2} \qquad b_{n+1} = \sqrt{a_n b_n} \qquad \pi_n = 2a_{n+1}^2 \Big/ \left(1 - \sum_{k=0}^{n} 2^k(a_k^2 - b_k^2)\right).$$

The number of correct digits more than doubles at every step. By $n = 9$ we are far beyond Shanks (the hand calculator). No end is in sight. Almost anyone can go past a billion digits, since with the Chudnovsky method we don't have to start over again.

It is time to stop. You may think (or hope) that nothing more could possibly be done with geometric series. We have gone a long way from $1/(1-x)$, but some functions can never be reached. One is e^x (and its relatives $\sin x$, $\cos x$, $\sinh x$, $\cosh x$). Another is $\sqrt{1-x}$ (and its relatives $1/\sqrt{1-x^2}$, $\sin^{-1}x$, $\sec^{-1}x$, ...). The exponentials are in 10.4, with series that converge for all x. The square roots are in 10.5, closer to geometric series and converging for $|x| < 1$. Before that we have to say what convergence means.

The series came fast, but I hope you see what can be done (subtract, multiply, differentiate, integrate). Addition is easy, division is harder, all are legal. Some unexpected numbers are the sums of infinite series.

Added in proof By e-mail I just learned that the record for π is back in Japan: 2^{30} digits which is more than 1.07 billion. The elapsed time was 100 hours (75 hours of CPU time on an NEC machine). The billionth digit after the decimal point is 9.

10.1 EXERCISES

Read-through questions

The geometric series $1 + x + x^2 + \cdots$ adds to __a__. It converges provided $|x| <$ __b__. The sum of n terms is __c__. The derivatives of the series match the derivatives of $1/(1-x)$ at the point $x =$ __d__, where the nth derivative is __e__. The decimal 1.111... is the geometric series at $x =$ __f__ and equals the fraction __g__. The decimal .666... multiplies this by __h__. The decimal .999... is the same as __i__.

The derivative of the geometric series is __j__ = __k__. This also comes from squaring the __l__ series. By choosing $x = .01$, the decimal 1.02030405 is close to __m__. The differential equation $dy/dx = y^2$ is solved by the geometric series, going term by term starting from $y(0) =$ __n__.

The integral of the geometric series is __o__ = __p__. At $x=1$ this becomes the __q__ series, which diverges. At $x=$ __r__ we find $\ln 2=$ __s__. The change from x to $-x$ produces the series $1/(1+x)=$ __t__ and $\ln(1+x)=$ __u__.

In the geometric series, changing to x^2 or $-x^2$ gives $1/(1-x^2)=$ __v__ and $1/(1+x^2)=$ __w__. Integrating the last one yields $x-\frac{1}{3}x^3+\frac{1}{5}x^5\cdots=$ __x__. The angle whose tangent is $x=1$ is $\tan^{-1}1=$ __y__. Then substituting $x=1$ gives the series $\pi=$ __z__.

1 The geometric series is $1+x+x^2+\cdots=G$. Another way to discover G is to multiply by x. Then $x+x^2+x^3+\cdots=xG$, and this can be subtracted from the original series. What does that leave, and what is G?

2 A basketball is dropped 10 feet and bounces back 6 feet. After every fall it recovers $\frac{3}{5}$ of its height. What total distance does the ball travel, bouncing forever?

3 Find the sums of $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$ and $1-\frac{1}{4}+\frac{1}{16}-\cdots$ and $10-1+.1-.01\ldots$ and $3.040404\ldots$.

4 Replace x by $1-x$ in the geometric series to find a series for $1/x$. Integrate to find a series for $\ln x$. These are power series "around the point $x=1$." What is their sum at $x=0$?

5 What is the *second derivative* of the geometric series, and what is its sum at $x=\frac{1}{2}$?

6 Multiply the series $(1+x+x^2+\cdots)(1-x+x^2-\cdots)$ and find the product by comparing with equation (14).

7 Start with the fraction $\frac{1}{7}$. Divide 7 into 1.000... (by long division or calculator) until the numbers start repeating. Which is the first number to repeat? How do you know that the next ________ digits will be the same as the first?

Note about the fractions $1/q$, $10/q$, $100/q$, ... All remainders are less than q so eventually two remainders are the same. By subtraction, q goes evenly into a power 10^N minus a smaller power 10^{N-n}. Thus $qc=10^N-10^{N-n}$ for some c and $1/q$ has a repeating decimal:

$$\frac{1}{q}=\frac{c}{10^N-10^{N-n}}=\frac{c}{10^N}\frac{1}{1-10^{-n}}$$
$$=\frac{c}{10^N}\left(1+\frac{1}{10^n}+\frac{1}{10^{2n}}+\cdots\right).$$

Conclusion: Every fraction equals a repeating decimal.

8 Find the repeating decimal for $\frac{1}{13}$ and read off c. What is the number n of digits before it repeats?

9 From the fact that every q goes evenly into a power 10^N minus a smaller power, show that all primes except 2 or 5 go evenly into 9 or 99 or 999 or $\cdots$.

10 Explain why .010010001... cannot be a fraction (the number of zeros increases).

11 Show that .123456789101112... is not a fraction.

12 From the geometric series, the repeating decimal 1.065065... equals what fraction? Explain why every repeating decimal equals a fraction.

13 Write .878787... and .123123... as fractions and as geometric series.

14 Find the square of 1.111... as an infinite series.

Find the functions which equal the sums 15–24.

15 $x+x^3+x^5+\cdots$

16 $1-2x+4x^2-\cdots$

17 $x^3+x^6+x^9+\cdots$

18 $\frac{1}{2}x-\frac{1}{4}x^2+\frac{1}{8}x^3-\cdots$

19 $\ln x+(\ln x)^2+(\ln x)^3+\cdots$

20 $x-2x^2+3x^3-\cdots$

21 $\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\cdots$

22 $x+\frac{x}{1+x}+\frac{x}{(1+x)^2}+\cdots$

23 $\tan x-\frac{1}{3}\tan^3 x+\frac{1}{5}\tan^5 x-\cdots$

24 $e^x+e^{2x}+e^{3x}+\cdots$

25 Multiply the series for $1/(1-x)$ and $1/(1+x)$ to find the coefficients of x, x^2, x^3 and x^n.

26 Compare the integral of $1+x^2+x^4+\cdots$ to equation (13) and find $\int dx/(1-x^2)$.

27 What fractions are close to .2468 and .987654321?

28 Find the first three terms in the series for $1/(1-x)^3$.

Add up the series 29–34. Problem 34 comes from (18).

29 $\frac{2}{3}+\frac{2}{3^2}+\frac{2}{3^3}+\cdots$

30 $.1+.02+.003+\cdots$

31 $.1+\frac{1}{2}(.01)+\frac{1}{3}(.001)+\cdots$

32 $.1-\frac{1}{2}(.01)+\frac{1}{3}(.001)-\cdots$

33 $.1+\frac{1}{3}(.001)+\frac{1}{5}(.00001)+\cdots$

34 $1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 3^2}-\cdots$

35 Compute the nth derivative of $1+2x+3x^2+\cdots$ at $x=0$. Compute also the nth derivative of $(1-x)^{-2}$.

36 The differential equation $dy/dx=y^2$ starts from $y(0)=b$. From the equation and its derivatives find y', y'', y''' at $x=0$, and construct the start of a series that matches those derivatives. Can you recognize $y(x)$?

37 The equation $dy/dx=y^2$ has the differential form $dy/y^2=dx$. Integrate both sides and choose the integration constant so that $y=b$ at $x=0$. Solve for $y(x)$ and compare with Problem 36.

38 In a bridge game, what is the average number μ of deals until you get the best hand? The probability on the first deal is $p_1=\frac{1}{4}$. Then $p_2=(\frac{3}{4})(\frac{1}{4})=$ (probability of missing on the first) times (probability of winning on the second). Generally $p_n=(\frac{3}{4})^{n-1}(\frac{1}{4})$. The mean value μ is $p_1+2p_2+3p_3+\cdots=$ ________.

39 Show that $(\Sigma a_n)(\Sigma b_n)=\Sigma a_n b_n$ is ridiculous.

40 Find a series for $\ln\frac{1}{3}$ by choosing x in (10b). Find a series for $\ln 3$ by choosing x in (13). How is $\ln\frac{1}{3}$ related to $\ln 3$, and which series converges faster?

41 Compute ln 3 to its second decimal place without a calculator (OK to check).

42 To four decimal places, find the angle whose tangent is $x=\frac{1}{10}$.

43 Two tennis players move to the net as they volley the ball. Starting together they each go forward 39 feet at 13 feet per second. The ball travels back and forth at 26 feet per second. How far does it travel before the collision at the net? (Look for an easy way and also an infinite series.)

44 How many terms of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ are needed before the first decimal place doesn't change? Which power of $\frac{1}{4}$ equals the 100th power of $\frac{1}{2}$? Which power $1/a^n$ equals $1/2^{100}$?

45 If $\tan y=\frac{1}{2}$ and $\tan z=\frac{1}{3}$, then the tangent of $y+z$ is $(\tan y+\tan z)/(1-\tan y \tan z)=1$. If $\tan y=\frac{1}{5}$ and $\tan z=$ ________, again $\tan(y+z)=1$. Why is this not as good as equation (20), to find $\pi/4$?

46 Find one decimal of π beyond 3.14 from the series for $4\tan^{-1}\frac{1}{2}$ and $4\tan^{-1}\frac{1}{3}$. How many terms are needed in each series?

47 (Calculator) In the same way find one decimal of π beyond 3.14159. How many terms did you take?

48 From equation (10a) what is $\Sigma e^{in}/n$?

49 Zeno's Paradox is that if you go half way, and then half way, and then half way..., you will never get there. In your opinion, does $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ add to 1 or not?

10.2 Convergence Tests: Positive Series

This is the third time we have stopped the calculations to deal with the definitions. Chapter 2 said what a derivative is. Chapter 5 said what an integral is. Now we say what the sum of a series is—*if it exists*. In all three cases ***a limit is involved***. That is the formal, careful, cautious part of mathematics, which decides if the active and progressive parts make sense.

The series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ converges to 1. The series $1+\frac{1}{2}+\frac{1}{3}+\cdots$ diverges to infinity. The series $1-\frac{1}{2}+\frac{1}{3}-\cdots$ converges to ln 2. When we speak about convergence or divergence of a series, we are really speaking about convergence or divergence of its "*partialsums*."

DEFINITION 1 The ***partial sum*** s_n of the series $a_1+a_2+a_3+\cdots$ stops at a_n:

$$s_n = \textit{\textbf{sum of the first n terms}} = a_1+a_2+\cdots+a_n.$$

Thus s_n is *part* of the total sum. The example $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ has partial sums

$$s_1=\frac{1}{2} \qquad s_2=\frac{3}{4} \qquad s_3=\frac{7}{8} \qquad s_n=1-\frac{1}{2^n}.$$

Those add up larger and larger parts of the series—what is the sum of the whole series? The answer is: ***The series*** $\frac{1}{2}+\frac{1}{4}+\ldots$ ***converges to*** 1 ***because its partial sums*** s_n ***converge to*** 1. The series $a_1+a_2+a_3+\ldots$ converges to s when its partial sums—going further and further out—approach this limit s. ***Add the*** a's, ***not the*** s's.

DEFINITION 2 ***The sum of a series is the limit of its partial sums*** s_n.

We repeat: *if the limit exists*. The numbers s_n may have no limit. When the partial sums jump around, the whole series *has no sum*. Then the series does not converge. When the partial sums approach s, the distant terms a_n are approaching zero. More than that, the *sum* of distant terms is approaching zero.

The new idea ($\Sigma a_n = s$) has been converted to the old idea ($s_n \to s$).

EXAMPLE 1 The geometric series $\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots$ converges to $s=\frac{1}{9}$.

The partial sums s_1, s_2, s_3, s_4 are .1, .11, .111, .1111. They are approaching $s=\frac{1}{9}$.

Note again the difference between the series of a's and the sequence of s's. The series $1+1+1+\cdots$ diverges because the sequence of s's is $1, 2, 3, \ldots$. A sharper example is the harmonic series: $1+\frac{1}{2}+\frac{1}{3}+\cdots$ diverges because its partial sums $1, 1\frac{1}{2}, \ldots$ eventually go past every number s. We saw that in 2.6 and will see it again here.

