# CHAPTER 12

# Motion Along a Curve

## 12.1 The Position Vector

This chapter is about "vector functions." The vector $2\mathbf{i}+4\mathbf{j}+8\mathbf{k}$ is constant. The vector $\mathbf{R}(t)=t\mathbf{i}+t^2\mathbf{j}+t^3\mathbf{k}$ is moving. It is a function of the parameter $t$, which often represents time. At each time $t$, the position vector $\mathbf{R}(t)$ locates the moving body:

$$\textbf{\textit{position vector}} = \mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}. \tag{1}$$

Our example has $x=t$, $y=t^2$, $z=t^3$. As $t$ varies, these points trace out a ***curve in space***. The parameter $t$ tells when the body passes each point on the curve. The constant vector $2\mathbf{i}+4\mathbf{j}+8\mathbf{k}$ is the position vector $\mathbf{R}(2)$ at the instant $t=2$.

What are the questions to be asked? Every student of calculus knows the first question: *Find the derivative*. If something moves, the Navy salutes it and we differentiate it. At each instant, the body moving along the curve has a speed and a direction. This information is contained in another vector function—the velocity vector $\mathbf{v}(t)$ which is the derivative of $\mathbf{R}(t)$:

$$\mathbf{v}(t) = \frac{d\mathbf{R}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}. \tag{2}$$

Since $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are fixed vectors, their derivatives are zero. In polar coordinates $\mathbf{i}$ and $\mathbf{j}$ are replaced by moving vectors. Then the velocity $\mathbf{v}$ has more terms from the product rule (Section 12.4).

Two important cases are uniform motion ***along a line and around a circle***. We study those motions in detail ($\mathbf{v}$ = constant on line, $\mathbf{v}$ = tangent to circle). This section also finds the speed and distance and acceleration for any motion $\mathbf{R}(t)$.

Equation (2) is the computing rule for the velocity $d\mathbf{R}/dt$. It is not the *definition* of $d\mathbf{R}/dt$, which goes back to basics and does not depend on coordinates:

$$\frac{d\mathbf{R}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{R}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{R}(t+\Delta t) - \mathbf{R}(t)}{\Delta t}.$$

We repeat: $\mathbf{R}$ is a vector so $\Delta\mathbf{R}$ is a vector so $d\mathbf{R}/dt$ is a vector. All three vectors are in Figure 12.1 ($t$ is not a vector!). This figure reveals the key fact about the geometry: ***The velocity $\mathbf{v} = d\mathbf{R}/dt$ is tangent to the curve.***

The vector $\Delta\mathbf{R}$ goes from one point on the curve to a nearby point. Dividing by $\Delta t$ changes its length, not its direction. That direction lines up with the tangent to the curve, as the points come closer.

**EXAMPLE 1** $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \qquad \mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$

This curve swings upward as $t$ increases. When $t = 0$ the velocity is $\mathbf{v} = \mathbf{i}$. The tangent is along the $x$ axis, since the $\mathbf{j}$ and $\mathbf{k}$ components are zero. When $t = 1$ the velocity is $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, and the curve is climbing.

For the shadow on the $xy$ plane, drop the $\mathbf{k}$ component. Position on the shadow is $t\mathbf{i} + t^2\mathbf{j}$. Velocity along the shadow is $\mathbf{i} + 2t\mathbf{j}$. The shadow is a plane curve.

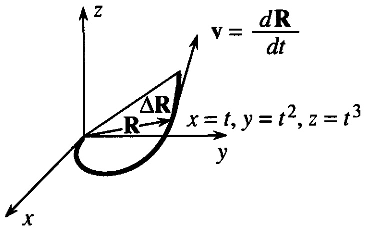



**Fig. 12.1** Position vector $\mathbf{R}$, change $\Delta\mathbf{R}$, velocity $d\mathbf{R}/dt$.

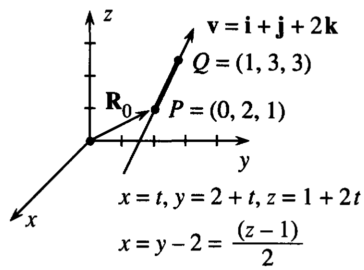



**Fig. 12.2** Equations of a line, with and without the parameter $t$.

**EXAMPLE 2** Uniform motion in a straight line: ***the velocity vector v is constant.***

The speed and direction don't change. The position vector moves with $d\mathbf{R}/dt = \mathbf{v}$:

$$\mathbf{R}(t) = \mathbf{R}_0 + t\mathbf{v} \quad (\mathbf{R}_0 \text{ fixed, } \mathbf{v} \text{ fixed, } t \text{ varying}) \tag{3}$$

That is the ***equation of a line*** in vector form. Certainly $d\mathbf{R}/dt = \mathbf{v}$. The starting point $\mathbf{R}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ is given. The velocity $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is also given. Separating the $x$, $y$ and $z$ components, equation (3) for a line is

$$\textit{line with parameter}: \quad x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3. \tag{4}$$

The speed along the line is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. The direction of the line is the unit vector $\mathbf{v}/|\mathbf{v}|$. We have three equations for $x, y, z$, and eliminating $t$ leaves two equations. The parameter $t$ equals $(x - x_0)/v_1$ from equation (4). It also equals $(y - y_0)/v_2$ and $(z - z_0)/v_3$. So these ratios equal each other, and $t$ is gone:

$$\textit{line without parameter}: \quad \frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}. \tag{5}$$

An example is $x = y/2 = z/3$. In this case $(x_0, y_0, z_0) = (0, 0, 0)$—the line goes through the origin. Another point on the line is $(x, y, z) = (2, 4, 6)$. Because $t$ is gone, we cannot say when we reach that point and how fast we are going. The equations $x/4 = y/8 = z/12$ give the same line. Without $t$ we can't know the velocity $\mathbf{v} = d\mathbf{R}/dt$.

**EXAMPLE 3** Find an equation for the line through $P = (0, 2, 1)$ and $Q = (1, 3, 3)$.

Solution We have choices! $\mathbf{R}_0$ can go to *any point* on the line. The velocity $\mathbf{v}$ can be *any multiple* of the vector from $P$ to $Q$. The decision on $\mathbf{R}_0$ controls where we start, and $\mathbf{v}$ controls our speed.

The vector from $P$ to $Q$ is $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. Those numbers 1, 1, 2 come from subtracting 0, 2, 1 from 1, 3, 3. We choose this vector $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ as a first $\mathbf{v}$, and double it for a second **v**. We choose the vector $\mathbf{R}_0 = \mathbf{P}$ as a first start and $\mathbf{R}_0 = \mathbf{Q}$ as a second start. Here are two different expressions for the same line—they are $\mathbf{P} + t\mathbf{v}$ and $\mathbf{Q} + t(2\mathbf{v})$:

$$\mathbf{R}(t) = (2\mathbf{j} + \mathbf{k}) + t(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \qquad \mathbf{R}^*(t) = (\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}) + t(2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}).$$

The vector $\mathbf{R}(t)$ gives $x = t$, $y = 2 + t$, $z = 1 + 2t$. The vector $\mathbf{R}^*$ is at a different point on the same line at the same time: $x^* = 1 + 2t$, $y^* = 3 + 2t$, $z^* = 3 + 4t$.

If I pick $t = 1$ in $\mathbf{R}$ and $t = 0$ in $\mathbf{R}^*$, the point is (1, 3, 3). We arrive there at different times. You are seeing how parameters work, to tell "where" and also "when." If $t$ goes from $-\infty$ to $+\infty$, all points on one line are also on the other line. The path is the same, but the "twins" are going at different speeds.

