12.3 Curvature and Normal Vector

A driver produces acceleration three ways—by the gas pedal, the brake, and steering wheel. The first two change the speed. Turning the wheel changes the direction. *All three change the velocity* (they give acceleration). For steady motion around a circle, the change is from steering—the acceleration $d\mathbf{v}/dt$ points to the center. We now look at motion along other curves, to separate change in the speed $|\mathbf{v}|$ from change in the direction $\mathbf{T}$.

The direction of motion is $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$. It depends on the path but not the speed (because we divide by $|\mathbf{v}|$). For turning we measure two things:

1. **How fast T turns: this will be the *curvature* κ (kappa).**
2. **Which direction T turns: this will be the *normal vector* N.**

κ and $\mathbf{N}$ depend, like s and $\mathbf{T}$, only on the shape of the curve. Replacing t by $2t$ or t^2 leaves them unchanged. For a circle we give the answers in advance. The normal vector $\mathbf{N}$ points to the center. The curvature κ is 1/radius.

A smaller turning circle means a larger curvature κ: more bending.

The curvature κ is ***change in direction*** $|d\mathbf{T}|$ ***divided by change in position*** $|ds|$. There are three formulas for κ—a direct one for graphs $y(x)$, a brutal but valuable one for any parametric curve $(x(t), y(t))$, and a neat formula that uses the vectors $\mathbf{v}$ and $\mathbf{a}$. We begin with the definition and the neat formula.

$$\textbf{DEFINITION} \quad \kappa = |d\mathbf{T}/ds| \qquad \textbf{FORMULA} \quad \kappa = |\mathbf{v} \times \mathbf{a}|/|\mathbf{v}|^3 \tag{1}$$

The definition does not involve the parameter t—but the calculations do. The position vector $\mathbf{R}(t)$ yields $\mathbf{v} = d\mathbf{R}/dt$ and $\mathbf{a} = d\mathbf{v}/dt$. If t is changed to $2t$, the velocity $\mathbf{v}$ is doubled and $\mathbf{a}$ is multiplied by 4. Then $|\mathbf{v} \times \mathbf{a}|$ and $|\mathbf{v}|^3$ are multiplied by 8, and their ratio κ is unchanged.

Proof of formula (1) Start from $\mathbf{v} = |\mathbf{v}|\mathbf{T}$ and compute its derivative $\mathbf{a}$:

$$\mathbf{a} = \frac{d|\mathbf{v}|}{dt}\mathbf{T} + |\mathbf{v}|\frac{d\mathbf{T}}{dt} \text{ by the product rule.}$$

Now take the cross product with $\mathbf{v} = |\mathbf{v}|\mathbf{T}$. Remember that $\mathbf{T} \times \mathbf{T} = \mathbf{0}$:

$$\mathbf{v} \times \mathbf{a} = |\mathbf{v}|\mathbf{T} \times |\mathbf{v}|\frac{d\mathbf{T}}{dt}. \tag{2}$$

We know that $|\mathbf{T}| = 1$. Equation (4) will show that $\mathbf{T}$ is perpendicular to $d\mathbf{T}/dt$. So $|\mathbf{v} \times \mathbf{a}|$ is the first length $|\mathbf{v}|$ times the second length $|\mathbf{v}|\,|d\mathbf{T}/dt|$. The factor $\sin\theta$ in the length of a cross product is 1 from the 90° angle. In other words

$$\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^2} \quad \text{and} \quad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}. \tag{3}$$

The chain rule brings the extra $|ds/dt| = |\mathbf{v}|$ into the denominator.

Before any examples, we show that $d\mathbf{T}/dt$ is perpendicular to $\mathbf{T}$. The reason is that $\mathbf{T}$ is a unit vector. Differentiate both sides of $\mathbf{T} \cdot \mathbf{T} = 1$:

$$\frac{d\mathbf{T}}{dt} \cdot \mathbf{T} + \mathbf{T} \cdot \frac{d\mathbf{T}}{dt} = 0 \qquad \textit{or} \qquad 2\mathbf{T} \cdot \frac{d\mathbf{T}}{dt} = 0. \tag{4}$$

That proof used the product rule $\mathbf{U}'\cdot\mathbf{V}+\mathbf{U}\cdot\mathbf{V}'$ for the derivative of $\mathbf{U}\cdot\mathbf{V}$ (Problem 23, with $\mathbf{U}=\mathbf{V}=\mathbf{T}$). Think of the vector $\mathbf{T}$ moving around the unit sphere. To keep a constant length $(\mathbf{T}+d\mathbf{T})\cdot(\mathbf{T}+d\mathbf{T})=1$, we need $2\mathbf{T}\cdot d\mathbf{T}=0$. Movement $d\mathbf{T}$ is perpendicular to radius vector $\mathbf{T}$.