Do not confuse $a_n \to 0$ with $s_n \to s$. You cannot be sure that a series converges, just on the basis that $a_n \to 0$. The harmonic series is the best example: $a_n = 1/n \to 0$ but still $s_n \to \infty$. This makes infinite series into a delicate game, which mathematicians enjoy. The line between divergence and convergence is hard to find and easy to cross. A slight push will speed up $a_n \to 0$ and make the s_n converge. Even though $a_n \to 0$ does not by itself guarantee convergence, it is the first requirement:

> **10A** If a series converges ($s_n \to s$) then its terms must approach zero ($a_n \to 0$).

Proof Suppose s_n approaches s (as required by convergence). Then also s_{n-1} approaches s, and the difference $s_n - s_{n-1}$ approaches zero. That difference is a_n. So $a_n \to 0$.

EXAMPLE 1 (continued) For the geometric series $1 + x + x^2 + \cdots$, the test $a_n \to 0$ is the same as $x^n \to 0$. The test is failed if $|x| \geqslant 1$, because the powers of x don't go to zero. Automatically the series diverges. The test is passed if $-1 < x < 1$. But to prove convergence, *we cannot rely on* $a_n \to 0$. It is the partial sums that must converge:

$$s_n = 1 + x + \cdots + x^{n-1} = \frac{1-x^n}{1-x} \quad \text{and} \quad s_n \to \frac{1}{1-x}. \quad \text{This is } s.$$

For other series, first check that $a_n \to 0$ (otherwise there is no chance of convergence). The a_n will not have the special form x^n—so we need sharper tests.

The geometric series stays in our mind for this reason. *Many convergence tests are comparisons with that series.* The right comparison gives enough information:

If $|a_1| < \frac{1}{2}$ and $|a_2| < \frac{1}{4}$ and $\ldots$, then $a_1 + a_2 + \ldots$ converges faster than $\frac{1}{2} + \frac{1}{4} + \ldots$.

More generally, the terms in $a_1 + a_2 + a_3 + \ldots$ may be smaller than $ax + ax^2 + ax^3 + \ldots$. Provided $x < 1$, the second series converges. Then $\sum a_n$ also converges. We move now to ***convergence by comparison*** or ***divergence by comparison.***

Throughout the rest of this section, all numbers a_n are assumed positive.

COMPARISON TEST AND INTEGRAL TEST

In practice it is rare to compute the partial sums $s_n = a_1 + \cdots + a_n$. Usually a simple formula can't be found. We may never know the exact limit s. But it is still possible to decide convergence—whether there *is* a sum—by comparison with another series that is known to converge.

> **10B** (*Comparison test*) Suppose that $0 \leqslant a_n \leqslant b_n$ and $\sum b_n$ converges. Then $\sum a_n$ converges.

The smaller terms a_n add to a smaller sum: $\sum a_n$ is below $\sum b_n$ and must converge. On the other hand suppose $a_n \geqslant c_n$ and $\sum c_n = \infty$. This comparison forces $\sum a_n = \infty$. ***A series diverges if it is above another divergent series.***

Note that a series of positive terms can only diverge "*to infinity.*" It cannot oscillate, because each term moves it forward. Either the s_n creep up on s, passing every number below it, or they pass all numbers and diverge. *If an increasing sequence s_n is bounded above, it must converge.* The line of real numbers is complete, and has no holes.

The harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges to infinity.

A comparison series is $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \ldots$. *The harmonic series is larger*. But this comparison series is really $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$, because $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$.

The comparison series diverges. The harmonic series, *above it*, must also diverge.

To apply the comparison test, we need something to compare with. In Example 2, we thought of another series. It was convenient because of those $\frac{1}{2}$'s. But a different series will need a different comparison, and where will it come from? There is an automatic way to think of a *comparison series*. It comes from the ***integral test***.

Allow me to apply the integral test to the same example. ***To understand the integral test, look at the areas in Figure 10.2***. The test compares rectangles with curved areas.

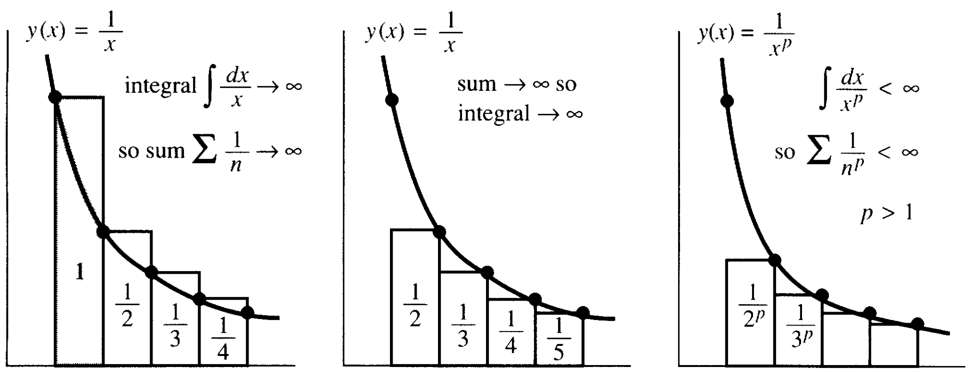

Fig. 10.2 ***Integral test***: Sums and integrals both diverge ($p = 1$) and both converge ($p > 1$).

EXAMPLE 2 (again) ***Compare*** $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ ***with the area under the curve*** $y = 1/x$.

Every term $a_n = 1/n$ is the area of a rectangle. We are comparing it with a curved area c_n. Both areas are between $x = n$ and $x = n + 1$, and ***the rectangle is above the curve***. So $a_n > c_n$:

$$\text{rectangular area } a_n = \frac{1}{n} \quad \text{exceeds curved area } c_n = \int_n^{n+1} \frac{dx}{x}.$$

Here is the point. Those c_n's look complicated, but *we can add them up*. The sum $c_1 + \ldots + c_n$ is the whole area, from 1 to $n + 1$. It equals $\ln(n + 1)$—we know the integral of $1/x$. We also know that the logarithm goes to infinity.

The rectangular area $1 + 1/2 + \ldots + 1/n$ is above the curved area. By comparison of areas, the harmonic series diverges to infinity—a little faster than $\ln(n + 1)$.

Remark The integral of $1/x$ has another advantage over the series with $\frac{1}{2}$'s. First, the integral test was automatic. From $1/n$ in the series, we went to $1/x$ in the integral. Second, the comparison is *closer*. Instead of adding only $\frac{1}{2}$ when the number of terms is doubled, the true partial sums grow like $\ln n$. To prove that, put rectangles ***under*** the curve.

Rectangles *below* the curve give an area *below* the integral. Figure 10.2b omits the first rectangle, to get under the curve. Then we have the opposite to the first comparison—the sum is now smaller than the integral:

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \int_1^n \frac{dx}{x} = \ln n.$$

Adding 1 to both sides, s_n *is below* $1 + \ln n$. From the previous test, s_n *is above* $\ln(n + 1)$. That is a narrow space—we have an excellent estimate of s_n. The sum of $1/n$

and the integral of $1/x$ diverge together. Problem 43 will show that the difference between s_n and $\ln n$ approaches "Euler's constant," which is $\gamma = .577\ldots$.

Main point: Rectangular area is s_n. Curved area is close. We are using integrals to help with sums (it used to be the opposite).

Question If a computer adds a million terms every second for a million years, how large is the partial sum of the harmonic series?
Answer The number of terms is $n = 60^2 \cdot 24 \cdot 365 \cdot 10^{12} < 3.2 \cdot 10^{19}$. Therefore $\ln n$ is less than $\ln 3.2 + 19 \ln 10 < 45$. By the integral test $s_n < 1 + \ln n$, the partial sum after a million years has not reached 46.

For other series, $1/x$ changes to a different function $y(x)$. At $x = n$ this function must equal a_n. Also $y(x)$ must be decreasing. Then a rectangle of height a_n is above the graph to the right of $x = n$, and below the graph to the left of $x = n$. ***The series and the integral box each other in***: ***left sum $\geqslant$ integral $\geqslant$ right sum***. The reasoning is the same as it was for $a_n = 1/n$ and $y(x) = 1/x$: There is finite area in the rectangles when there is finite area under the curve.

When we can't add the a's, we integrate $y(x)$ and compare areas:

10C ***(Integral test)*** If $y(x)$ is decreasing and $y(n)$ agrees with a_n, then

$$a_1 + a_2 + a_3 + \cdots \quad \text{and} \quad \int_1^\infty y(x)\,dx \quad \text{both converge or both diverge.}$$

EXAMPLE 3 The "p-series" $\frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$ converges if $p > 1$. Integrate $y = \frac{1}{x^p}$:

$$\frac{1}{n^p} < \int_{n-1}^{n} \frac{dx}{x^p} \quad \text{so by addition} \quad \sum_{n=2}^{\infty} \frac{1}{n^p} < \int_1^\infty \frac{dx}{x^p}.$$

In Figure 10.2c, the area is finite if $p > 1$. The integral equals $[x^{1-p}/(1-p)]_1^\infty$, which is $1/(p-1)$. ***Finite area means convergent series.*** If $1/1^p$ is the first term, add 1 to the curved area:

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \;<\; 1 + \frac{1}{p-1} \;=\; \frac{p}{p-1}.$$

The borderline case $p = 1$ is the harmonic series (divergent). By the comparison test, every $p < 1$ also produces divergence. Thus $\Sigma 1/\sqrt{n}$ diverges by comparison with $\int dx/\sqrt{x}$ (and also by comparison with $\Sigma 1/n$). Section 7.5 on improper integrals runs parallel to this section on "improper sums" (infinite series).

Notice the special cases $p = 2$ and $p = 3$. The series $1 + \frac{1}{4} + \frac{1}{9} + \cdots$ converges. Euler found $\pi^2/6$ as its sum. The series $1 + \frac{1}{8} + \frac{1}{27} + \cdots$ also converges. That is proved by comparing $\Sigma 1/n^3$ with $\Sigma 1/n^2$ or with $\int dx/x^3$. But the sum for $p = 3$ is unknown.

Extra credit problem The sum of the p-series leads to the most important problem in pure mathematics. The "zeta function" is $Z(p) = \Sigma 1/n^p$, so $Z(2) = \pi^2/6$ and $Z(3)$ is unknown. Riemann studied the complex numbers p where $Z(p) = 0$ (there are infinitely many). He conjectured that *the real part of those p is always $\frac{1}{2}$*. That has been tested for the first billion zeros, but never proved.

COMPARISON WITH THE GEOMETRIC SERIES

We can compare any new series $a_1 + a_2 + \cdots$ with $1 + x + \cdots$. Remember that the first million terms have nothing to do with convergence. It is further out, as $n \to \infty$, that the comparison stands or falls. We still assume that $a_n > 0$.

10D (*Ratio test*) If a_{n+1}/a_n approaches a limit $L < 1$, the series converges.

10E (*Root test*) If the nth root $(a_n)^{1/n}$ approaches $L < 1$, the series converges.

Roughly speaking, these tests make a_n comparable with L^n—therefore convergent. The tests also establish divergence if $L > 1$. They give no decision when $L = 1$. Unfortunately $L = 1$ is the most important and the hardest case.

On the other hand, you will now see that the ratio test is fairly easy.

EXAMPLE 4 The geometric series $x + x^2 + \cdots$ has ratio exactly x. The nth root is also exactly x. So $L = x$. There is convergence if $x < 1$ (known) and divergence if $x > 1$ (also known). The divergence of $1 + 1 + \cdots$ is too delicate (!) for the ratio test and root test, because $L = 1$.

EXAMPLE 5 The p-series has $a_n = 1/n^p$ and $a_{n+1}/a_n = n^p/(n+1)^p$. The limit as $n \to \infty$ is $L = 1$, for every p. The ratio test does not feel the difference between $p = 2$ (convergence) and $p = 1$ (divergence) or even $p = -1$ (extreme divergence). Neither does the root test. So the integral test is sharper.

EXAMPLE 6 A combination of p-series and geometric series can now be decided:

$$\frac{x}{1^p} + \frac{x^2}{2^p} + \cdots + \frac{x^n}{n^p} + \cdots \text{ has ratio } \frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)^p}\frac{n^p}{x^n} \text{ approaching } L = x.$$

It is $|x| < 1$ *that decides convergence, not* p. ***The powers*** x^n ***are stronger than any*** n^p. The factorials $n!$ will now prove stronger than any x^n.

EXAMPLE 7 *The exponential series* $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$ ***converges for all*** x.