**Question 1** When do these twins meet? When does $\mathbf{R}(t) = \mathbf{R}^*(t)$?
Answer They meet at $t = -1$, when $\mathbf{R} = \mathbf{R}^* = -\mathbf{i} + \mathbf{j} - \mathbf{k}$.

**Question 2** What is an equation for the segment between $P$ and $Q$ (not beyond)?
Answer In the equation for $\mathbf{R}(t)$, let $t$ go from 0 to 1 (not beyond):

$$x = t \quad y = 2 + t \quad z = 1 + 2t \quad [0 \leqslant t \leqslant 1 \text{ for segment}]. \tag{6}$$

At $t = 0$ we start from $P = (0, 2, 1)$. At $t = 1$ we reach $Q = (1, 3, 3)$.

**Question 3** What is an equation for the line without the parameter $t$?
Answer Solve equations (6) for $t$ or use (5): $x/1 = (y - 2)/1 = (z - 1)/2$.

**Question 4** Which point on the line is closest to the origin?
Answer The derivative of $x^2 + y^2 + z^2 = t^2 + (2 + t)^2 + (1 + 2t)^2$ is $8 + 8t$. This derivative is zero at $t = -1$. So the closest point is $(-1, 1, -1)$.

**Question 5** Where does the line meet the plane $x + y + z = 11$?
Answer Equation (6) gives $x + y + z = 3 + 4t = 11$. So $t = 2$. The meeting point is $x = t = 2$, $y = t + 2 = 4$, $z = 1 + 2t = 5$.

**Question 6** What line goes through (3, 1, 1) perpendicular to the plane $x - y - z = 1$?
Answer The normal vector to the plane is $\mathbf{N} = \mathbf{i} - \mathbf{j} - \mathbf{k}$. *That is* **v**. The position vector to (3, 1, 1) is $\mathbf{R}_0 = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$. Then $\mathbf{R} = \mathbf{R}_0 + t\mathbf{v}$.

## COMPARING LINES AND PLANES

A line has one parameter or two equations. We give the starting point and velocity: $(x, y, z) = (x_0, y_0, z_0) + t(v_1, v_2, v_3)$. That tells directly which points are on the line. Or we eliminate $t$ to find the two equations in (5).

A plane has one equation or two parameters! The equation is $ax + by + cz = d$. That tells us *in*directly which points are on the plane. (Instead of knowing $x, y, z$, we know the equation they satisfy. Instead of directions **v** and **w** in the plane, we are told the perpendicular direction $\mathbf{N} = (a, b, c)$.) With parameters, the line contains $\mathbf{R}_0 + t\mathbf{v}$ and the plane contains $\mathbf{R}_0 + t\mathbf{v} + s\mathbf{w}$. A plane looks worse with parameters ($t$ and $s$), a line looks better.

Questions 5 and 6 connected lines to planes. Here are two more. See Problems 41–44:

**Question 7** When is the line $\mathbf{R}_0 + t\mathbf{v}$ parallel to the plane? When is it perpendicular?
Answer The test is $\mathbf{v} \cdot \mathbf{N} = 0$. The test is $\mathbf{v} \times \mathbf{N} = 0$.

**EXAMPLE 4** Find the plane containing $P_0 = (1, 2, 1)$ and the line of points $(1, 0, 0) + t(2, 0, -1)$. That vector **v** will be in the plane.

Solution The vector $\mathbf{v} = 2\mathbf{i} - \mathbf{k}$ goes along the line. The vector $\mathbf{w} = 2\mathbf{j} + \mathbf{k}$ goes from $(1, 0, 0)$ to $(1, 2, 1)$. Their cross product is

$$\mathbf{N} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}.$$

The plane $2x - 2y + 4z = 2$ has this normal $\mathbf{N}$ and contains the point $(1, 2, 1)$.

## SPEED, DIRECTION, DISTANCE, ACCELERATION

We go back to the curve traced out by $\mathbf{R}(t)$. The derivative $\mathbf{v}(t) = d\mathbf{R}/dt$ is the velocity vector along that curve. The ***speed*** is the magnitude of $\mathbf{v}$:

$$\text{speed} = |\mathbf{v}| = \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}. \tag{7}$$

The ***direction*** of the velocity vector is $\mathbf{v}/|\mathbf{v}|$. This is a unit vector, since $\mathbf{v}$ is divided by its length. ***The unit tangent vector*** $\mathbf{v}/|\mathbf{v}|$ ***is denoted by*** $\mathbf{T}$.

The tangent vector is constant for lines. It changes direction for curves.

**EXAMPLE 5** (important) Find $\mathbf{v}$ and $|\mathbf{v}|$ and $\mathbf{T}$ for steady motion around a circle:

$$x = r \cos \omega t, \quad y = r \sin \omega t, \quad z = 0.$$

Solution The position vector is $\mathbf{R} = r \cos \omega t\, \mathbf{i} + r \sin \omega t\, \mathbf{j}$. The velocity is

$$\mathbf{v} = d\mathbf{R}/dt = -\omega r \sin \omega t\, \mathbf{i} + \omega r \cos \omega t\, \mathbf{j} \quad \text{(tangent, not unit tangent)}$$

The speed is the radius $r$ times the angular velocity $\omega$:

$$|\mathbf{v}| = \sqrt{(-\omega r \sin \omega t)^2 + (\omega r \cos \omega t)^2} = \omega r.$$

The unit tangent vector is $\mathbf{v}$ divided by $|\mathbf{v}|$:

$$\mathbf{T} = -\sin \omega t\, \mathbf{i} + \cos \omega t\, \mathbf{j} \qquad \text{(length 1 since } \sin^2 \omega t + \cos^2 \omega t = 1).$$

Think next about the *distance traveled*. Distance along a curve is always denoted by $s$ (called ***arc length***). I don't know why we use $s$—certainly not as the initial for speed. In fact speed is distance divided by time. The ratio $s/t$ gives average speed; $ds/dt$ is instantaneous speed. We are back to Chapter 1 and Section 8.3, the relation of speed to distance:

$$\text{speed } |\mathbf{v}| = ds/dt \qquad \text{distance } s = \int (ds/dt)\, dt = \int |\mathbf{v}(t)|\, dt.$$

Notice that $|\mathbf{v}|$ and $s$ and $t$ are scalars. The direction vector is $\mathbf{T}$:

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{d\mathbf{R}/dt}{ds/dt} = \frac{d\mathbf{R}}{ds} = \textbf{\textit{unit tangent vector.}} \tag{8}$$

In Figure 12.3, the chord length (straight) is $|\Delta \mathbf{R}|$. The arc length (curved) is $\Delta s$. As $\Delta \mathbf{R}$ and $\Delta s$ approach zero, the ratio $|\Delta \mathbf{R}/\Delta s|$ approaches $|\mathbf{T}| = 1$.