Our first examples will be **plane curves**. The position vector $\mathbf{R}(t)$ has components $x(t)$ and $y(t)$ but no $z(t)$. Look at the components of $\mathbf{v}$ and $\mathbf{a}$ and $\mathbf{v}\times\mathbf{a}$ (x' means dx/dt):

$$\begin{array}{cccc} \mathbf{R} & x(t) & y(t) & 0 \\ \mathbf{v} & x'(t) & y'(t) & 0 \\ \mathbf{a} & x''(t) & y''(t) & 0 \\ \mathbf{v}\times\mathbf{a} & 0 & 0 & x'y''-y'x'' \end{array} \qquad \begin{array}{l} |\mathbf{v}|=\sqrt{|x'|^2+|y'|^2} \\ \kappa=\dfrac{|x'y''-y'x''|}{((x')^2+(y')^2)^{3/2}} \end{array} \tag{5}$$

Equation (5) is the brutal but valuable formula for κ. Apply it to movement around a circle. We should find $\kappa=1/\text{radius } a$:

EXAMPLE 1 When $x=a\cos\omega t$ and $y=a\sin\omega t$ we substitute x', y', x'', y'' into (5):

$$\kappa=\frac{(-\omega a\sin\omega t)(-\omega^2 a\sin\omega t)-(\omega a\cos\omega t)(-\omega^2 a\cos\omega t)}{[(\omega a\sin\omega t)^2+(\omega a\cos\omega t)^2]^{3/2}}=\frac{\omega^3a^2}{[\omega^2a^2]^{3/2}}.$$

This is ω^3a^2/ω^3a^3 and ω cancels. The speed makes no difference to $\kappa=1/a$.

The third formula for κ applies to an ordinary plane curve given by $y(x)$. The parameter t is x! You see the square root in the speed $|\mathbf{v}|=ds/dx$:

$$\begin{array}{cccc} \mathbf{R} & x & y(x) & 0 \\ \mathbf{v} & 1 & dy/dx & 0 \\ \mathbf{a} & 0 & d^2y/dx^2 & 0 \\ \mathbf{v}\times\mathbf{a} & 0 & 0 & d^2y/dx^2 \end{array} \qquad \begin{array}{l} |\mathbf{v}|=\sqrt{1+(dy/dx)^2} \\ \kappa=\dfrac{|d^2y/dx^2|}{(1+(dy/dx)^2)^{3/2}} \end{array} \tag{6}$$

In practice this is the most popular formula for κ. The most popular approximation is $|d^2y/dx^2|$. (The denominator is omitted.) For the bending of a beam, the nonlinear equation uses κ and the linear equation uses d^2y/dx^2. We can see the difference for a parabola:

EXAMPLE 2 The curvature of $y=\frac{1}{2}x^2$ is $\kappa=|y''|/(1+(y')^2)^{3/2}=1/(1+x^2)^{3/2}$.