The terms of this series are $x^n/n!$ The ratio between neighboring terms is

$$\frac{x^{n+1}/(n+1)!}{x^n/n!} = \frac{x}{n+1}, \text{ which approaches } L = 0 \text{ as } n \to \infty.$$

With $x = 1$, this ratio test gives convergence of $\sum 1/n!$ *The sum is* e. With $x = 4$, the larger series $\sum 4^n/n!$ also converges. We know this sum too—it is e^4. Also the sum of $x^n n^p/n!$ converges for any x and p. Again $L = 0$—the ratio test is not even close. ***The factorials take over, and give convergence.***

Here is the proof of convergence when the ratios approach $L < 1$. Choose x halfway from L to 1. Then $x < 1$. Eventually the ratios go below x and stay below:

$$a_{N+1}/a_N < x \qquad a_{N+2}/a_{N+1} < x \qquad a_{N+3}/a_{N+2} < x \quad \cdots$$

Multiply the first two inequalities. Then multiply all three:

$$a_{N+1}/a_N < x \qquad a_{N+2}/a_N < x^2 \qquad a_{N+3}/a_N < x^3 \quad \cdots$$

Therefore $a_{N+1} + a_{N+2} + a_{N+3} + \cdots$ is less than $a_N(x + x^2 + x^3 + \cdots)$. Since $x < 1$, comparison with the geometric series gives convergence.

EXAMPLE 8 The series $\sum 1/n^n$ is ideal for the root test. The nth root is $1/n$. Its limit is $L = 0$. Convergence is even faster than for $e = \sum 1/n!$ The root test is easily explained, since $(a_n)^{1/n} < x$ yields $a_n < x^n$ and x is close to $L < 1$. So we compare with the geometric series.

SUMMARY FOR POSITIVE SERIES

The convergence of geometric series and p-series and exponential series is settled. I will put these a_n's in a line, going from most divergent to most convergent. The crossover to convergence is after $1/n$:

$1+1+\cdots$	$(p<1)\ \frac{1}{n^p}\ \ \frac{1}{n}\ \ \frac{1}{n^p}\ (p>1)$	$\frac{n}{2^n}\ \ \frac{1}{2^n}\ \ \frac{4^n}{n!}\ \ \frac{1}{n!}\ \ \frac{1}{n^n}$
10A	**10B** and **10C**	**10D** and **10E**
$(a_n \nrightarrow 0)$	(comparison and integral)	(ratio and root)

You should know that this crossover is not as sharp as it looks. On the convergent side, $1/n(\ln n)^2$ comes before all those p-series. On the divergent side, $1/n(\ln n)$ and $1/n(\ln n)(\ln \ln n)$ belong after $1/n$. For any divergent (or convergent) series, there is another that diverges (or converges) more slowly.

Thus there is no hope of an ultimate all-purpose comparison test. But comparison is the best method available. Every series in that line can be compared with its neighbors, and other series can be placed in between. It is a topic that is understood best by examples.

EXAMPLE 9 $\sum \frac{1}{\ln n}$ diverges because $\sum \frac{1}{n}$ diverges. The comparison uses $\ln n < n$.

EXAMPLE 10 $\sum \frac{1}{n(\ln n)^2} \approx \int \frac{dx}{x(\ln x)^2} < \infty \qquad \sum \frac{1}{n(\ln n)} \approx \int \frac{dx}{x(\ln x)} = \infty.$

The indefinite integrals are $-1/\ln x$ and $\ln(\ln x)$. The first goes to zero as $x \to \infty$; the integral and series both converge. The second integral $\ln(\ln x)$ goes to infinity—very slowly but it gets there. So the second series diverges. These examples squeeze new series into the line, closer to the crossover.

EXAMPLE 11 $\frac{1}{n^2+1} < \frac{1}{n^2}$ so $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \cdots < \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots$ (*convergence*).

The constant 1 in this denominator has no effect—and again in the next example.

EXAMPLE 12 $\frac{1}{2n-1} > \frac{1}{2n}$ so $\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots > \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots.$

$\sum 1/2n$ is 1/2 times $\sum 1/n$, so both series diverge. ***Two series behave in the same way if the ratios a_n/b_n approach*** $L > 0$. Examples 11–12 have $n^2/(n^2+1) \to 1$ and $2n/(2n-1) \to 1$. That leads to our final test:

> **10F** (*Limit comparison test*) **If the ratio a_n/b_n approaches a positive limit L, then $\sum a_n$ and $\sum b_n$ either both diverge or both converge.**

Reason: a_n is smaller than $2Lb_n$ and larger than $\frac{1}{2}Lb_n$, at least when n is large. So the two series behave in the same way. For example $\sum \sin(7/n^p)$ converges for $p > 1$, not for $p \leqslant 1$. It behaves like $\sum 1/n^p$ (here $L = 7$). The tail end of a series (large n) controls convergence. The front end (small n) controls most of the sum.

There are many more series to be investigated by comparison.

10.2 EXERCISES

Read-through questions

The convergence of $a_1 + a_2 + \cdots$ is decided by the partial sums $s_n =$ __a__. If the s_n approach s, then $\Sigma a_n =$ __b__. For the __c__ series $1 + x + \cdots$ the partial sums are $s_n =$ __d__. In that case $s_n \to 1/(1-x)$ if and only if __e__. In all cases the limit $s_n \to s$ requires that $a_n \to$ __f__. But the harmonic series $a_n = 1/n$ shows that we can have $a_n \to$ __g__ and still the series __h__.

The comparison test says that if $0 \leqslant a_n \leqslant b_n$ then __i__. In case a decreasing $y(x)$ agrees with a_n at $x = n$, we can apply the __j__ test. The sum Σa_n converges if and only if __k__. By this test the p-series $\Sigma 1/n^p$ converges if and only if p is __l__. For the harmonic series ($p = 1$), $s_n = 1 + \cdots + 1/n$ is close to the integral $f(n) =$ __m__.

The __n__ test applies when $a_{n+1}/a_n \to L$. There is convergence if __o__, divergence if __p__, and no decision if __q__. The same is true for the __r__ test, when $(a_n)^{1/n} \to L$. For a geometric-p-series combination $a_n = x^n/n^p$, the ratio a_{n+1}/a_n equals __s__. Its limit is $L =$ __t__ so there is convergence if __u__. For the exponential $e^x = \Sigma x^n/n!$ the limiting ratio a_{n+1}/a_n is $L =$ __v__. This series always __w__ because $n!$ grows faster than any x^n or n^p.

There is no sharp line between __x__ and __y__. But if Σb_n converges and $a_n/b_n \to L$, it follows from the __z__ test that Σa_n also converges.

1 Here is a quick proof that a finite sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots = s$ is impossible. Division by 2 would give $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots = \frac{1}{2}s$. Subtraction would leave $1 + \frac{1}{3} + \frac{1}{5} + \cdots = \frac{1}{2}s$. Those last two series cannot both add to $\frac{1}{2}s$ because ______.

2 Behind every decimal $s = .abc\ldots$ is a convergent series $a/10 + b/100 +$ ______ $+ \cdots$. By a comparison test prove convergence.

3 From these partial sums s_n, find a_n and also $s = \Sigma_1^\infty a_n$:

(a) $s_n = 1 - \dfrac{1}{n}$ (b) $s_n = 4n$ (c) $s_n = \ln \dfrac{2n}{n+1}$.

4 Find the partial sums $s_n = a_1 + a_2 + \cdots + a_n$:

(a) $a_n = 1/3^{n-1}$ (b) $a_n = \ln \dfrac{n}{n+1}$ (c) $a_n = n$

5 Suppose $0 < a_n < b_n$ and Σa_n converges. What can be deduced about Σb_n? Give examples.

6 (a) Suppose $b_n + c_n < a_n$ (all positive) and Σa_n converges. What can you say about Σb_n and Σc_n?
(b) Suppose $a_n < b_n + c_n$ (all positive) and Σa_n diverges. What can you say about Σb_n and Σc_n?

Decide convergence or divergence in 7–10 (and give a reason).

7 $\frac{1}{100} + \frac{1}{200} + \frac{1}{300} + \cdots$ **8** $\frac{1}{100} + \frac{1}{105} + \frac{1}{110} + \cdots$

9 $\frac{1}{101} + \frac{1}{104} + \frac{1}{109} + \cdots$ **10** $\frac{1}{101} + \frac{2}{108} + \frac{3}{127} + \cdots$

Establish convergence or divergence in 11–20 by a comparison test.

11 $\sum \dfrac{1}{n^2 + 10}$ **12** $\sum \dfrac{1}{\sqrt{n^2 + 10}}$

13 $\sum \dfrac{1}{n + \sqrt{n}}$ **14** $\sum \dfrac{\sqrt{n}}{n^2 + 4}$

15 $\sum \dfrac{n^3}{n^2 + n^4}$ **16** $\sum \dfrac{1}{n^2} \cos\left(\dfrac{1}{n}\right)$

17 $\sum \dfrac{1}{2^n - 1}$ **18** $\sum \sin^2\left(\dfrac{1}{n}\right)$

19 $\sum \dfrac{1}{3^n - 2^n}$ **20** $\sum \dfrac{1}{e^n - n^e}$

For 21–28 find the limit L in the ratio test or root test.

21 $\sum \dfrac{3^n}{n!}$ **22** $\sum \dfrac{1}{n^2}$

23 $\sum \dfrac{n^2 2^n}{n!}$ **24** $\sum \left(\dfrac{n-1}{n}\right)^n$

25 $\sum \dfrac{n}{2^n}$ **26** $\sum \dfrac{n!}{e^{n^2}}$

27 $\sum \left(\dfrac{n-1}{n}\right)^{n^2}$ **28** $\sum \dfrac{n!}{n^n}$

29 $(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4})$ is "telescoping" because $\frac{1}{2}$ and $\frac{1}{3}$ cancel $-\frac{1}{2}$ and $-\frac{1}{3}$. Add the infinite telescoping series

$$s = \sum_1^\infty \left(\frac{1}{n} - \frac{1}{n+1}\right) = \sum_1^\infty \left(\frac{1}{n(n+1)}\right).$$

30 Compute the sum s for other "telescoping series":

(a) $\left(\dfrac{1}{1} - \dfrac{1}{3}\right) + \left(\dfrac{1}{2} - \dfrac{1}{4}\right) + \left(\dfrac{1}{3} - \dfrac{1}{5}\right) \cdots$

(b) $\ln \frac{1}{2} + \ln \frac{2}{3} + \ln \frac{3}{4} + \cdots$

31 In the integral test, what sum is larger than $\int_1^n y(x)\,dx$ and what sum is smaller? Draw a figure to illustrate.

32 Comparing sums with integrals, find numbers larger and smaller than

$$s_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} \text{ and } s_n = 1 + \frac{1}{8} + \cdots + \frac{1}{n^3}.$$

33 Which integral test shows that $\sum_1^\infty 1/e^n$ converges? What is the sum?

34 Which integral test shows that $\sum_1^\infty n/e^n$ converges? What is the sum?

Decide for or against convergence in 35–42, based on $\int y(x)\,dx$.

35 $\sum \frac{1}{n^2+1}$ 36 $\sum \frac{1}{3n+5}$

37 $\sum \frac{n}{n^2+1}$ 38 $\sum \frac{\ln n}{n}$ $\left(\text{is } \frac{\ln x}{x} \text{ decreasing?}\right)$

39 $\sum n^e/n^\pi$ 40 $\sum_2^\infty \frac{1}{n(\ln n)(\ln \ln n)}$

41 $\sum e^n/\pi^n$ 42 $\sum n/e^{n^2}$

43 (a) Explain why $D_n = \left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) - \ln n$ is positive by using rectangles as in Figure 10.2.

(b) Show that D_{n+1} is less than D_n by proving that

$$\frac{1}{n+1} < \int_n^{n+1} \frac{dx}{x}.$$

(c) (Calculator) The decreasing D_n's must approach a limit. Compute them until they go below .6 and below .58 (when?). The limit of the D_n is *Euler's constant* $\gamma = .577\ldots$.

44 In the harmonic series, use $s_n \approx .577 + \ln n$ to show that $s_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$ needs more than 600 terms to reach $s_n > 7$. How many terms for $s_n > 10$?

45 (a) Show that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots-\frac{1}{2n} = \frac{1}{n+1}+\cdots+\frac{1}{2n}$ by adding $2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)$ to both sides.

(b) Why is the right side close to $\ln 2n - \ln n$? Deduce that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ approaches $\ln 2$.

46 Every second a computer adds a million terms of $\sum 1/(n \ln n)$. By comparison with $\int dx/(x \ln x)$, estimate the partial sum after a million years (see Question in text).