Think finally about the ***acceleration vector*** $\mathbf{a}(t)$. It is the rate of change of velocity (not the rate of change of speed):

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{R}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}. \tag{9}$$

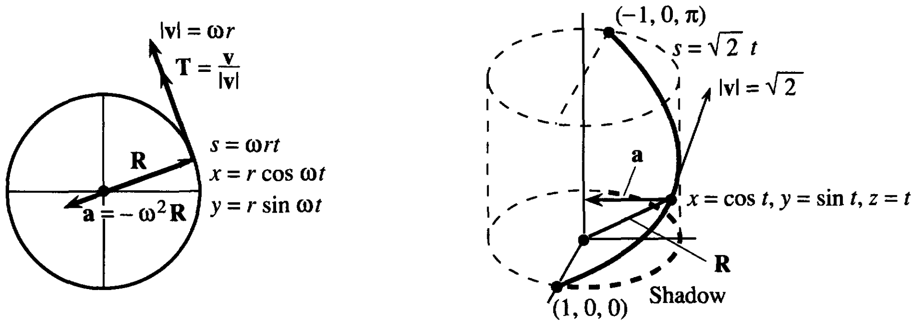



**Fig. 12.3** Steady motion around a circle. Half turn up a helix.

For steady motion along a line, as in $x = t$, $y = 2 + t$, $z = 1 + 2t$, there is no acceleration. The second derivatives are all zero. For steady motion around a circle, there *is* acceleration. In driving a car, you accelerate with the gas pedal or the brake. ***You also accelerate by turning the wheel.*** It is the velocity vector that changes, not the speed.

**EXAMPLE 6** Find the distance $s(t)$ and acceleration $\mathbf{a}(t)$ for circular motion.

Solution The speed in Example 5 is $ds/dt = \omega r$. After integrating, the distance is $s = \omega rt$. At time $t$ we have gone through an angle of $\omega t$. The radius is $r$, so the distance traveled agrees with $\omega t$ times $r$. Note that the dimension of $\omega$ is 1/time. (Angles are dimensionless.) At time $t = 2\pi/\omega$ we have gone once around the circle—to $s = 2\pi r$ not back to $s = 0$.

The acceleration is $\mathbf{a} = d^2\mathbf{R}/dt^2$. Remember $\mathbf{R} = r \cos \omega t\, \mathbf{i} + r \sin \omega t\, \mathbf{j}$:

$$\mathbf{a}(t) = -\omega^2 r \cos \omega t\, \mathbf{i} - \omega^2 r \sin \omega t\, \mathbf{j}. \tag{10}$$

That direction is opposite to $\mathbf{R}$. This is a special motion, with no action on the gas pedal or the brake. All the acceleration is from turning. The magnitude is $|\mathbf{a}| = \omega^2 r$, with the correct dimension of distance/(time)$^2$.

**EXAMPLE 7** Find $\mathbf{v}$ and $s$ and $\mathbf{a}$ around the helix $\mathbf{R} = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + t\, \mathbf{k}$.

Solution The velocity is $\mathbf{v} = -\sin t\, \mathbf{i} + \cos t\, \mathbf{j} + \mathbf{k}$. The speed is

$$ds/dt = |\mathbf{v}| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2} \text{ (constant)}.$$

Then distance is $s = \sqrt{2}\, t$. At time $t = \pi$, a half turn is complete. The distance along the shadow is $\pi$ (a half circle). The distance along the helix is $\sqrt{2}\, \pi$, because of its 45° slope.

The unit tangent vector is velocity/speed, and the acceleration is $d\mathbf{v}/dt$:

$$\mathbf{T} = (-\sin t\, \mathbf{i} + \cos t\, \mathbf{j} + \mathbf{k})/\sqrt{2} \qquad \mathbf{a} = -\cos t\, \mathbf{i} - \sin t\, \mathbf{j}.$$

**EXAMPLE 8** Find $\mathbf{v}$ and $s$ and $\mathbf{a}$ around the ellipse $x = \cos t$, $y = 2 \sin t$, $z = 0$.

Solution Take derivatives: $\mathbf{v} = -\sin t\, \mathbf{i} + 2 \cos t\, \mathbf{j}$ and $|\mathbf{v}| = \sqrt{\sin^2 t + 4 \cos^2 t}$. This is the speed $ds/dt$. For the distance $s$, something bad happens (or something normal). The speed is not simplified by $\sin^2 t + \cos^2 t = 1$. We cannot integrate $ds/dt$ to find a formula for $s$. The square root defeats us.

The acceleration $-\cos t\, \mathbf{i} - 2 \sin t\, \mathbf{j}$ still points to the center. This is *not* the Earth going around the sun. The path is an ellipse but the speed is wrong. See Section 12.4 (the pound note) for a terrible error in the position of the sun.

**12A** The basic formulas for motion along a curve are

$$\mathbf{v}=\frac{d\mathbf{R}}{dt} \quad \mathbf{a}=\frac{d\mathbf{v}}{dt} \quad |\mathbf{v}|=\frac{ds}{dt} \quad \mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{d\mathbf{R}/dt}{ds/dt}=\frac{d\mathbf{R}}{ds}.$$

Suppose we know the acceleration $\mathbf{a}(t)$ and the initial velocity $\mathbf{v}_0$ and position $\mathbf{R}_0$. Then $\mathbf{v}(t)$ and $\mathbf{R}(t)$ are also known. We integrate each component:

$$\mathbf{a}(t)=\text{constant} \Rightarrow \mathbf{v}(t)=\mathbf{v}_0+\mathbf{a}t \qquad \Rightarrow \mathbf{R}(t)=\mathbf{R}_0+\mathbf{v}_0t+\tfrac{1}{2}\mathbf{a}t^2$$

$$\mathbf{a}(t)=\cos t\,\mathbf{k} \quad \Rightarrow \mathbf{v}(t)=\mathbf{v}_0+\sin t\,\mathbf{k} \Rightarrow \mathbf{R}(t)=\mathbf{R}_0+\mathbf{v}_0t-\cos t\,\mathbf{k}.$$

### THE CURVE OF A BASEBALL

There is a nice discussion of curve balls in the calculus book by Edwards and Penney. We summarize it here (optionally). The ball leaves the pitcher's hand five feet off the ground: $\mathbf{R}_0=0\mathbf{i}+0\mathbf{j}+5\mathbf{k}$. The initial velocity is $\mathbf{v}_0=120\mathbf{i}-2\mathbf{j}+2\mathbf{k}$ (120 ft/sec is more than 80 miles per hour). The acceleration is $-32\mathbf{k}$ from gravity, plus a new term from ***spin***. If the spin is around the $z$ axis, and the ball goes along the $x$ axis, then this acceleration is in the $y$ direction. (It comes from the cross product $\mathbf{k}\times\mathbf{i}$—there is a pressure difference on the sides of the ball.) A good pitcher can achieve $\mathbf{a}=16\mathbf{j}-32\mathbf{k}$. The batter integrates as fast as he can:

$$\mathbf{v}(t)=\mathbf{v}_0+\mathbf{a}t=120\mathbf{i}+(-2+16t)\mathbf{j}+(2-32t)\mathbf{k}$$

$$\mathbf{R}(t)=\mathbf{R}_0+\mathbf{v}_0t+\tfrac{1}{2}\mathbf{a}t^2=120t\,\mathbf{i}+(-2t+8t^2)\mathbf{j}+(5+2t-16t^2)\mathbf{k}.$$

Notice the $t^2$. The effect of spin is small at first, then suddenly bigger (as every batter knows). So is the effect of gravity—the ball starts to dive. At $t=\frac{1}{2}$, the $\mathbf{i}$ component is 60 feet and the ball reaches the batter. The $\mathbf{j}$ component is 1 foot and the $\mathbf{k}$ component is 2 feet—the curve goes low over the outside corner.

At $t=\frac{1}{4}$, when the batter saw the ball halfway, the $\mathbf{j}$ component was zero. It looked as if it was coming right over the plate.