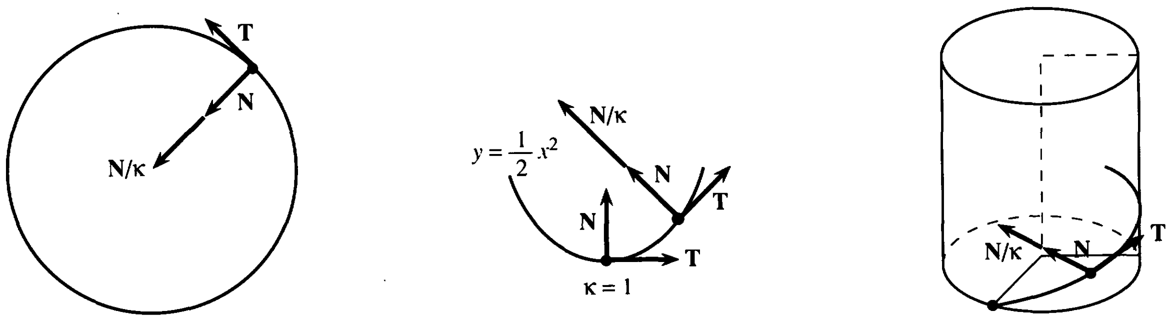

Fig. 12.7 Normal $\mathbf{N}$ divided by curvature κ for circle and parabola and unit helix.

The approximation is $y''=1$. This agrees with κ at $x=0$, where the parabola turns the corner. But for large x, the curvature approaches zero. Far out on the parabola, we go a long way for a small change in direction.

The parabola $y=-\frac{1}{2}x^2$, opening down, has the same κ. Now try a space curve.

EXAMPLE 3 Find the curvature of the unit helix $\mathbf{R}=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}+t\mathbf{k}$.

Take the cross product of $\mathbf{v}=-\sin t\,\mathbf{i}+\cos t\,\mathbf{j}+\mathbf{k}$ and $\mathbf{a}=-\cos t\,\mathbf{i}-\sin t\,\mathbf{j}$:

$$\mathbf{v}\times\mathbf{a}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\sin t & \cos t & 1\\ -\cos t & -\sin t & 0\end{vmatrix}=\sin t\,\mathbf{i}-\cos t\,\mathbf{j}+\mathbf{k}.$$

This cross product has length $\sqrt{2}$. Also the speed is $|\mathbf{v}|=\sqrt{\sin^2 t+\cos^2 t+1}=\sqrt{2}$:

$$\kappa=|\mathbf{v}\times\mathbf{a}|/|\mathbf{v}|^3=\sqrt{2}/(\sqrt{2})^3=\tfrac{1}{2}.$$

Compare with a unit circle. Without the climbing term $t\mathbf{k}$, the curvature would be 1. Because of climbing, each turn of the helix is longer and $\kappa=\frac{1}{2}$.

That makes one think: Is the helix twice as long as the circle? No. The length of a turn is only increased by $|\mathbf{v}|=\sqrt{2}$. The other $\sqrt{2}$ is because the tangent $\mathbf{T}$ slopes upward. The shadow in the base turns a full 360°, but $\mathbf{T}$ turns less.

THE NORMAL VECTOR N

The discussion is bringing us to an important vector. Where κ measures the ***rate*** of turning, the unit vector $\mathbf{N}$ gives the ***direction*** of turning. $\mathbf{N}$ is perpendicular to $\mathbf{T}$, and in the plane that leaves practically no choice. Turn left or right. For a space curve, follow $d\mathbf{T}$. Remember equation (4), which makes $d\mathbf{T}$ perpendicular to $\mathbf{T}$.

***The normal vector* $\mathbf{N}$ *is a unit vector along* $d\mathbf{T}/dt$. *It is perpendicular to* $\mathbf{T}$:**

$$\textbf{DEFINITION}\quad \mathbf{N}=\frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|}=\frac{1}{\kappa}\frac{d\mathbf{T}}{ds}\qquad \textbf{FORMULA}\quad \mathbf{N}=\frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}. \tag{7}$$

EXAMPLE 4 Find the normal vector $\mathbf{N}$ for the same helix $\mathbf{R}=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}+t\mathbf{k}$.

Solution Copy $\mathbf{v}$ from Example 3, divide by $|\mathbf{v}|$, and compute $d\mathbf{T}/dt$:

$$\mathbf{T}=\mathbf{v}/|\mathbf{v}|=(-\sin t\,\mathbf{i}+\cos t\,\mathbf{j}+\mathbf{k})/\sqrt{2}\quad\text{and}\quad d\mathbf{T}/dt=(-\cos t\,\mathbf{i}-\sin t\,\mathbf{j})/\sqrt{2}.$$

To change $d\mathbf{T}/dt$ into a unit vector, cancel the $\sqrt{2}$. ***The normal vector is*** $\mathbf{N}=-\cos t\,\mathbf{i}-\sin t\,\mathbf{j}$. It is perpendicular to $\mathbf{T}$. Since the $\mathbf{k}$ component is zero, $\mathbf{N}$ is horizontal. The tangent $\mathbf{T}$ slopes up at 45°—it goes around the circle at that latitude. The normal $\mathbf{N}$ is tangent to this circle ($\mathbf{N}$ is tangent to the path of the tangent!). So $\mathbf{N}$ stays horizontal as the helix climbs.