47 Estimate $\sum_{100}^{1000} \frac{1}{n^2}$ by comparison with an integral.

48 If Σa_n converges (all $a_n > 0$) show that Σa_n^2 converges.

49 If Σa_n converges (all $a_n > 0$) show that $\Sigma \sin a_n$ converges. How could $\Sigma \sin a_n$ converge when Σa_n diverges?

50 The nth prime number p_n satisfies $p_n/n \ln n \to 1$. Prove that

$$\sum \frac{1}{p_n} = \frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\cdots \text{ diverges.}$$

Construct a series Σa_n that converges faster than Σb_n but slower than Σc_n (meaning $a_n/b_n \to 0$, $a_n/c_n \to \infty$).

51 $b_n = 1/n^2$, $c_n = 1/n^3$ 52 $b_n = n(\frac{1}{2})^n$, $c_n = (\frac{1}{2})^n$

53 $b_n = 1/n!$, $c_n = 1/n^n$ 54 $b_n = 1/n^e$, $c_n = 1/e^n$

In Problem 53 use Stirling's formula $\sqrt{2\pi n}\, n^n/e^n n! \to 1$.

55 For the series $\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\cdots$ show that the ratio test fails. The roots $(a_n)^{1/n}$ do approach a limit L. Find L from the even terms $a_{2k} = 1/2^k$. Does the series converge?

56 (For instructors) If the ratios a_{n+1}/a_n approach a positive limit L show that the roots $(a_n)^{1/n}$ also approach L.

Decide convergence in 57–66 and name your test.

57 $\sum \frac{1}{(\ln n)^n}$ 58 $\sum \frac{1}{n^{\ln n}}$

59 $\sum \frac{1}{10^n}$ 60 $\sum \frac{1}{\ln (10^n)}$

61 $\sum \ln \frac{n+2}{n+1}$ 62 $\sum n^{-1/n}$

63 $\sum \frac{1}{(\ln n)^p}$ (test all p) 64 $\sum \frac{\ln n}{n^p}$ (test all p)

65 $\sum \frac{3^n}{4^n - 2^n}$ 66 $\sum \frac{n^p}{(n!)^q}$ (test all p, q)

67 Suppose $a_n/b_n \to 0$ in the limit comparison test. Prove that Σa_n converges if Σb_n converges.

68 Can you invent a series whose convergence you and your instructor cannot decide?

10.3 Convergence Tests: All Series

This section finally allows the numbers a_n to be negative. The geometric series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots = \frac{2}{3}$ is certainly allowed. So is the series $\pi = 4-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\cdots$. If we change all signs to $+$, the geometric series would still converge (to the larger sum 2). This is the first test, to bring back a positive series by taking the ***absolute value*** $|a_n|$ of every term.

DEFINITION The series Σa_n is "***absolutely convergent***" if $\Sigma |a_n|$ is convergent.

Changing a negative number from a_n to $|a_n|$ increases the sum. Main point: The smaller series $\Sigma\, a_n$ is sure to converge if $\Sigma\, |a_n|$ converges.

10G **If $\Sigma\, |a_n|$ converges then $\Sigma\, a_n$ converges (absolutely). But $\Sigma\, a_n$ might converge, as in the series for π, even if $\Sigma\, |a_n|$ diverges to infinity.**

EXAMPLE 1 Start with the positive series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$. *Change any signs to minus.* Then the new series converges (absolutely). The right choice of signs will make it converge to any number between -1 and 1.

EXAMPLE 2 Start with the alternating series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ which converges to $\ln 2$. Change to plus signs. The new series $1+\frac{1}{2}+\frac{1}{3}+\cdots$ diverges to infinity. The original alternating series was not absolutely convergent. It was only "***conditionally convergent***." A series can converge (conditionally) by a careful choice of signs—even if $\Sigma\, |a_n| = \infty$.

If $\Sigma\, |a_n|$ converges then $\Sigma\, a_n$ converges. Here is a quick proof. The numbers $a_n + |a_n|$ are either zero (if a_n is negative) or $2|a_n|$. By comparison with $\Sigma\, 2|a_n|$, which converges, $\Sigma\,(a_n + |a_n|)$ must converge. Now subtract the convergent series $\Sigma\, |a_n|$. The difference $\Sigma\, a_n$ also converges, completing the proof. All tests for positive series (integral, ratio, comparison, ...) apply immediately to absolute convergence, because we switch to $|a_n|$.

EXAMPLE 3 Start with the geometric series $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$ which converges to $\frac{1}{2}$. *Change any of those signs to minus.* Then the new series must converge (absolutely). But the sign changes cannot achieve all sums between $-\frac{1}{2}$ and $\frac{1}{2}$. This time the sums belong to the famous (and very thin) *Cantor set* of Section 3.7.

EXAMPLE 4 (looking ahead) Suppose $\Sigma\, a_n x^n$ converges for a particular number x. Then for every x nearer to zero, it converges absolutely. This will be proved and used in Section 10.6 on power series, where it is the most important step in the theory.

EXAMPLE 5 Since $\Sigma\, 1/n^2$ converges, so does $\Sigma\,(\cos n)/n^2$. That second series has irregular signs, but it converges absolutely by comparison with the first series (since $|\cos n| < 1$). Probably $\Sigma\,(\tan n)/n^2$ does not converge, because the tangent does not stay bounded like the cosine.

ALTERNATING SERIES

The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ converges to $\ln 2$. That was stated without proof. This is an example of an ***alternating series***, in which the signs alternate between plus and minus. There is the additional property that the absolute values $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ decrease to zero. Those two facts—decrease to zero with alternating signs—guarantee convergence.

10H **An alternating series $a_1 - a_2 + a_3 - a_4 \cdots$ converges (at least conditionally, maybe not absolutely) if every $a_{n+1} \leqslant a_n$ and $a_n \to 0$.**

The best proof is in Figure 10.3. Look at $a_1 - a_2 + a_3$. It is below a_1, because a_3 (with plus sign) is smaller than a_2 (with minus sign). The sum of five terms is less than the

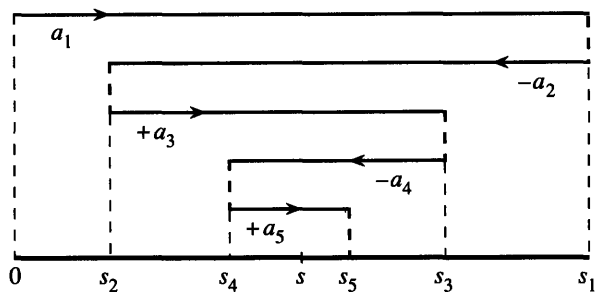

Fig. 10.3 An alternating series converges when the absolute values decrease to zero.

sum of three terms, because a_5 is smaller than a_4. These partial sums $s_1, s_3, s_5, \ldots$ with an odd number of terms are *decreasing*.

Now look at two terms $a_1 - a_2$, then four terms, then six terms. Adding on $a_3 - a_4$ increases the sum (because $a_3 \geqslant a_4$). Similarly s_6 is greater than s_4 (because it includes $a_5 - a_6$ which is positive). So the sums $s_2, s_4, s_6, \ldots$ are *increasing*.

The difference between s_{n-1} and s_n is the single number $\pm a_n$. It is required by 10H to approach zero. Therefore the decreasing sequence $s_1, s_3, \ldots$ approaches the *same* limit s as the increasing sequence $s_2, s_4, \ldots$. The series converges to s, which always lies between s_{n-1} and s_n.

This plus-minus pattern is special but important. The positive series Σa_n may not converge. *The alternating series is* $\Sigma(-1)^{n+1} a_n$.

EXAMPLE 6 The alternating series $4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} \cdots$ is conditionally convergent (to π). The absolute values decrease to zero. Is this series absolutely convergent? *No.* With plus signs, $4(1 + \frac{1}{3} + \frac{1}{5} + \cdots)$ diverges like the harmonic series.

EXAMPLE 7 The alternating series $1 - 1 + 1 - 1 + \cdots$ is not convergent at all. *Which requirement in* 10H *is not met*? The partial sums $s_1, s_3, s_5, \ldots$ all equal 1 and $s_2, s_4, s_6, \ldots$ all equal 0—but they don't approach the same limit s.

MULTIPLYING AND REARRANGING SERIES

In Section 10.1 we added and subtracted and multiplied series. Certainly addition and subtraction are safe. If one series has partial sums $s_n \to s$ and the other has partial sums $t_n \to t$, then addition gives partial sums $s_n + t_n \to s + t$. But multiplication is more dangerous, because the *order* of the multiplication can make a difference. More exactly, ***the order of terms is important when the series are conditionally convergent.*** For absolutely convergent series, the order makes no difference. We can rearrange their terms and multiply them in any order, and the sum and product comes out right:

10I Suppose Σa_n converges absolutely. If $A_1, A_2, \ldots$ is any reordering of the a's, then $\Sigma A_n = \Sigma a_n$. In the new order ΣA_n also converges absolutely.

10J Suppose $\Sigma a_n = s$ and $\Sigma b_n = t$ converge absolutely. Then the infinitely many terms $a_i b_j$ in their product add (in any order) to st.

Rather than proving 10I and 10J, we show what happens when there is only conditional convergence. Our favorite is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, converging conditionally to ln 2. By rearranging, it will converge conditionally to *anything*! Suppose the desired sum is 1000. Take positive terms $1+\frac{1}{3}+\cdots$ until they pass 1000. Then add negative terms $-\frac{1}{2}-\frac{1}{4}-\cdots$ until the subtotal drops below 1000. Then new positive terms bring it above 1000, and so on. All terms are eventually used, since at least one new term is needed at each step. The limit is $s=1000$.

We also get strange products, when series fail to converge absolutely:

$$\left(1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\cdots\right)\left(1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\cdots\right)\overset{?}{=}1-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{3}}\right)\cdots.$$

On the left the series converge (conditionally). The alternating terms go to zero. On the right the series diverges. Its terms in parentheses don't even approach zero, and the product is completely wrong.

I close by emphasizing that it is absolute convergence that matters. ***The most important series are power series*** $\Sigma a_n x^n$. Like the geometric series (with all $a_n=1$) there is absolute convergence over an interval of x's. They give *functions* of x, which is what calculus needs and wants.

We go next to the series for e^x, which is absolutely convergent everywhere. From the viewpoint of convergence tests it is too easy—the danger is gone. But from the viewpoint of calculus and its applications, e^x is unconditionally the best.

10.3 EXERCISES

Read-through questions

The series Σa_n is absolutely convergent if the series __a__ is convergent. Then the original series Σa_n is also __b__. But the series Σa_n can converge without converging absolutely. That is called __c__ convergence, and the series __d__ is an example.

For alternating series, the sign of each a_{n+1} is __e__ to the sign of a_n. With the extra conditions that __f__ and __g__, the series converges (at least conditionally). The partial sums $s_1, s_3, \ldots$ are __h__ and the partial sums $s_2, s_4, \ldots$ are __i__. The difference between s_n and s_{n-1} is __j__. Therefore the two series converge to the same number s. An alternating series that converges absolutely [conditionally] (not at all) is __k__ [__l__] (__m__). With absolute [conditional] convergence a reordering (can or cannot?) change the sum.

Do the series 1–12 converge *absolutely* or *conditionally*?

1 $\sum(-1)^{n+1}\dfrac{n}{n+3}$

2 $\sum(-1)^{n-1}/\sqrt{n+3}$

3 $\sum(-1)^{n+1}\dfrac{1}{n!}$

4 $\sum(-1)^{n+1}\dfrac{3^n}{n!}$

5 $\sum(-1)^{n+1}3\sqrt{n}/(n+1)$

6 $\sum(-1)^{n+1}\sin^2 n$

7 $\sum(-1)^{n+1}\ln\left(\dfrac{1}{n}\right)$

8 $\sum(-1)^{n+1}\dfrac{\sin^2 n}{n}$

9 $\sum(-1)^{n+1}n^2/(1+n^4)$

10 $\sum(-1)^{n+1}2^{1/n}$

11 $\sum(-1)^{n+1}n^{1/n}$

12 $\sum(-1)^{n+1}(1-n^{1/n})$

13 Suppose Σa_n converges absolutely. Explain why keeping the positive a's gives another convergent series.

14 Can Σa_n converge absolutely if all a_n are negative?

15 Show that the alternating series $1-\frac{1}{2}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}+\cdots$ does not converge, by computing the partial sums $s_2, s_4, \ldots$. Which requirement of 10H is not met?

16 Show that $\frac{2}{3}-\frac{3}{5}+\frac{4}{7}-\frac{5}{9}+\cdots$ does not converge. Which requirement of 10H is not met?

17 (a) For an alternating series with terms decreasing to zero, why does the sum s always lie between s_{n-1} and s_n?