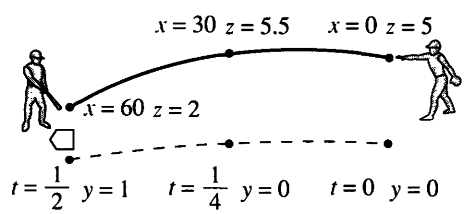



**Fig. 12.4** A curve ball approaches home plate. Halfway it is on line.

## 12.1 EXERCISES

**Read-through questions**

The position vector __a__ along the curve changes with the parameter $t$. The velocity is __b__. The acceleration is __c__. If the position is $\mathbf{i}+t\mathbf{j}+t^2\mathbf{k}$, then $\mathbf{v}=$ __d__ and $\mathbf{a}=$ __e__. In that example the speed is $|\mathbf{v}|=$ __f__. This equals $ds/dt$, where $s$ measures the __g__. Then $s=\int$ __h__. The tangent vector is in the same direction as the __i__, but $\mathbf{T}$ is a __j__ vector. In general $\mathbf{T}=$ __k__ and in the example $\mathbf{T}=$ __l__.

Steady motion along a line has $\mathbf{a}=$ __m__. If the line is $x=y=z$, the unit tangent vector is $\mathbf{T}=$ __n__. If the speed is

$|\mathbf{v}| = \sqrt{3}$, the velocity vector is $\mathbf{v}$ = __o__. If the initial position is (1, 0, 0), the position vector is $\mathbf{R}(t)$ = __p__. The general equation of a line is $x = x_0 + tv_1$, $y$ = __q__, $z$ = __r__. In vector notation this is $\mathbf{R}(t)$ = __s__. Eliminating $t$ leaves the equations $(x - x_0)/v_1 = (y - y_0)/v_2$ = __t__. A line in space needs __u__ equations where a plane needs __v__. A line has one parameter where a plane has __w__. The line from $\mathbf{R}_0$ = (1, 0, 0) to (2, 2, 2) with $|\mathbf{v}| = 3$ is $\mathbf{R}(t)$ = __x__.

Steady motion around a circle (radius $r$, angular velocity $\omega$) has $x$ = __y__, $y$ = __z__, $z = 0$. The velocity is $\mathbf{v}$ = __A__. The speed is $|\mathbf{v}|$ = __B__. The acceleration is $\mathbf{a}$ = __C__, which has magnitude __D__ and direction __E__. Combining upward motion $\mathbf{R} = t\mathbf{k}$ with this circular motion produces motion around a __F__. Then $\mathbf{v}$ = __G__ and $|\mathbf{v}|$ = __H__.

**1** Sketch the curve with parametric equations $x = t$, $y = t^3$. Find the velocity vector and the speed at $t = 1$.

**2** Sketch the path with parametric equations $x = 1 + t$, $y = 1 - t$. Find the $xy$ equation of the path and the speed along it.

**3** On the circle $x = \cos t$, $y = \sin t$ explain by the chain rule and then by geometry why $dy/dx = -\cot t$.

**4** Locate the highest point on the curve $x = 6t$, $y = 6t - t^2$. This curve is a ______. What is the acceleration $\mathbf{a}$?

**5** Find the velocity vector and the $xy$ equation of the tangent line to $x = e^t$, $y = e^{-t}$ at $t = 0$. What is the $xy$ equation of the curve?

**6** Describe the shapes of these curves: (a) $x = 2^t$, $y = 4^t$; (b) $x = 4^t$, $y = 8^t$; (c) $x = 4^t$, $y = 4t$.

**Note:** *To find* "parametric equations" *is to find* $x(t)$, $y(t)$, *and possibly* $z(t)$.

**7** Find parametric equations for the line through $P$ = (1, 2, 4) and $Q$ = (5, 5, 4). Probably your speed is 5; change the equations so the speed is 10. Probably your $\mathbf{R}_0$ is $P$; change the start to $Q$.

**8** Find an equation for any one plane that is perpendicular to the line in Problem 7. Also find equations for any one line that is perpendicular.

**9** On a straight line from (2, 3, 4) with velocity $\mathbf{v} = \mathbf{i} - \mathbf{k}$, the position vector is $\mathbf{R}(t)$ = ______. If the velocity vector is changed to $t\mathbf{i} - t\mathbf{k}$, then $\mathbf{R}(t)$ = ______. The path is still ______.

**10** Find parametric equations for steady motion from $P$ = (3, 1, −2) at $t = 0$ on a line to $Q$ = (0, 0, 0) at $t = 3$. What is the speed? Change parameters so the speed is $e^t$.

**11** The equations $x - 1 = \frac{1}{2}(y - 2) = \frac{1}{3}(z - 2)$ describe a ______. The same path is given parametrically by $x = 1 + t$, $y$ = ______, $z$ = ______. The same path is also given by $x = 1 + 2t$, $y$ = ______, $z$ = ______.

**12** Find parametric equations to go around the unit circle with speed $e^t$ starting from $x = 1$, $y = 0$. When is the circle completed?

**13** The path $x = 2y = 3z = 6t$ is a ______ traveled with speed ______. If $t$ is restricted by $t \geq 1$ the path starts at ______. If $t$ is restricted by $0 \leq t \leq 1$ the path is a ______.

**14** Find the closest point to the origin on the line $x = 1 + t$, $y = 2 - t$. When and where does it cross the 45° line through the origin? Find the equation of a line it never crosses.

**15** (a) How far apart are the two parallel lines $x = y$ and $x = y + 1$? (b) How far is the point $x = t$, $y = t$ from the point $x = t$, $y = t + 1$? (c) What is the closest distance if their speeds are different: $x = t$, $y = t$ and $x = 2t$, $y = 2t + 1$?

**16** Which vectors follow the same path as $\mathbf{R} = t\mathbf{i} + t^2\mathbf{j}$? The speed along the path may be different.

(a) $2t\mathbf{i} + 2t^2\mathbf{j}$ (b) $2t\mathbf{i} + 4t^2\mathbf{j}$ (c) $-t\mathbf{i} + t^2\mathbf{j}$ (d) $t^3\mathbf{i} + t^6\mathbf{j}$

**17** Find a parametric form for the straight line $y = mx + b$.

**18** The line $x = 1 + v_1 t$, $y = 2 + v_2 t$ passes through the origin provided ______ $v_1$ + ______ $v_2 = 0$. This line crosses the 45° line $y = x$ unless ______ $v_1$ + ______ $v_2 = 0$.

**19** Find the velocity $\mathbf{v}$ and speed $|\mathbf{v}|$ and tangent vector $\mathbf{T}$ for these motions: (a) $\mathbf{R} = t\mathbf{i} + t^{-1}\mathbf{j}$ (b) $\mathbf{R} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j}$ (c) $\mathbf{R} = (t + 1)\mathbf{i} + (2t + 1)\mathbf{j} + (2t + 2)\mathbf{k}$.

**20** If the velocity $dx/dt\,\mathbf{i} + dy/dt\,\mathbf{j}$ is always perpendicular to the position vector $x\mathbf{i} + y\mathbf{j}$, show from their dot product that $x^2 + y^2$ is constant. The point stays on a circle.

**21** Find two paths $\mathbf{R}(t)$ with the same $\mathbf{v} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$. Find a third path with a different $\mathbf{v}$ but the same acceleration.

**22** If the acceleration is a constant vector, the path must be ______. If the path is a straight line, the acceleration vector must be ______.