There is also a third direction, perpendicular to $\mathbf{T}$ and $\mathbf{N}$. It is the ***binormal*** vector $\mathbf{B}=\mathbf{T}\times\mathbf{N}$, computed in Problems 25–30. The unit vectors $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ provide the natural coordinate system for the path—along the curve, in the plane of the curve, and out of that plane. The theory is beautiful but the computations are not often done—we stop here.

TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

May I return a last time to the gas pedal and brake and steering wheel? The first two give acceleration along **T**. Turning gives acceleration along **N**. The rate of turning (curvature κ) and the direction **N** are established. We now ask about the *force* required. Newton's Law is $\mathbf{F} = m\mathbf{a}$, so we need the acceleration **a**—especially its component along **T** and its component along **N**.

$$\textit{\textbf{The acceleration is }} \mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left[\frac{ds}{dt}\right]^2\mathbf{N}. \tag{8}$$

For a straight path, d^2s/dt^2 is the only acceleration—the ordinary second derivative. The term $\kappa(ds/dt)^2$ is the acceleration in turning. Both have the dimension of length/(time)^2.

The force to steer around a corner depends on curvature and speed—as all drivers know. Acceleration is the derivative of $\mathbf{v} = |\mathbf{v}|\mathbf{T} = (ds/dt)\mathbf{T}$:

$$\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{ds}\frac{ds}{dt}. \tag{9}$$

That last term is $\kappa(ds/dt)^2\mathbf{N}$, since $d\mathbf{T}/ds = \kappa\mathbf{N}$ by formula (7). So (8) is proved.

EXAMPLE 5 A fixed speed $ds/dt = 1$ gives $d^2s/dt^2 = 0$. The only acceleration is $\kappa\mathbf{N}$.

EXAMPLE 6 Find the components of **a** for circular speed-up $\mathbf{R}(t) = \cos t^2\,\mathbf{i} + \sin t^2\,\mathbf{j}$.

Without stopping to think, compute $d\mathbf{R}/dt = \mathbf{v}$ and $ds/dt = |\mathbf{v}|$ and $\mathbf{v}/|\mathbf{v}| = \mathbf{T}$:

$$\mathbf{v} = -2t\sin t^2\,\mathbf{i} + 2t\cos t^2\,\mathbf{j}, \quad |\mathbf{v}| = 2t, \quad \mathbf{T} = -\sin t^2\,\mathbf{i} + \cos t^2\,\mathbf{j}.$$

The derivative of $ds/dt = |\mathbf{v}|$ is $d^2s/dt^2 = 2$. The derivative of **v** is **a**:

$$\mathbf{a} = -2\sin t^2\,\mathbf{i} + 2\cos t^2\,\mathbf{j} - 4t^2\cos t^2\,\mathbf{i} - 4t^2\sin t^2\,\mathbf{j}.$$

In the first terms of **a** we see 2**T**. In the last terms we must be seeing $\kappa|\mathbf{v}|^2\mathbf{N}$. Certainly $|\mathbf{v}|^2 = 4t^2$ and $\kappa = 1$, because the circle has radius 1. Thus $\mathbf{a} = 2\mathbf{T} + 4t^2\mathbf{N}$ has the tangential component 2 and normal component $4t^2$—acceleration along the circle and in to the center.