(b) Is $s-s_n$ positive or negative if s_n stops at a positive a_n?

18 Use Problem 17 to give a bound on the difference between $s_5=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$ and the sum $s=\ln 2$ of the infinite series.

19 Find the sum $1-\dfrac{1}{2!}+\dfrac{1}{3!}-\dfrac{1}{4!}+\cdots=s$. The partial sum s_4 is (above s)(below s) by less than ________.

20 Give a bound on the difference between $s_{100}=\dfrac{1}{1^2}-\dfrac{1}{2^2}+\dfrac{1}{3^2}\cdots-\dfrac{1}{100^2}$ and $s=\sum(-1)^{n+1}/n^2$.

21 Starting from $\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\cdots=\dfrac{\pi^2}{6}$, with plus signs, show that the alternating series in Problem 20 has $s=\pi^2/12$.

22 Does the alternating series in 20 or the positive series in 21 give π^2 more quickly? Compare $1/101^2-1/102^2+\cdots$ with $1/101^2+1/102^2+\cdots$.

23 If Σa_n does not converge show that $\Sigma|a_n|$ does not converge.

24 Find conditions which guarantee that $a_1 + a_2 - a_3 + a_4 + a_5 - a_6 + \cdots$ will converge (negative term follows two positive terms).

25 If the terms of $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ are rearranged into $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots$, show that this series now adds to $\frac{1}{2}\ln 2$. (Combine each positive term with the following negative term.)

26 Show that the series $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$ converges to $\frac{3}{2}\ln 2$.

27 What is the sum of $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \frac{1}{6} + \cdots$?

28 Combine $1 + \cdots + \dfrac{1}{n} - \ln n \to \gamma$ and $1 - \frac{1}{2} + \frac{1}{3} - \cdots \to \ln 2$ to prove $1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \cdots = \ln 2$.

29 (a) Prove that this alternating series converges:

$$1 - \int_1^2 \frac{dx}{x} + \frac{1}{2} - \int_2^3 \frac{dx}{x} + \frac{1}{3} - \int_3^4 \frac{dx}{x} + \cdots$$

(b) Show that its sum is Euler's constant γ.

30 Prove that this series converges. Its sum is $\pi/2$.

$$\int_0^\pi \frac{\sin x}{x}dx + \int_\pi^{2\pi} \frac{\sin x}{x}dx + \cdots = \int_0^\infty \frac{\sin x}{x}dx.$$

31 The cosine of $\theta = 1$ radian is $1 - \dfrac{1}{2!} + \dfrac{1}{4!} - \cdots$. Compute $\cos 1$ to five correct decimals (how many terms?).

32 The sine of $\theta = \pi$ radians is $\pi - \dfrac{\pi^3}{3!} + \dfrac{\pi^5}{5!} - \cdots$. Compute $\sin \pi$ to eight correct decimals (how many terms?).

33 If Σa_n^2 and Σb_n^2 are convergent show that $\Sigma a_n b_n$ is absolutely convergent.
Hint: $(a \pm b)^2 \geqslant 0$ yields $2|ab| \leqslant a^2 + b^2$.

34 Verify the *Schwarz inequality* $(\Sigma a_n b_n)^2 \leqslant (\Sigma a_n^2)(\Sigma b_n^2)$ if $a_n = (\frac{1}{2})^n$ and $b_n = (\frac{1}{3})^n$.

35 Under what condition does $\sum\limits_0^\infty (a_{n+1} - a_n)$ converge and what is its sum?

36 For a conditionally convergent series, explain how the terms could be rearranged so that the sum is $+\infty$. All terms must eventually be included, even negative terms.

37 Describe the terms in the product $(1 + \frac{1}{2} + \frac{1}{4} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots)$ and find their sum.

38 *True or false*:

(a) Every alternating series converges.

(b) Σa_n converges conditionally if $\Sigma|a_n|$ diverges.

(c) A convergent series with positive terms is absolutely convergent.

(d) If Σa_n and Σb_n both converge, so does $\Sigma(a_n + b_n)$.

39 Every number x between 0 and 2 equals $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ with suitable terms deleted. Why?

40 Every number s between -1 and 1 equals $\pm\frac{1}{2} \pm \frac{1}{4} \pm \frac{1}{8} \pm \cdots$ with a suitable choice of signs. (Add $1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ to get Problem 39.) Which signs give $s = -1$ and $s = 0$ and $s = \frac{1}{3}$?

41 Show that no choice of signs will make $\pm\frac{1}{3} \pm \frac{1}{9} \pm \frac{1}{27} \pm \cdots$ equal to zero.

42 The sums in Problem 41 form a *Cantor set* centered at zero. What is the smallest positive number in the set? Choose signs to show that $\frac{1}{4}$ is in the set.

***43** Show that the tangent of $\theta = \frac{1}{2}(\pi - 1)$ is $\sin 1/(1 - \cos 1)$. This is the imaginary part of $s = -\ln(1 - e^i)$. From $s = \Sigma\, e^{in}/n$ deduce the remarkable sum $\Sigma(\sin n)/n = \frac{1}{2}(\pi - 1)$.

44 Suppose Σa_n converges and $|x| < 1$. Show that $\Sigma a_n x^n$ converges absolutely.

10.4 The Taylor Series for e^x, $\sin x$, and $\cos x$

This section goes back from numbers to functions. Instead of $\Sigma a_n = s$ it deals with $\Sigma a_n x^n = f(x)$. ***The sum is a function of*** x. The geometric series has all $a_n = 1$ (including a_0, the constant term) and its sum is $f(x) = 1/(1 - x)$. The derivatives of $1 + x + x^2 + \cdots$ match the derivatives of f. Now we choose the a_n differently, to match a different function.

The new function is e^x. All its derivatives are e^x. At $x = 0$, this function and its derivatives equal 1. To match these 1's, we move factorials into the denominators.

Term by term the series is

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{1}$$

$x^n/n!$ has the correct nth derivative (= 1). ***From the derivatives at*** $x = 0$, ***we have built back the function***! At $x = 1$ the right side is $1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots$ and the left side is $e = 2.71828\ldots$. At $x = -1$ the series gives $1 - 1 + \frac{1}{2} - \frac{1}{6} + \cdots$, which is e^{-1}.

The same term-by-term idea works for differential equations, as follows.

EXAMPLE 1 Solve $dy/dx = -y$ starting from $y = 1$ at $x = 0$.

Solution *The zeroth derivative at* $x = 0$ *is the function itself*: $y = 1$. Then the equation $y' = -y$ gives $y' = -1$ and $y'' = -y' = +1$. The alternating derivatives $1, -1, 1, -1, \ldots$ are matched by the alternating series for e^{-x}:

$$y = 1 - x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \cdots = e^{-x} \text{ (the correct solution to } y' = -y).$$

EXAMPLE 2 Solve $d^2y/dx^2 = -y$ starting from $y = 1$ and $y' = 0$ (the answer is $\cos x$).

Solution The equation gives $y'' = -1$ (again at $x = 0$). The derivative of the equation gives $y''' = -y' = 0$. Then $y'''' = -y'' = +1$. The even derivatives are alternately $+1$ and -1, the odd derivatives are zero. This is matched by a series of even powers, which constructs $\cos x$:

$$y = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots = \cos x.$$

The first terms $1 - \frac{1}{2}x^2$ came earlier in the book. Now we have the whole alternating series. It converges absolutely for all x, by comparison with the series for e^x (odd powers are dropped). The partial sums in Figure 10.4 reach further and further before they lose touch with $\cos x$.

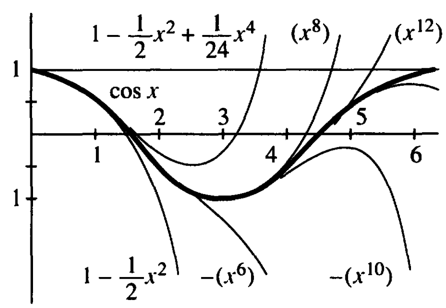

Fig. 10.4 The partial sums $1 - x^2/2 + x^4/24 - \cdots$ of the cosine series.

If we wanted plus signs instead of plus-minus, we could average e^x and e^{-x}. The differential equation for $\cosh x$ is $d^2y/dx^2 = +y$, to give plus signs:

$$\frac{1}{2}(e^x + e^{-x}) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots \text{ (which is } \cosh x).$$

TAYLOR SERIES

The idea of ***matching derivatives by powers*** is becoming central to this chapter. The derivatives are given at a basepoint (say $x = 0$). They are numbers $f(0), f'(0), \ldots$. The derivative $f^{(n)}(0)$ will be the nth derivative of $a_n x^n$, if we choose a_n to be $f^{(n)}(0)/n!$

Then the series $\Sigma a_n x^n$ has the same derivatives at the basepoint as $f(x)$:

10K The *Taylor series* that matches $f(x)$ and all its derivatives at $x = 0$ is

$$f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

The first terms give the linear and quadratic approximations that we know well. The x^3 term was mentioned earlier (but not used). Now we have *all* the terms—an "infinite approximation" that is intended to equal $f(x)$.

Two things are needed. First, the series must converge. Second, the function must do what the series predicts, away from $x = 0$. Those are true for e^x and cos x and sin x; the series equals the function. We proceed on that basis.

The Taylor series with special basepoint $x = 0$ is also called the "*Maclaurin series.*"

EXAMPLE 3 Find the Taylor series for $f(x) = \sin x$ around $x = 0$.

Solution The numbers $f^{(n)}(0)$ are the values of $f = \sin x$, $f' = \cos x$, $f'' = -\sin x$, ... at $x = 0$. Those values are $0, 1, 0, -1, 0, 1, \ldots$. All even derivatives are zero. To find the coefficients in the Taylor series, divide by the factorials:

$$\sin x = x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \cdots. \tag{2}$$

EXAMPLE 4 Find the Taylor series for $f(x) = (1 + x)^5$ around $x = 0$.

Solution This function starts at $f(0) = 1$. Its derivative is $5(1 + x)^4$, so $f'(0) = 5$. The second derivative is $5 \cdot 4 \cdot (1 + x)^3$, so $f''(0) = 5 \cdot 4$. The next three derivatives are $5 \cdot 4 \cdot 3$, $5 \cdot 4 \cdot 3 \cdot 2$, $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. *After that all derivatives are zero.* Therefore the Taylor series ***stops*** after the x^5 term:

$$1 + 5x + \frac{5 \cdot 4}{2!}x^2 + \frac{5 \cdot 4 \cdot 3}{3!}x^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{4!}x^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5!}x^5. \tag{3}$$

You may recognize 1, 5, 10, 10, 5, 1. They are the ***binomial coefficients***, which appear in Pascal's triangle (Section 2.2). By matching derivatives, we see why 0!, 1!, 2!, ... are needed in the denominators.

There is no doubt that $x = 0$ is the nicest basepoint. But Taylor series can be constructed around other points $x = a$. The principle is the same—*match derivatives by powers*—but now the powers to use are $(x - a)^n$. The derivatives $f^{(n)}(a)$ are computed at the new basepoint $x = a$.

The Taylor series begins with $f(a) + f'(a)(x - a)$. This is the tangent approximation at $x = a$. The whole "infinite approximation" is centered at a—at that point it has the same derivatives as $f(x)$.

10L The *Taylor series* for $f(x)$ around the basepoint $x = a$ is

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n. \tag{4}$$

EXAMPLE 5 Find the Taylor series for $f(x) = (1 + x)^5$ around $x = a = 1$.

Solution At $x = 1$, the function is $(1 + 1)^5 = 32$. Its first derivative $5(1 + x)^4$ is $5 \cdot 16 = 80$. We compute the nth derivative, divide by $n!$, and multiply by $(x - 1)^n$:

$$32 + 80(x - 1) + 80(x - 1)^2 + 40(x - 1)^3 + 10(x - 1)^4 + (x - 1)^5. \tag{5}$$

That Taylor series (which stops at $n = 5$) should agree with $(1 + x)^5$. It does. We could rewrite $1 + x$ as $2 + (x - 1)$, and take its fifth power directly. Then 32, 16, 8, 4, 2, 1 will multiply the usual coefficients 1, 5, 10, 10, 5, 1 to give our Taylor coefficients 32, 80, 80, 40, 10, 1. The series stops as it will stop for any polynomial—because the high derivatives are zero.

EXAMPLE 6 Find the Taylor series for $f(x) = e^x$ around the basepoint $x = 1$.