**23** Find the minimum and maximum speed if $x = t + \cos t$, $y = t - \sin t$. Show that $|\mathbf{a}|$ is constant but not $\mathbf{a}$. The point is going around a circle while the center is moving on what line?

**24** Find $x(t)$, $y(t)$ so that the point goes around the circle $(x - 1)^2 + (y - 3)^2 = 4$ with speed 1.

**25** A ball that is circling with $x = \cos 2t$, $y = \sin 2t$ flies off on a tangent at $t = \pi/8$. Find its departure point and its position at a later time $t$ (linear motion; compute its constant velocity $\mathbf{v}$).

**26** Why is $|\mathbf{a}|$ generally different from $d^2s/dt^2$? Give an example of the difference, and an example where they are equal.

**27** Change $t$ so that the speed along the helix $\mathbf{R} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ is 1 instead of $\sqrt{2}$. Call the new parameter $s$.

**28** Find the speed $ds/dt$ on the line $x = 1 + 6t$, $y = 2 + 3t$, $z = 2t$. Integrate to find the length $s$ from (1, 2, 0) to (13, 8, 4). Check by using $12^2 + 6^2 + 4^2$.

**29** Find $\mathbf{v}$ and $|\mathbf{v}|$ and $\mathbf{a}$ for the curve $x = \tan t$, $y = \sec t$. What is this curve? At what time does it go to infinity, and along what line?

**30** Construct parametric equations for travel on a helix with speed $t$.

**31** Suppose the unit tangent vector $\mathbf{T}(t)$ is the derivative of $\mathbf{R}(t)$. What does that say about the speed? Give a noncircular example.

**32** For travel on the path $y = f(x)$, with no parameter, it is impossible to find the ________ but still possible to find the ________ at each point of the path.

**Find $x(t)$ and $y(t)$ for paths 33–36.**

**33** Around the square bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, with speed 2. The formulas have four parts.

**34** Around the unit circle with speed $e^{-t}$. Do you get all the way around?

**35** Around a circle of radius 4 with acceleration $|\mathbf{a}| = 1$.

**36** Up and down the $y$ axis with constant acceleration $-\mathbf{j}$, returning to $(0, 0)$ at $t = 10$.

**37** True (with reason) or false (with example):

(a) If $|\mathbf{R}| = 1$ for all $t$ then $|\mathbf{v}| = \text{constant}$.

(b) If $\mathbf{a} = 0$ then $\mathbf{R} = \text{constant}$.

(c) If $\mathbf{v} \cdot \mathbf{v} = \text{constant}$ then $\mathbf{v} \cdot \mathbf{a} = 0$.

(d) If $\mathbf{v} \cdot \mathbf{R} = 0$ then $\mathbf{R} \cdot \mathbf{R} = \text{constant}$.

(e) There is no path with $\mathbf{v} = \mathbf{a}$.

**38** Find the position vector to the shadow of $t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ on the $xz$ plane. Is the curve ever parallel to the line $x = y = z$?

**39** On the ellipse $x = a\cos t$, $y = b\sin t$, the angle $\theta$ from the center is not the same as $t$ because ________.

**40** Two particles are racing from $(1, 0)$ to $(0, 1)$. One follows $x = \cos t$, $y = \sin t$, the other follows $x = 1 + v_1 t$, $y = v_2 t$. Choose $v_1$ and $v_2$ so that the second particle goes slower but wins.

**41** Two lines in space are given by $\mathbf{R}(t) = \mathbf{P} + t\mathbf{v}$ and $\mathbf{R}(t) = \mathbf{Q} + t\mathbf{w}$. Four possibilities: The lines are parallel or the same or intersecting or skew. Decide which is which based on the vectors $\mathbf{v}$ and $\mathbf{w}$ and $\mathbf{u} = \mathbf{Q} - \mathbf{P}$ (which goes between the lines):

(a) The lines are parallel if ________ are parallel.

(b) The lines are the same if ________ are parallel.

(c) The lines intersect if ________ are not parallel but ________ lie in the same plane.

(d) The lines are skew if the triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is ________.

**42** If the lines are skew (not in the same plane), find a formula based on $\mathbf{u}, \mathbf{v}, \mathbf{w}$ for the distance between them. The vector $\mathbf{u}$ may not be perpendicular to the two lines, so project it onto a vector that is.

**43** The distance from $\mathbf{Q}$ to the line $\mathbf{P} + t\mathbf{v}$ is the projection of $\mathbf{u} = \mathbf{Q} - \mathbf{P}$ perpendicular to $\mathbf{v}$. How far is $\mathbf{Q} = (9, 4, 5)$ from the line $x = 1 + t$, $y = 1 + 2t$, $z = 3 + 2t$?

**44** Solve Problem 43 by calculus: substitute for $x, y, z$ in $(x-9)^2 + (y-4)^2 + (z-5)^2$ and minimize. Which $(x, y, z)$ on the line is closest to $(9, 4, 5)$?

**45** Practice with parameters, starting from $x = F(t)$, $y = G(t)$.

(a) The mirror image across the 45° line is $x =$ ________, $y =$ ________.

(b) Write the curve $x = t^3$, $y = t^2$ as $y = f(x)$.

(c) Why can't $x = t^2$, $y = t^3$ be written as $y = f(x)$?

(d) If $F$ is invertible then $t = F^{-1}(x)$ and $y =$ ________$(x)$.

**46** From 12:00 to 1:00 a snail crawls steadily out the minute hand (one meter in one hour). Find its position at time $t$ starting from $(0, 0)$.

## 12.2 Plane Motion: Projectiles and Cycloids

The previous section started with $\mathbf{R}(t)$. From this position vector we computed $\mathbf{v}$ and $\mathbf{a}$. Now we find $\mathbf{R}(t)$ itself, from more basic information. The laws of physics govern projectiles, and the motion of a wheel produces a cycloid (which enters problems in robotics). The projectiles fly without friction, so the only force is gravity.

These motions occur in a plane. The two components of position will be $x$ (across) and $y$ (up). A projectile moves as $t$ changes, so we look for $x(t)$ and $y(t)$. We are shooting a basketball or firing a gun or peacefully watering the lawn, and we have to aim in the right direction (not directly at the target). If the hose delivers water at 10 meters/second, can you reach the car 12 meters away?

The usual initial position is (0, 0). Some flights start higher, at (0, $h$). The initial velocity is ($v_0 \cos \alpha$, $v_0 \sin \alpha$), where $v_0$ is the speed and $\alpha$ is the angle with the horizontal. The acceleration from gravity is purely vertical: $d^2y/dt^2 = -g$. So the horizontal velocity stays at its initial value. The upward velocity decreases by $-gt$:

$$dx/dt = v_0 \cos \alpha, \; dy/dt = v_0 \sin \alpha - gt.$$

The horizontal distance $x(t)$ is steadily increasing. The height $y(t)$ increases and then decreases. To find the position, integrate the velocities (for a high start add $h$ to $y$):

$$\textbf{\textit{The projectile path is}} \; x(t) = (v_0 \cos \alpha)t, \; y(t) = (v_0 \sin \alpha)t - \tfrac{1}{2}gt^2. \qquad (1)$$

This path is a *parabola*. But it is not written as $y = ax^2 + bx + c$. It could be, if we eliminated $t$. Then we would lose track of time. The parabola is $y(x)$, with no parameter, where we have $x(t)$ and $y(t)$.

Basic question: ***Where does the projectile hit the ground***? For the parabola, we solve $y(x) = 0$. That gives the position $x$. *For the projectile we solve* $y(t) = 0$. That gives the *time* it hits the ground, not the place. If that time is $T$, then $x(T)$ gives the place. The information is there. It takes two steps instead of one, but we learn more.