Solution At $x = 1$ the function and all its derivatives equal e. Therefore the Taylor series has that constant factor (note the powers of $x - 1$, not x):

$$e^x = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \cdots. \tag{6}$$

DEFINING THE FUNCTION BY ITS SERIES

Usually, we define $\sin x$ and $\cos x$ from the sides of a triangle. But we could start instead with the series. Define $\sin x$ by equation (2). The logic goes backward, but it is still correct:

First, prove that the series converges.
Second, prove properties like $(\sin x)' = \cos x$.
Third, connect the definitions by series to the sides of a triangle.

We don't plan to do all this. The usual definition was good enough. But note first: There is no problem with convergence. The series for $\sin x$ and $\cos x$ and e^x all have terms $\pm x^n/n!$. The factorials make the series converge for all x. The general rule for e^x times e^y can be based on the series. Equation (6) is typical: e is multiplied by powers of $(x - 1)$. Those powers add to e^{x-1}. So the series proves that $e^x = ee^{x-1}$. That is just one example of the multiplication $(e^x)(e^y) = e^{x+y}$:

$$\left(1 + x + \frac{x^2}{2} + \ldots\right)\left(1 + y + \frac{y^2}{2} + \ldots\right) = 1 + x + y + \frac{x^2}{2} + xy + \frac{y^2}{2} + \ldots. \tag{7}$$

Term by term, multiplication gives the series for e^{x+y}. Term by term, differentiating the series for e^x gives e^x. Term by term, the derivative of $\sin x$ is $\cos x$:

$$\frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots. \tag{8}$$

We don't need the famous limit $(\sin x)/x \to 1$, by which geometry gave us the derivative. The identities of trigonometry become identities of infinite series. We could even define π as the first positive x at which $x - \frac{1}{6}x^3 + \cdots$ equals zero. But it is certainly not obvious that this sine series returns to zero—much less that the point of return is near 3.14.

*The function that **will** be defined by infinite series is $e^{i\theta}$.* This is the exponential of the ***imaginary number*** $i\theta$ (a multiple of $i = \sqrt{-1}$). The result $e^{i\theta}$ is a ***complex number***, and our goal is to identify it. (We will be confirming Section 9.4.) The technique is to treat $i\theta$ like all other numbers, real or complex, and simply put it into the series:

DEFINITION $e^{i\theta}$ is the sum of $1 + (i\theta) + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \cdots$. (9)

Now use $i^2 = -1$. The even powers are $i^4 = +1$, $i^6 = -1$, $i^8 = +1, \ldots$. We are just multiplying -1 by -1 to get 1. The odd powers are $i^3 = -i$, $i^5 = +i, \ldots$. Therefore $e^{i\theta}$ splits into a *real part* (with no i's) and an *imaginary part* (multiplying i):

$$e^{i\theta}=\left(1-\frac{1}{2!}\theta^2+\frac{1}{4!}\theta^4-\cdots\right)+i\left(\theta-\frac{1}{3!}\theta^3+\frac{1}{5!}\theta^5-\cdots\right). \tag{10}$$

You recognize those series. They are $\cos\theta$ and $\sin\theta$. Therefore:

Euler's formula is $e^{i\theta}=\cos\theta+i\sin\theta$. ***Note that*** $e^{2\pi i}=1$.

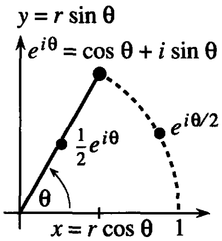

Fig. 10.5

The real part is $x=\cos\theta$ and the imaginary part is $y=\sin\theta$. Those coordinates pick out the point $e^{i\theta}$ in the "complex plane." Its distance from the origin (0, 0) is $r=1$, because $(\cos\theta)^2+(\sin\theta)^2=1$. Its angle is θ, as shown in Figure 10.5. The number -1 is $e^{i\pi}$, at the distance $r=1$ and the angle π. It is on the real axis to the left of zero. If $e^{i\theta}$ is multiplied by $r=2$ or $r=\frac{1}{2}$ or any $r\geqslant 0$, the result is a complex number at a distance r from the origin:

Complex numbers: $re^{i\theta}=r(\cos\theta+i\sin\theta)=r\cos\theta+ir\sin\theta=x+iy$.

With $e^{i\theta}$, a negative number has a logarithm. ***The logarithm of*** -1 ***is imaginary*** (it is $i\pi$, since $e^{i\pi}=-1$). A negative number also has fractional powers. The fourth root of -1 is $(-1)^{1/4}=e^{i\pi/4}$. More important for calculus: *The derivative of* $x^{5/4}$ *is* $\frac{5}{4}x^{1/4}$. That sounds old and familiar, but at $x=-1$ it was never allowed.

Complex numbers tie up the loose ends left by the limitations of real numbers.

The formula $e^{i\theta}=\cos\theta+i\sin\theta$ has been called "one of the greatest mysteries of undergraduate mathematics." Writers have used desperate methods to avoid infinite series. That proof in (10) may be the clearest (I remember sending it to a prisoner studying calculus) but here is a way to start from $d/dx(e^{ix})=ie^{ix}$.

A different proof of Euler's formula Any complex number is $e^{ix}=r(\cos\theta+i\sin\theta)$ for *some* r and θ. Take the x derivative of both sides, and substitute for ie^{ix}:

$$(\cos\theta+i\sin\theta)dr/dx+r(-\sin\theta+i\cos\theta)d\theta/dx=ir(\cos\theta+i\sin\theta).$$

Comparing the real parts and also the imaginary parts, we need $dr/dx=0$ and $d\theta/dx=1$. The starting values $r=1$ and $\theta=0$ are known from $e^{i0}=1$. Therefore r is always 1 and θ is x. Substituting into the first sentence of the proof, we have Euler's formula $e^{i\theta}=1(\cos\theta+i\sin\theta)$.

10.4 EXERCISES

Read-through questions

The ____a____ series is chosen to match $f(x)$ and all its ____b____ at the basepoint. Around $x=0$ the series begins with $f(0)+$ ____c____ $x+$ ____d____ x^2. The coefficient of x^n is ____e____. For $f(x)=e^x$ this series is ____f____. For $f(x)=\cos x$ the series is ____g____. For $f(x)=\sin x$ the series is ____h____. If the signs were all positive in those series, the functions would be $\cosh x$ and ____i____. Addition gives $\cosh x+\sinh x=$ ____j____.

In the Taylor series for $f(x)$ around $x=a$, the coefficient of $(x-a)^n$ is $b_n=$ ____k____. Then $b_n(x-a)^n$ has the same ____l____ as f at the basepoint. In the example $f(x)=x^2$, the Taylor coefficients are $b_0=$ ____m____, $b_1=$ ____n____, $b_2=$ ____o____. The series $b_0+b_1(x-a)+b_2(x-a)^2$ agrees with the original ____p____. The series for e^x around $x=a$ has $b_n=$ ____q____. Then the Taylor series reproduces the identity $e^x=($ ____r____ $)($ ____s____ $)$.

We define e^x, $\sin x$, $\cos x$, and also $e^{i\theta}$ by their series. The derivative $d/dx(1+x+\frac{1}{2}x^2+\cdots)=1+x+\cdots$ translates to ____t____. The derivative of $1-\frac{1}{2}x^2+\cdots$ is ____u____. Using $i^2=-1$ the series $1+i\theta+\frac{1}{2}(i\theta)^2+\cdots$ splits into $e^{i\theta}=$ ____v____. Its square gives $e^{2i\theta}=$ ____w____. Its reciprocal is $e^{-i\theta}=$ ____x____. Multiplying by r gives $re^{i\theta}=$ ____y____ $+i$ ____z____, which connects the polar and rectangular forms of a ____A____ number. The logarithm of $e^{i\theta}$ is ____B____.

1 Write down the series for e^{2x} and compute all derivatives at $x=0$. Give a series of numbers that adds to e^2.

2 Write down the series for $\sin 2x$ and check the third derivative at $x=0$. Give a series of numbers that adds to $\sin 2\pi=0$.

In 3–8 find the derivatives of $f(x)$ at $x=0$ and the Taylor series (powers of x) with those derivatives.

3 $f(x)=e^{ix}$

4 $f(x)=1/(1+x)$

5 $f(x)=1/(1-2x)$

6 $f(x)=\cosh x$

7 $f(x)=\ln(1-x)$

8 $f(x)=\ln(1+x)$

Problems 9–14 solve differential equations by series.

9 From the equation $dy/dx=y-2$ find all the derivatives of y at $x=0$ starting from $y(0)=1$. Construct the infinite series for y, identify it as a known function, and verify that the function satisfies $y'=y-2$.

10 Differentiate the equation $y'=cy+s$ (c and s constant) to find all derivatives of y at $x=0$. If the starting value is $y_0=0$, construct the Taylor series for y and identify it with the solution of $y'=cy+s$ in Section 6.3.

11 Find the infinite series that solves $y''=-y$ starting from $y=0$ and $y'=1$ at $x=0$.

12 Find the infinite series that solves $y'=y$ starting from $y=1$ at $x=3$ (use powers of $x-3$). Identify y as a known function.

13 Find the infinite series (powers of x) that solves $y''=2y'-y$ starting from $y=0$ and $y'=1$ at $x=0$.

14 Solve $y''=y$ by a series with $y=1$ and $y'=0$ at $x=0$ and identify y as a known function.

15 Find the Taylor series for $f(x)=(1+x)^2$ around $x=a=0$ and around $x=a=1$ (powers of $x-1$). Check that both series add to $(1+x)^2$.

16 Find all derivatives of $f(x)=x^3$ at $x=a$ and write out the Taylor series around that point. Verify that it adds to x^3.

17 What is the series for $(1-x)^5$ with basepoint $a=1$?

18 Write down the Taylor series for $f=\cos x$ around $x=2\pi$ and also for $f=\cos(x-2\pi)$ around $x=0$.

In 19–24 compute the derivatives of f and its Taylor series around $x=1$.

19 $f(x)=1/x$

20 $f(x)=1/(2-x)$

21 $f(x)=\ln x$

22 $f(x)=x^4$

23 $f(x)=e^{-x}$

24 $f(x)=e^{2x}$

In 25–33 write down the first three nonzero terms of the Taylor series around $x=0$, from the series for e^x, $\cos x$, and $\sin x$.

25 xe^{2x}

26 $\cos\sqrt{x}$

27 $(1-\cos x)/x^2$

28 $\dfrac{\sin x}{x}$

29 $\displaystyle\int_0^x \frac{\sin x}{x}\,dx$

30 $\sin x^2$

31 e^{x^2}

32 $b^x=e^{x\ln b}$

33 $e^x\cos x$

*34 For $x<0$ the derivative of x^n is still nx^{n-1}:

$$\frac{d}{dx}(x^n)=\frac{d}{dx}(|x|^n e^{in\pi})=n|x|^{n-1}e^{in\pi}\frac{d|x|}{dx}.$$

What is $d|x|/dx$? Rewrite this answer as nx^{n-1}.

35 Why doesn't $f(x)=\sqrt{x}$ have a Taylor series around $x=0$? Find the first two terms around $x=1$.

36 Find the Taylor series for 2^x around $x=0$.

In 37–44 find the first three terms of the Taylor series around $x=0$.

37 $f(x)=\tan^{-1}x$

38 $f(x)=\sin^{-1}x$

39 $f(x)=\tan x$

40 $f(x)=\ln(\cos x)$

41 $f(x)=e^{\sin x}$

42 $f(x)=\tanh^{-1}x$

43 $f(x)=\cos^2 x$

44 $f(x)=\sec^2 x$

45 From $e^{i\theta}=\cos\theta+i\sin\theta$ and $e^{-i\theta}=\cos\theta-i\sin\theta$, add and subtract to find $\cos\theta$ and $\sin\theta$.

46 Does $(e^{i\theta})^2$ equal $\cos^2\theta+i\sin^2\theta$ or $\cos\theta^2+i\sin\theta^2$?

47 Find the real and imaginary parts and the 99th power of $e^{i\pi}$, $e^{i\pi/2}$, $e^{i\pi/4}$, and $e^{-i\pi/6}$.

48 The three cube roots of 1 are 1, $e^{2\pi i/3}$, $e^{4\pi i/3}$.
(a) Find the real and imaginary parts of $e^{2\pi i/3}$.
(b) Explain why $(e^{2\pi i/3})^3=1$.
(c) Check this statement in rectangular coordinates.

49 The cube roots of $-1=e^{i\pi}$ are $e^{i\pi/3}$ and ________ and ________. Find their sum and their product.

50 Find the squares of $2e^{i\pi/3}=1+\sqrt{3}\,i$ and $4e^{i\pi/4}=2\sqrt{2}+i2\sqrt{2}$ in both polar and rectangular coordinates.