**EXAMPLE 1** Water leaves the hose at 10 meters/second (this is $v_0$). It starts up at the angle $\alpha$. Find the time $T$ when $y$ is zero again, and find where the projectile lands.

Solution The flight ends when $y = (10 \sin \alpha)T - \frac{1}{2}gT^2 = 0$. The flight time is $T = (20 \sin \alpha)/g$. At that time, the horizontal distance is

$$x(T) = (10 \cos \alpha)T = (200 \cos \alpha \sin \alpha)/g. \text{ This is the } \textbf{\textit{range}} \; R.$$

The projectile (or water from the hose) hits the ground at $x = R$. To simplify, replace $200 \cos \alpha \sin \alpha$ by $100 \sin 2\alpha$. Since $g = 9.8$ meters/sec$^2$, ***we can't reach the car***:

The range $R = (100 \sin 2\alpha)/9.8$ is at most 100/9.8. This is less than 12.

The range is greatest when $\sin 2\alpha = 1$ ($\alpha$ is 45°). To reach 12 meters we could stand on a ladder (Problem 14). To hit a baseball against air resistance, the best angle is nearer to 35°. Figure 12.5 shows symmetric parabolas (no air resistance) and unsymmetric flight paths that drop more steeply.

> **12B** The flight time $T$ and the horizontal range $R = x(T)$ are reached when $y = 0$, which means $(v_0 \sin \alpha)T = \frac{1}{2}gT^2$:
>
> $$T = (2v_0 \sin \alpha)/g \text{ and } R = (v_0 \cos \alpha)T = (v_0^2 \sin 2\alpha)/g.$$

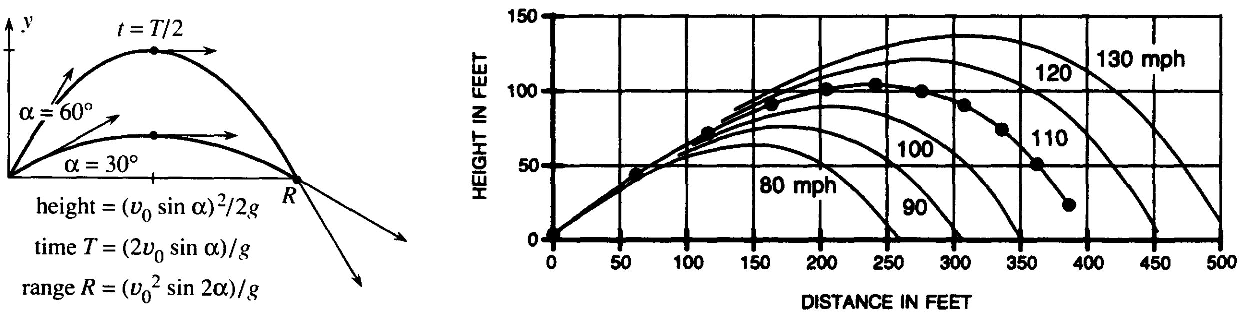



**Fig. 12.5** Equal range $R$, different times $T$. Baseballs hit at 35° with increasing $v_0$. The dots are at half-seconds (from *The Physics of Baseball* by Robert Adair: Harper and Row 1990).

**EXAMPLE 2** What are the correct angles $\alpha$ for a given range $R$ and given $v_0$?

Solution The range is $R = (v_0^2 \sin 2\alpha)/g$. This determines the sine of $2\alpha$—***but two angles can have the same sine.*** We might find $2\alpha = 60°$ or $120°$. The starting angles $\alpha = 30°$ and $\alpha = 60°$ in Figure 12.5 give the same $\sin 2\alpha$ and the same range $R$. The flight times contain $\sin \alpha$ and are different.

By calculus, the maximum height occurs when $dy/dt = 0$. Then $v_0 \sin \alpha = gt$, which means that $t = (v_0 \sin \alpha)/g$. This is half of the total flight time $T$—the time going up equals the time coming down. The value of $y$ at this halfway time $t = \frac{1}{2}T$ is

$$y_{max} = (v_0 \sin \alpha)(v_0 \sin \alpha)/g - \tfrac{1}{2}g(v_0 \sin \alpha/g)^2 = (v_0 \sin \alpha)^2/2g. \qquad (2)$$

**EXAMPLE 3** If a ski jumper goes 90 meters down a 30° slope, after taking off at 28 meters/second, find equations for the flight time and the ramp angle $\alpha$.

Solution The jumper lands at the point $x = 90 \cos 30°$, $y = -90 \sin 30°$ (minus sign for obvious reasons). The basic equation (2) is $x = (28 \cos \alpha)t$, $y = (28 \sin \alpha)t - \frac{1}{2}gt^2$. Those are two equations for $\alpha$ and $t$. Note that $t$ is not $T$, the flight time to $y = 0$.

***Conclusion*** The position of a projectile involves three parameters $v_0$, $\alpha$, and $t$. ***Three pieces of information determine the flight*** (almost). The reason for the word *almost* is the presence of $\sin \alpha$ and $\cos \alpha$. Some flight requirements cannot be met (reaching a car at 12 meters). Other requirements can be met in two ways (when the car is close). The equation $\sin \alpha = c$ is more likely to have no solution or two solutions than exactly one solution.

Watch for the three pieces of information in each problem. When a football starts at $v_0 = 20$ meters/second and hits the ground at $x = 40$ meters, the third fact is _______. This is like a lawyer who is asked the fee and says \$1000 for three questions. "Isn't that steep?" says the client. "Yes," says the lawyer, "now what's your last question?"

## CYCLOIDS

A projectile's path is a parabola. To compute it, eliminate $t$ from the equations for $x$ and $y$. Problem 5 finds $y = ax^2 + bx$, a parabola through the origin. The path of a point on a wheel seems equally simple, but eliminating $t$ is virtually impossible. The cycloid is a curve that really needs and uses a parameter.

To trace out a cycloid, ***roll a circle of radius a along the x axis.*** Watch the point that starts at the bottom of the circle. It comes back to the bottom at $x = 2\pi a$, after a complete turn of the circle. The path in between is shown in Figure 12.6. After a century of looking for the $xy$ equation, a series of great scientists (Galileo, Christopher Wren, Huygens, Bernoulli, even Newton and l'Hôpital) found the right way to study a cycloid—by introducing a parameter. We will call it $\theta$; it could also be $t$.

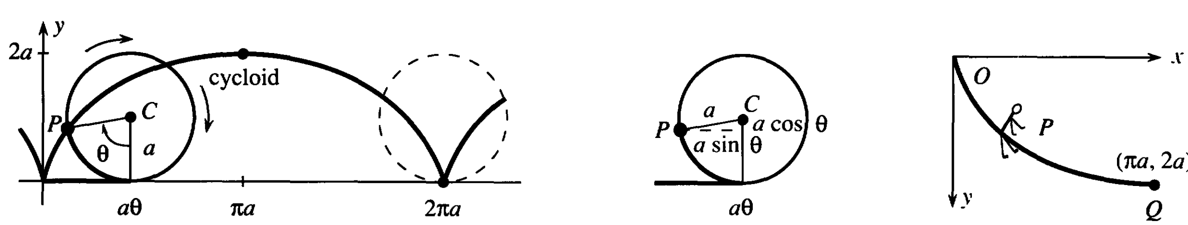



**Fig. 12.6** Path of $P$ on a rolling circle is a cycloid. Fastest slide to Q.

The parameter is the angle $\theta$ through which the circle turns. (This angle is not at the origin, like $\theta$ in polar coordinates.) The circle rolls a distance $a\theta$, radius times angle, along the $x$ axis. So the center of the circle is at $x = a\theta$, $y = a$. To account for the segment $CP$, subtract $a \sin \theta$ from $x$ and $a \cos \theta$ from $y$:

$$\textbf{\textit{The point P has }} x = a(\theta - \sin \theta) \textbf{\textit{ and }} y = a(1 - \cos \theta). \tag{3}$$

At $\theta = 0$ the position is $(0, 0)$. At $\theta = 2\pi$ the position is $(2\pi a, 0)$. In between, the slope of the cycloid comes from the chain rule:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a \sin \theta}{a(1 - \cos \theta)}. \tag{4}$$

This is infinite at $\theta = 0$. The point on the circle starts straight upward and the cycloid has a *cusp*. Note how all calculations use the parameter $\theta$. We go quickly:

**Question 1** Find the area under one arch of the cycloid ($\theta = 0$ to $\theta = 2\pi$).
Answer The area is $\int y\, dx = \int_0^{2\pi} a(1 - \cos \theta)a(1 - \cos \theta)d\theta$. This equals $3\pi a^2$.

**Question 2** Find the length of the arch, using $ds = \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta$.
Answer $\int ds = \int_0^{2\pi} a\sqrt{(1 - \cos \theta)^2 + (\sin \theta)^2}\, d\theta = \int_0^{2\pi} a\sqrt{2 - 2 \cos \theta}\, d\theta$.
Now substitute $1 - \cos \theta = 2 \sin^2 \frac{1}{2}\theta$. The square root is $2 \sin \frac{1}{2}\theta$. The length is $8a$.

**Question 3** If the cycloid is turned over ($y$ is downward), find the time to slide to the bottom. The slider starts with $v = 0$ at $y = 0$.
Answer Kinetic plus potential energy is $\frac{1}{2}mv^2 - mgy = 0$ (it starts from zero and can't change). So the speed is $v = \sqrt{2gy}$. This is $ds/dt$ and we know $ds$:

$$\text{sliding time} = \int dt = \int \frac{ds}{\sqrt{2gy}} = \int_0^{\pi} \frac{a\sqrt{2 - 2 \cos \theta}\, d\theta}{\sqrt{2ga(1 - \cos \theta)}} = \pi\sqrt{a/g}.$$

Check dimensions: $a$ = distance, $g$ = distance/(time)$^2$, $\pi\sqrt{a/g}$ = time. ***That is the shortest sliding time for any curve.*** The cycloid solves the "brachistochrone problem," which minimizes the time down curves from $O$ to $Q$ (Figure 12.6). You might think a straight path would be quicker—it is certainly shorter. A straight line has the equation $x = \pi y/2$, so the sliding time is

$$\int dt = \int ds/\sqrt{2gy} = \int_0^{2a} \sqrt{(\pi/2)^2 + 1}\, dy/\sqrt{2gy} = \sqrt{\pi^2 + 4}\sqrt{a/g}. \tag{5}$$

This is larger than the cycloid time $\pi\sqrt{a/g}$. It is better to start out vertically and pick up speed early, even if the path is longer.

Instead of publishing his solution, John Bernoulli turned this problem into an international challenge: *Prove that the cycloid gives the fastest slide.* Most mathematicians couldn't do it. The problem reached Isaac Newton (this was later in his life). As you would expect, Newton solved it. For some reason he sent back his proof with no name. But when Bernoulli received the answer, he was not fooled for a moment: "I recognize the lion by his claws."

What is also amazing is a further property of the cycloid: ***The time to Q is the same if you begin anywhere along the path.*** Starting from rest at $P$ instead of $O$, the bottom is reached at the same time. This time Bernoulli got carried away: "You will be petrified with astonishment when I say...".

There are other beautiful curves, closely related to the cycloid. For an ***epicycloid***, the circle rolls around the outside of another circle. For a ***hypocycloid***, the rolling circle is inside the fixed circle. The *astroid* is the special case with radii in the ratio 1 to 4. It is the curved star in Problem 34, where $x = a \cos^3\theta$ and $y = a \sin^3\theta$.

The cycloid even solves the old puzzle: *What point moves backward when a train starts forward*? The train wheels have a flange that extends below the track, and $dx/dt < 0$ at the bottom of the flange.

## 12.2 EXERCISES

**Read-through questions**

A projectile starts with speed $v_0$ and angle $\alpha$. At time $t$ its velocity is $dx/dt =$ __a__, $dy/dt =$ __b__ (the downward acceleration is $g$). Starting from (0, 0), the position at time $t$ is $x =$ __c__, $y =$ __d__. The flight time back to $y = 0$ is $T =$ __e__. At that time the horizontal range is $R =$ __f__. The flight path is a __g__.

The three quantities $v_0$, __h__, __i__ determine the projectile's motion. Knowing $v_0$ and the position of the target, we (can) (cannot) solve for $\alpha$. Knowing $\alpha$ and the position of the target, we (can) (cannot) solve for $v_0$.

A __j__ is traced out by a point on a rolling circle. If the radius is $a$ and the turning angle is $\theta$, the center of the circle is at $x =$ __k__, $y =$ __l__. The point is at $x =$ __m__, $y =$ __n__, starting from (0, 0). It travels a distance __o__ in a full turn of the circle. The curve has a __p__ at the end of every turn. An upside-down cycloid gives the __q__ slide between two points.

**Problems 1–18 and 41 are about projectiles**

**1** Find the time of flight $T$, the range $R$, and the maximum height $Y$ of a projectile with $v_0 = 16$ ft/sec and
(a) $\alpha = 30°$ (b) $\alpha = 60°$ (c) $\alpha = 90°$.

**2** If $v_0 = 32$ ft/sec and the projectile returns to the ground at $T = 1$, find the angle $\alpha$ and the range $R$.

**3** A ball is thrown at 60° with $v_0 = 20$ meters/sec to clear a wall 2 meters high. How far away is the wall?

**4** If $\mathbf{v}(0) = 3\mathbf{i} + 3\mathbf{j}$ find $\mathbf{v}(t)$, $\mathbf{v}(1)$, $\mathbf{v}(2)$ and $\mathbf{R}(t)$, $\mathbf{R}(1)$, $\mathbf{R}(2)$.

**5** (a) Eliminate $t$ from $x = t$, $y = t - \frac{1}{2}t^2$ to find the $xy$ equation of the path. At what $x$ is $y = 0$?
(b) Do the same for any $v_0$ and $\alpha$.

**6** Find the angle $\alpha$ for a ball kicked at 30 meters/second if it clears 6 meters traveling horizontally.

**7** How far out does a stone hit the water $h$ feet below, starting with velocity $v_0$ at angle $\alpha = 0$?

**8** How far out does the same stone go, starting at angle $\alpha$? Find an equation for the angle that maximizes the range.

**9** A ball starting from (0, 0) passes through (5, 2) after 2 seconds. Find $v_0$ and $\alpha$. (The units are meters.)

***10** With $x$ and $y$ from equation (1), show that

$$v_0^2 \geq (gx/v_0)^2 + 2gy.$$

If a fire is at height $H$ and the water velocity is $v_0$, how far can the fireman put the hose back from the fire? (The parabola in this problem is the "envelope" enclosing all possible paths.)