51 Multiply $e^{is}=\cos s+i\sin s$ times $e^{it}=\cos t+i\sin t$ to find formulas for $\cos(s+t)$ and $\sin(s+t)$.

52 Multiply e^{is} times e^{-it} to find formulas for $\cos(s-t)$ and $\sin(s-t)$.

53 Find the logarithm of i. Then find another logarithm of i. (What can you add to the exponent of $e^{\ln i}$ without changing the result?)

54 (Proof that e is irrational) If $e=p/q$ then

$$N=p!\left[\frac{1}{e}-\left(1-\frac{1}{1!}+\frac{1}{2!}-\cdots\pm\frac{1}{p!}\right)\right]$$

would be an integer. (Why?) The number in brackets—the distance from the alternating series to its sum $1/e$—is less than the last term which is $1/p!$ Deduce that $|N|<1$ and reach a contradiction, which proves that e cannot equal p/q.

55 Solve $dy/dx=y$ by infinite series starting from $y=2$ at $x=0$.

10.5 Power Series

This section studies the properties of a power series. When the basepoint is zero, the powers are x^n. The series is $\Sigma a_n x^n$. When the basepoint is $x = a$, the powers are $(x-a)^n$. We want to know when and where (and how quickly) the series converges to the underlying function. For e^x and $\cos x$ and $\sin x$ there is convergence for all x—but that is certainly not true for $1/(1-x)$. The convergence is best when the function is smooth.

First I emphasize that power series are not the only series. For many applications they are not the best choice. An alternative is a sum of sines, $f(x) = \Sigma b_n \sin nx$. That is a "*Fourier sine series*", which treats all x's equally instead of picking on a basepoint. A Fourier series allows jumps and corners in the graph—it takes the rough with the smooth. By contrast a power series is terrific near its basepoint, and gets worse as you move away. The Taylor coefficients a_n are totally determined *at the basepoint*—where all derivatives are computed. Remember the rule for Taylor series:

$$a_n = (n\text{th derivative at the basepoint})/n! = f^{(n)}(a)/n! \tag{1}$$

A remarkable fact is the convergence in a *symmetric interval around* $x = a$.

10M A power series $\Sigma a_n x^n$ either converges for all x, or it converges only at the basepoint $x = 0$, or else it has a ***radius of convergence r***:

$$\Sigma a_n x^n \textit{ converges absolutely if } |x| < r \textit{ and diverges if } |x| > r.$$

The series $\Sigma x^n/n!$ converges for all x (the sum is e^x). The series $\Sigma n! x^n$ converges for no x (except $x = 0$). The geometric series Σx^n converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. ***Its radius of convergence is*** $r = 1$. Note that its sum $1/(1-x)$ is perfectly good for $|x| > 1$—the function is all right but the series has given up. If something goes wrong at the distance r, a power series can't get past that point.

When the basepoint is $x = a$, the interval of convergence shifts over to $|x - a| < r$. The series converges for x between $a - r$ and $a + r$ (symmetric around a). We cannot say in advance whether the endpoints $a \pm r$ give divergence or convergence (absolute or conditional). *Inside* the interval, an easy comparison test will now prove convergence.

PROOF OF 10M Suppose $\Sigma a_n X^n$ converges at a particular point X. The proof will show that $\Sigma a_n x^n$ converges when $|x|$ is less than the number $|X|$. Thus convergence at X gives convergence at all closer points x (I mean closer to the basepoint 0). Proof: Since $\Sigma\, a_n X^n$ converges, its terms approach zero. Eventually $|a_n X^n| < 1$ and then

$$|a_n x^n| = |a_n X^n|\,|x/X|^n < |x/X|^n.$$

Our series $\Sigma a_n x^n$ is absolutely convergent by comparison with the geometric series for $|x/X|$, which converges since $|x/X| < 1$.

EXAMPLE 1 The series $\Sigma n x^n/4^n$ has radius of convergence $r = 4$.

The ratio test and root test are best for power series. The ratios between terms approach $x/4$ (and so does the nth root of $nx^n/4^n$):

$$\frac{(n+1)x^{n+1}}{4^{n+1}} \bigg/ \frac{nx^n}{4^n} = \frac{x}{4}\frac{n+1}{n} \text{ approaches } L = \frac{x}{4}.$$

The ratio test gives convergence if $L < 1$, which means $|x| < 4$.

EXAMPLE 2 The sine series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ has $r = \infty$ (it converges everywhere).

The ratio of $x^{n+2}/(n+2)!$ to $x^n/n!$ is $x^2/(n+2)(n+1)$. This approaches $L = 0$.

EXAMPLE 3 The series $\Sigma(x-5)^n/n^2$ has radius $r = 1$ *around its basepoint* $a = 5$.

The ratios between terms approach $L = x - 5$. (The fractions $n^2/(n+1)^2$ go toward 1.) There is absolute convergence if $|x-5| < 1$. This is the interval $4 < x < 6$, symmetric around the basepoint. This series happens to converge at the endpoints 4 and 6, because of the factor $1/n^2$. That factor decides the delicate question—convergence at the endpoints—but all powers of n give the same ***interval of convergence*** $4 < x < 6$.

CONVERGENCE TO THE FUNCTION: REMAINDER TERM AND RADIUS r

Remember that a Taylor series starts with a function $f(x)$. The derivatives at the basepoint produce the series. Suppose the series converges: ***Does it converge to the function?*** This is a question about the *remainder* $R_n(x) = f(x) - s_n(x)$, which is the difference between f and the partial sum $s_n = a_0 + \cdots + a_n(x-a)^n$. ***The remainder*** R_n ***is the error if we stop the series***, ending with the nth derivative term $a_n(x-a)^n$.

> **10N** Suppose f has an $(n+1)$st derivative from the basepoint a out to x. Then for some point c in between (position not known) the remainder at x equals
>
> $$R_n(x) = f(x) - s_n(x) = f^{(n+1)}(c)(x-a)^{n+1}/(n+1)! \qquad (2)$$

The error in stopping at the nth derivative is controlled by the $(n+1)$st derivative.

You will guess, correctly, that the unknown point c comes from the Mean Value Theorem. For $n = 1$ the proof is at the end of Section 3.8. That was the error $e(x)$ in *linear* approximation:

$$R_1(x) = f(x) - f(a) - f'(a)(x-a) = \tfrac{1}{2}f''(c)(x-a)^2.$$

For every n, the proof compares R_n with $(x-a)^{n+1}$. Their $(n+1)$st derivatives are $f^{(n+1)}$ and $(n+1)!$ The generalized Mean Value Theorem says that the ratio of R_n to $(x-a)^{n+1}$ equals the ratio of those derivatives, at the right point c. That is equation (2). The details can stay in Section 3.8 and Problem 23, because the main point is what we want. ***The error is exactly like the next term*** $a_{n+1}(x-a)^{n+1}$, except that the $(n+1)$st derivative is at c instead of the basepoint a.

EXAMPLE 4 When f is e^x, the $(n+1)$st derivative is e^x. Therefore the error is

$$R_n = e^x - \left(1 + x + \cdots + \frac{x^n}{n!}\right) = e^c \frac{x^{n+1}}{(n+1)!}. \qquad (3)$$

At $x = 1$ and $n = 2$, the error is $e - (1 + 1 + \frac{1}{2}) \approx .218$. The right side is $e^c/6$. The unknown point is $c = \ln(.218 \cdot 6) = .27$. Thus c lies between the basepoint $a = 0$ and the error point $x = 1$, as required. The series converges to the function, because $R_n \to 0$.

In practice, n is the number of derivatives to be calculated. We may aim for an error $|R_n|$ below 10^{-6}. Unfortunately, the high derivative in formula (2) is awkward to estimate (except for e^x). And high derivatives in formula (1) are difficult to compute. Most real calculations use only a *few terms* of a Taylor series. For more accuracy we move the basepoint closer, or switch to another series.

There is a direct connection between the function and the convergence radius r. A hint came for $f(x) = 1/(1-x)$. The function blows up at $x = 1$—which also ends the convergence interval for the series. Another hint comes for $f = 1/x$, if we expand around $x = a = 1$:

$$\frac{1}{x} = \frac{1}{1-(1-x)} = 1 + (1-x) + (1-x)^2 + \cdots. \tag{4}$$

This geometric series converges for $|1-x| < 1$. Convergence stops at the end point $x = 0$—exactly where $1/x$ blows up. ***The failure of the function stops the convergence of the series.*** But note that $1/(1+x^2)$, which never seems to fail, also has convergence radius $r = 1$:

$$1/(1+x^2) = 1 - x^2 + x^4 - x^6 + \cdots \text{ converges only for } |x| < 1.$$

When you see the reason, you will know why r is a "radius." There is a circle, and the function fails at the edge of the circle. The circle contains complex numbers as well as real numbers. The imaginary points i and $-i$ are at the edge of the circle. ***The function fails at those points because*** $1/(1+i^2) = \infty$.

Complex numbers are pulling the strings, out of sight. The circle of convergence reaches out to the nearest "singularity" of $f(x)$, real or imaginary or complex. For $1/(1+x^2)$, the singularities at i and $-i$ make $r = 1$. If we expand around $a = 3$, the distance to i and $-i$ is $r = \sqrt{10}$. If we change to $\ln(1+x)$, which blows up at $x = -1$, the radius of convergence of $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$ is $r = 1$.

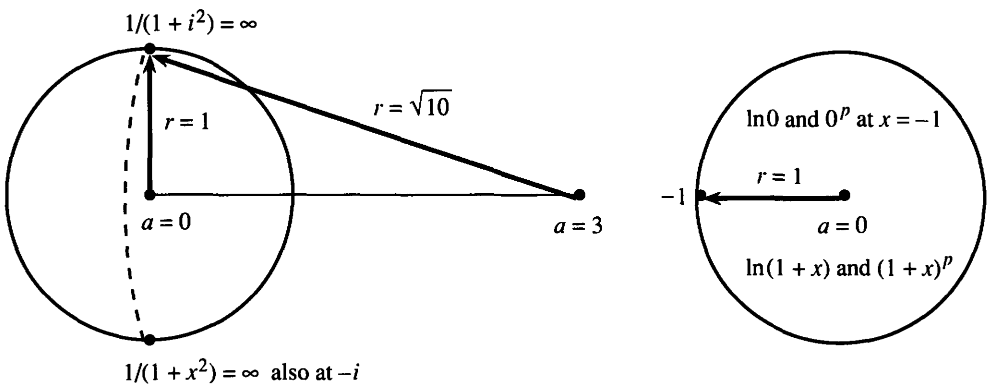

Fig. 10.6 Convergence radius r is distance from basepoint a to nearest singularity.

THE BINOMIAL SERIES

We close this chapter with one more series. It is the Taylor series for $(1+x)^p$, around the basepoint $x = 0$. A typical power is $p = \frac{1}{2}$, where we want the terms in

$$\sqrt{1+x} = 1 + \tfrac{1}{2}x + a_2x^2 + \cdots.$$

The slow way is to square both sides, which gives $1 + x + (2a_2 + \frac{1}{4})x^2$ on the right. Since $1 + x$ is on the left, $a_2 = -\frac{1}{8}$ is needed to remove the x^2 term. Eventually a_3 can be found. The fast way is to match the derivatives of $f = (1+x)^{1/2}$:

$$f' = \tfrac{1}{2}(1+x)^{-1/2} \qquad f'' = (\tfrac{1}{2})(-\tfrac{1}{2})(1+x)^{-3/2} \qquad f''' = (\tfrac{1}{2})(-\tfrac{1}{2})(-\tfrac{3}{2})(1+x)^{-5/2}.$$

At $x = 0$ those derivatives are $\frac{1}{2}$, $-\frac{1}{4}$, $\frac{3}{8}$. Dividing by 1!, 2!, 3! gives

$$a_1 = \frac{1}{2} \quad a_2 = -\frac{1}{8} \quad a_3 = \frac{1}{16} \qquad a_n = \frac{1}{n!}\left(\frac{1}{2}\right)\left(\frac{1}{2} - 1\right)\cdots\left(\frac{1}{2} - n + 1\right).$$

These are the ***binomial coefficients*** when the power is $p = \frac{1}{2}$.

Notice the difference from the binomials in Chapter 2. For those, the power p was a positive integer. The series $(1 + x)^2 = 1 + 2x + x^2$ stopped at x^2. The coefficients for $p = 2$ were $1, 2, 1, 0, 0, 0, \ldots$. For fractional p or negative p those later coefficients are *not* zero, and we find them from the derivatives of $(1 + x)^p$:

$$(1 + x)^p \quad p(1 + x)^{p-1} \quad p(p-1)(1 + x)^{p-2} \qquad f^{(n)} = p(p-1)\cdots(p - n + 1)(1 + x)^{p-n}.$$

Dividing by $0!, 1!, 2!, \ldots, n!$ at $x = 0$, the binomial coefficients are

$$1 \quad p \quad \frac{p(p-1)}{2} \quad \cdots \quad \frac{f^{(n)}(0)}{n!} = \frac{p(p-1)\cdots(p-n+1)}{n!}. \tag{5}$$

For $p = n$ that last binomial coefficient is $n!/n! = 1$. It gives the final x^n at the end of $(1 + x)^n$. For other values of p, the binomial series never stops. ***It converges for*** $|x| < 1$:

$$(1 + x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \cdots = \sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!}x^n. \tag{6}$$

When $p = 1, 2, 3, \ldots$ ***the binomial coefficient*** $p!/n!(n-p)!$ ***counts the number of ways to select a group of n friends out of a group of p friends***. If you have 20 friends, you can choose 2 of them in $(20)(19)/2 = 190$ ways.

Suppose p is not a positive integer. What goes wrong with $(1 + x)^p$, to stop the convergence at $|x| = 1$? The failure is at $x = -1$. If p is negative, $(1 + x)^p$ blows up. If p is positive, as in $\sqrt{1 + x}$, the higher derivatives blow up. Only for a positive integer $p = n$ does the convergence radius move out to $r = \infty$. In that case the series for $(1 + x)^n$ stops at x^n, and f never fails.

A power series is a function in a new form. It is not a simple form, but sometimes it is the only form. To compute f we have to sum the series. To square f we have to multiply series. But the operations of calculus—derivative and integral—are easier. That explains why power series help to solve differential equations, which are a rich source of new functions. (Numerically the series are not always so good.) I should have said that the derivative and integral are easy *for each separate term* $a_n x^n$—and fortunately the convergence radius of the whole series is not changed.

If $f(x) = \Sigma a_n x^n$ ***has convergence radius r, so do its derivative and its integral:***

$df/dx = \Sigma n a_n x^{n-1}$ and $\int f(x)dx = \Sigma a_n x^{n+1}/(n+1)$ also converge for $|x| < r$.

EXAMPLE 5 The series for $1/(1-x)$ and its derivative $1/(1-x)^2$ and its integral $-\ln(1-x)$ all have $r = 1$ (because they all have trouble at $x = 1$). The series are Σx^n and $\Sigma n x^{n-1}$ and $\Sigma x^{n+1}/(n+1)$.

EXAMPLE 6 We can integrate e^{x^2} (previously impossible) by integrating every term in its series:

$$\int e^{x^2}\,dx = \int\left(1 + x^2 + \frac{1}{2!}x^4 + \cdots\right)dx = x + \frac{x^3}{3} + \frac{1}{2!}\left(\frac{x^5}{5}\right) + \frac{1}{3!}\left(\frac{x^7}{7}\right) + \cdots.$$

This always converges ($r = \infty$). The derivative of e^{x^2} was never a problem.

10.5 EXERCISES

Read-through questions

If $|x| < |X|$ and $\Sigma a_n X^n$ converges, then the series $\Sigma a_n x^n$ also __a__. There is convergence in a __b__ interval around the __c__. For $\Sigma(2x)^n$ the convergence radius is $r =$ __d__. For $\Sigma x^n/n!$ the radius is $r =$ __e__. For $\Sigma(x-3)^n$ there is convergence for $|x-3| <$ __f__. Then x is between __g__ and __h__.

Starting with $f(x)$, its Taylor series $\Sigma a_n x^n$ has $a_n =$ __i__. With basepoint a, the coefficient of $(x-a)^n$ is __j__. The error after the x^n term is called the __k__ $R_n(x)$. It is equal to __l__ where the unknown point c is between __m__. Thus the error is controlled by the __n__ derivative.

The circle of convergence reaches out to the first point where $f(x)$ fails. For $f = 4/(2-x)$, that point is $x =$ __o__. Around the basepoint $a = 5$, the convergence radius would be $r =$ __p__. For $\sin x$ and $\cos x$ the radius is $r =$ __q__.

The series for $\sqrt{1+x}$ is the __r__ series with $p = \frac{1}{2}$. Its coefficients are $a_n =$ __s__. Its convergence radius is __t__. Its square is the very short series $1 + x$.

In 1–6 find the Taylor series for $f(x)$ around $x = 0$ and its radius of convergence r. At what point does $f(x)$ blow up?

1 $f(x) = 1/(1-4x)$

2 $f(x) = 1/(1-4x^2)$

3 $f(x) = e^{1-x}$

4 $f(x) = \tan x$ (through x^3)

5 $f(x) = \ln(e + x)$

6 $f(x) = 1/(1+4x^2)$

Find the interval of convergence and the function in 7–10.

7 $f(x) = \sum_0^\infty \left(\frac{x-1}{2}\right)^n$

8 $f(x) = \sum_0^\infty n(x-a)^{n-1}$

9 $f(x) = \sum_0^\infty \frac{1}{n+1}(x-a)^{n+1}$

10 $f(x) = (x - 2\pi) - \frac{(x-2\pi)^3}{3!} + \cdots$

11 Write down the Taylor series for $(e^x - 1)/x$, based on the series for e^x. At $x = 0$ the function is $0/0$. Evaluate the series at $x = 0$. Check by l'Hôpital's Rule on $(e^x - 1)/x$.

12 Write down the Taylor series for xe^x around $x = 0$. Integrate and substitute $x = 1$ to find the sum of $1/n!(n+2)$.

13 If $f(x)$ is an even function, so $f(-x) = f(x)$, what can you say about its Taylor coefficients in $f = \Sigma a_n x^n$?

14 Puzzle out the sums of the following series:

(a) $x + x^2 - x^3 + x^4 + x^5 - x^6 + \cdots$

(b) $1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \cdots$

(c) $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$

15 From the series for $(1 - \cos x)/x^2$ find the limit as $x \to 0$ faster than l'Hôpital's rule.

16 Construct a power series that converges for $0 < x < 2\pi$.

17–24 are about remainders and 25–36 are about binomials.

17 If the cosine series stops before $x^8/8!$ show from (2) that the remainder R_7 is less than $x^8/8!$ Does this also follow because the series is alternating?

18 If the sine series around $x = 2\pi$ stops after the terms in problem 10, estimate the remainder from equation (2).

19 Estimate by (2) the remainder $R_n = x^{n+1} + x^{n+2} + \cdots$ in the geometric series. Then compute R_n exactly and find the unknown point c for $n = 2$ and $x = \frac{1}{2}$.

20 For $-\ln(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + R_3$, use equation (2) to show that $R_3 \leqslant \frac{1}{8}$ at $x = \frac{1}{2}$.

21 Find R_n in Problem 20 and show that the series converges to the function at $x = \frac{1}{2}$ (prove that $R_n \to 0$).

22 By estimating R_n prove that the Taylor series for e^x around $x = 1$ converges to e^x as $n \to \infty$.

23 (Proof of the remainder formula when $n = 2$)

(a) At $x = a$ find R_2, R_2', R_2'', R_2'''.

(b) At $x = a$ evaluate $g(x) = (x-a)^3$ and g', g'', g'''.

(c) What rule gives $\frac{R_2(x) - R_2(a)}{g(x) - g(a)} = \frac{R_2'(c_1)}{g'(c_1)}$?

(d) In $\frac{R_2'(c_1) - R_2'(a)}{g'(c_1) - g'(a)} = \frac{R_2''(c_2)}{g''(c_2)}$ and

$\frac{R_2''(c_2) - R_2''(a)}{g''(c_2) - g''(a)} = \frac{R_2'''(c)}{g'''(c)}$ where are c_1 and c_2 and c?

(e) Combine (a-b-c-d) into the remainder formula (2).

24 All derivatives of $f(x) = e^{-1/x^2}$ are zero at $x = 0$, including $f(0) = 0$. What is $f(.1)$? What is the Taylor series around $x = 0$? What is the radius of convergence? Where does the series converge to $f(x)$? For $x = 1$ and $n = 1$ what is the remainder estimate in (2)?

25 (a) Find the first three terms in the binomial series for $1/\sqrt{1-x^2}$.

(b) Integrate to find the first three terms in the Taylor series for $\sin^{-1} x$.

26 Show that the binomial coefficients in $1/\sqrt{1-x} = \Sigma a_n x^n$ are $a_n = 1 \cdot 3 \cdot 5 \cdots (2n-1)/2^n n!$

27 For $p = -1$ and $p = -2$ find nice formulas for the binomial coefficients.

28 Change the dummy variable and add lower limits to make $\Sigma^\infty nx^{n-1} = \Sigma^\infty (n+1)x^n$.

29 In $(1-x)^{-1}=\Sigma x^n$ the coefficient of x^n is the number of groups of n friends that can be formed from 1 friend (not binomial—repetition is allowed!). The coefficient is 1 and there is only one group—the same friend n times.

(a) Describe all groups of n friends that can be formed from 2 friends. (There are $n+1$ groups.)

(b) How many groups of 5 friends can be formed from 3 friends?

30 (a) What is the coefficient of x^n when $1+x+x^2+\cdots$ multiplies $1+x+x^2+\cdots$? Write the first three terms.

(b) What is the coefficient of x^5 in $(\Sigma x^k)^3$?

31 Show that the binomial series for $\sqrt{1+4x}$ has integer coefficients. (Note that x^n changes to $(4x)^n$. These coefficients are important in counting trees, paths, parentheses...)

32 In the series for $1/\sqrt{1-4x}$, show that the coefficient of x^n is $(2n)!$ divided by $(n!)^2$.

Use the binomial series to compute 33–36 with error less than 1/1000.

33 $(15)^{1/4}$

34 $(1001)^{1/3}$

35 $(1.1)^{1.1}$

36 $e^{1/1000}$

37 From $\sec x = 1/[1-(1-\cos x)]$ find the Taylor series of $\sec x$ up to x^6. What is the radius of convergence r (distance to blowup point)?

38 From $\sec^2 x = 1/[1-\sin^2 x]$ find the Taylor series up to x^2. Check by squaring the secant series in Problem 37. Check by differentiating the tangent series in Problem 39.

39 (Division of series) Find $\tan x$ by long division of $\sin x/\cos x$:

$$\left(x-\frac{x^3}{6}+\frac{x^5}{120}\cdots\right)\Big/\left(1-\frac{x^2}{2}+\frac{x^4}{24}\cdots\right)=x+\frac{x^3}{3}+\frac{2x^5}{15}+\cdots.$$

40 (Composition of series) If $f=a_0+a_1x+a_2x^2+\cdots$ and $g=b_1x+b_2x^2+\cdots$ find the $1, x, x^2$ coefficients of $f(g(x))$. Test on $f=1/(1+x)$, $g=x/(1-x)$, with $f(g(x))=1-x$.

41 (Multiplication of series) From the series for $\sin x$ and $1/(1-x)$ find the first four terms for $f=\sin x/(1-x)$.

42 (Inversion of series) If $f=a_1x+a_2x^2+\cdots$ find coefficients b_1, b_2 in $g=b_1x+b_2x^2+\cdots$ so that $f(g(x))=x$. Compute b_1, b_2 for $f=e^x-1$, $g=f^{-1}=\ln(1+x)$.

43 From the multiplication $(\sin x)(\sin x)$ or the derivatives of $f(x)=\sin^2 x$ find the first three terms of the series. Find the first four terms for $\cos^2 x$ by an easy trick.

44 Somehow find the first six nonzero terms for $f=(1-x)/(1-x^3)$.

45 Find four terms of the series for $1/\sqrt{1-x}$. Then square the series to reach a geometric series.

46 Compute $\int_0^1 e^{-x^2}dx$ to 3 decimals by integrating the power series.

47 Compute $\int_0^1 \sin^2 t\, dt$ to 4 decimals by power series.

48 Show that $\Sigma x^n/n$ converges at $x=-1$, even though its derivative Σx^{n-1} diverges. How can they have the same convergence radius?

49 Compute $\lim_{x\to 0}(\sin x-\tan x)/x^3$ from the series.

50 If the nth root of a_n approaches $L>0$, explain why $\Sigma a_n x^n$ has convergence radius $r=1/L$.

51 Find the convergence radius r around basepoints $a=0$ and $a=1$ from the blowup points of $(1+\tan x)/(1+x^2)$.