**11** Estimate the initial speed of a 100-meter golf shot hit at $\alpha = 45°$. Is the true $v_0$ larger or smaller, when air friction is included?

**12** $T = 2v_0(\sin\alpha)/g$ is in seconds and $R = (v_0^2 \sin 2\alpha)/g$ is in meters if $v_0$ and $g$ are in ________.

**13** (a) What is the greatest height a ball can be thrown? Aim straight up with $v_0 = 28$ meters/sec.

**14** If a baseball goes 100 miles per hour for 60 feet, how long does it take (in seconds) and how far does it fall from gravity (in feet)? Use $\frac{1}{2}gt^2$.

**15** If you double $v_0$, what happens to the range and maximum height? If you change the angle by $d\alpha$, what happens to those numbers?

**16** At what point on the path is the speed of the projectile (a) least (b) greatest?

**17** If the hose with $v_0 = 10$m/sec is at a 45° angle, $x$ reaches 12 meters when $t =$ ________ and $y =$ ________. From a ladder of height ________ the water will reach the car (12 meters).

**18** Describe the two trajectories a golf ball can take to land right in the hole, if it starts with a large known velocity $v_0$. In reality (with air resistance) which of those shots would fall closer?

**Problems 19–34 are about cycloids and related curves**

**19** Find the unit tangent vector $\mathbf{T}$ to the cycloid. Also find the speed at $\theta = 0$ and $\theta = \pi$, if the wheel turns at $d\theta/dt = 1$.

**20** The slope of the cycloid is infinite at $\theta = 0$:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta}{1 - \cos\theta}.$$

By whose rule? Estimate the slope at $\theta = \frac{1}{10}$ and $\theta = -\frac{1}{10}$. Where does the slope equal one?

**21** Show that the tangent to the cycloid at $P$ in Figure 12.6a goes through $x = a\theta$, $y = 2a$. Where is this point on the rolling circle?

**22** For a **trochoid**, the point $P$ is a distance $d$ from the center of the rolling circle. Redraw Figure 12.6b to find $x = a\theta - d\sin\theta$ and $y =$ ________.

**23** If a circle of radius $a$ rolls inside a circle of radius $2a$, show that one point on the small circle goes across on a straight line.

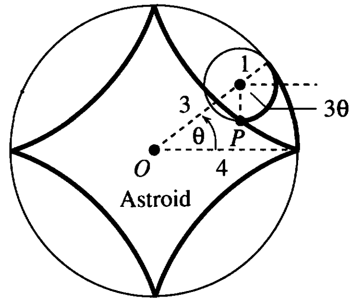
**24** Find $d^2y/dx^2$ for the cycloid, which is concave ________.

**25** If $d\theta/dt = c$, find the velocities $dx/dt$ and $dy/dt$ along the cycloid. Where is $dx/dt$ greatest and where is $dy/dt$ greatest?

**26** Experiment with graphs of $x = a\cos\theta + b\sin\theta$, $y = c\cos\theta + d\sin\theta$ using a computer. What kind of curves are they? Why are they closed?

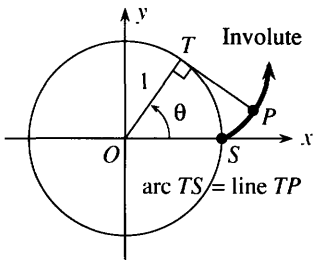
**27** A stone in a bicycle tire goes along a cycloid. Find equations for the stone's path if it flies off at the top (a projectile).

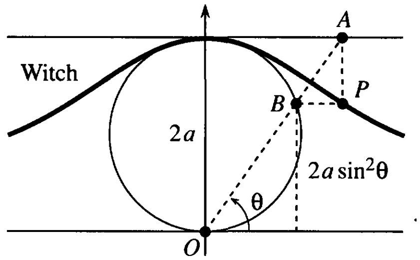
**28** Draw curves on a computer with $x = a\cos\theta + b\cos 3\theta$ and $y = c\sin\theta + d\sin 3\theta$. Is there a limit to the number of loops?

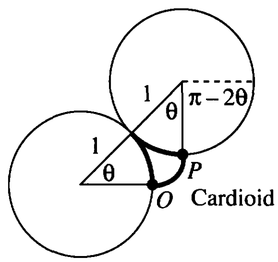



**35** Find the area inside the astroid.

**36** Explain why $x = 2a\cot\theta$ and $y = 2a\sin^2\theta$ for the point $P$ on the **witch of Agnesi**. Eliminate $\theta$ to find the $xy$ equation. **Note**: Maria Agnesi wrote the first three-semester calculus text (l'Hôpital didn't do integral calculus). The word "witch" is a total mistranslation, nothing to do with her or the curve.




**29** When a penny rolls completely around another penny, the head makes ________ turns. When it rolls inside a circle four times larger (for the astroid), the head makes ________ turns.

**30** Display the cycloid family with computer graphics:

(a) **cycloid**

(b) **epicycloid** $x = C\cos\theta - \cos C\theta$, $y = C\sin\theta + \sin C\theta$

(c) **hypocycloid** $x = c\cos\theta + \cos c\theta$, $y = c\sin\theta - \sin c\theta$

(d) **astroid** $(c = 3)$

(e) **deltoid** $(c = 2)$.

**31** If one arch of the cycloid is revolved around the $x$ axis, find the surface area and volume.

**32** For a hypocycloid the fixed circle has radius $c + 1$ and the circle rolling inside has radius 1. There are $c + 1$ cusps if $c$ is an integer. How many cusps (use computer graphics if possible) for $c = 1/2$? $c = 3/2$? $c = \sqrt{2}$? *What curve for* $c = 1$?

**33** When a string is unwound from a circle find $x(\theta)$ and $y(\theta)$ for point $P$. Its path is the "**involute**" of the circle.

**34** For the point $P$ on the **astroid**, explain why $x = 3\cos\theta + \cos 3\theta$ and $y = 3\sin\theta - \sin 3\theta$. The angle in the figure is $3\theta$ because both circular arcs have length ________. Convert to $x = 4\cos^3\theta$, $y = 4\sin^3\theta$ by triple-angle formulas.

**37** For a **cardioid** the radius $C - 1$ of the fixed circle equals the radius 1 of the circle rolling outside (epicycloid with $C = 2$). (a) The coordinates of $P$ are $x = -1 + 2\cos\theta - \cos 2\theta$, $y =$ ________. (b) The double-angle formulas yield $x = 2\cos\theta(1 - \cos\theta)$, $y =$ ________. (c) $x^2 + y^2 =$ ________ so its square root is $r =$ ________.

**38** Explain the last two steps in equation (5) for the sliding time down a straight path.

**39** On an upside-down cycloid the slider takes the same time $T$ to reach bottom *wherever it starts.* Starting at $\theta = \alpha$, write $1 - \cos\theta = 2\sin^2\theta/2$ and $1 - \cos\alpha = 2\sin^2\alpha/2$ to show that

$$T = \int_\alpha^\pi \frac{\sqrt{2a^2(1-\cos\theta)}\,d\theta}{\sqrt{2ag(\cos\alpha - \cos\theta)}} = \pi\sqrt{\frac{a}{g}}.$$

**40** Suppose a heavy weight is attached to the top of the rolling circle. What is the path of the weight?

**41** The wall in Fenway Park is 37 feet high and 315 feet from home plate. A baseball hit 3 feet above the ground at $\alpha = 22.5°$ will just go over if $v_0 =$ ________. The time to reach the wall is ________.