Table of Formulas

$\mathbf{v} = d\mathbf{R}/dt \quad \mathbf{a} = d\mathbf{v}/dt$

$|\mathbf{v}| = ds/dt \quad \mathbf{T} = \mathbf{v}/|\mathbf{v}| = |d\mathbf{R}/ds|$

Curvature $\kappa = |d\mathbf{T}/ds| = |\mathbf{v} \times \mathbf{a}|/|\mathbf{v}|^3$

Plane curves $\kappa = \dfrac{|x'y'' - y'x''|}{((x')^2 + (y')^2)^{3/2}} = \dfrac{|d^2y/dx^2|}{(1 + (dy/dx)^2)^{3/2}}$

Normal vector $\mathbf{N} = \dfrac{1}{\kappa}\dfrac{d\mathbf{T}}{ds} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

Acceleration $\mathbf{a} = (d^2s/dt^2)\mathbf{T} + \kappa|\mathbf{v}|^2\mathbf{N}$

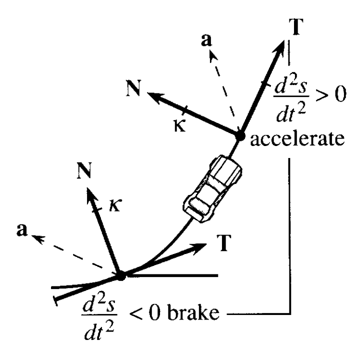

Fig. 12.8 Components of **a** as car turns corner.

12.3 EXERCISES

Read-through questions

The curvature tells how fast the curve __a__. For a circle of radius a, the direction changes by 2π in a distance __b__, so $\kappa =$ __c__. For a plane curve $y = f(x)$ the formula is $\kappa = |y''|/$ __d__. The curvature of $y = \sin x$ is __e__. At a point where $y'' = 0$ (an __f__ point) the curve is momentarily straight and $\kappa =$ __g__. For a space curve $\kappa = |\mathbf{v} \times \mathbf{a}|/$ __h__.

The normal vector $\mathbf{N}$ is perpendicular to __i__. It is a __j__ vector along the derivative of $\mathbf{T}$, so $\mathbf{N} =$ __k__. For motion around a circle $\mathbf{N}$ points __l__. Up a helix $\mathbf{N}$ also points __m__. Moving at unit speed on any curve, the time t is the same as the __n__ s. Then $|\mathbf{v}| =$ __o__ and $d^2s/dt^2 =$ __p__ and $\mathbf{a}$ is in the direction of __q__.

Acceleration equals __r__ $\mathbf{T} +$ __s__ $\mathbf{N}$. At unit speed around a unit circle, those components are __t__. An astronaut who spins once a second in a radius of one meter has $|\mathbf{a}| =$ __u__ meters/sec^2, which is about __v__ g.

Compute the curvature κ in Problems 1–8.

1 $y = e^x$

2 $y = \ln x$ (where is κ largest?)

3 $x = 2 \cos t$, $y = 2 \sin t$

4 $x = \cos t^2$, $y = \sin t^2$

5 $x = 1 + t^2$, $y = 3t^2$ (the path is a ________).

6 $x = \cos^3 t$, $y = \sin^3 t$

7 $r = \theta = t$ (so $x = t \cos t$, $y =$ ________)

8 $x = t$, $y = \ln \cos t$

9 Find $\mathbf{T}$ and $\mathbf{N}$ in Problem 4.

10 Show that $\mathbf{N} = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}$ in Problem 6.

11 Compute $\mathbf{T}$ and $\mathbf{N}$ in Problem 8.

12 Find the speed $|\mathbf{v}|$ and curvature κ of a projectile:

$$x = (v_0 \cos \alpha)t, \; y = (v_0 \sin \alpha)t - \tfrac{1}{2}gt^2.$$

13 Find $\mathbf{T}$ and $|\mathbf{v}|$ and κ for the helix $\mathbf{R} = 3 \cos t\,\mathbf{i} + 3 \sin t\,\mathbf{j} + 4t\,\mathbf{k}$. How much longer is a turn of the helix than the corresponding circle? What is the upward slope of $\mathbf{T}$?

14 When $\kappa = 0$ the path is a ________. This happens when $\mathbf{v}$ and $\mathbf{a}$ are ________. Then $\mathbf{v} \times \mathbf{a} =$ ________.

15 Find the curvature of a cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

16 If all points of a curve are moved twice as far from the origin ($x \to 2x$, $y \to 2y$), what happens to κ? What happens to $\mathbf{N}$?

17 Find κ and $\mathbf{N}$ at $\theta = \pi$ for the hypocycloid $x = 4 \cos \theta + \cos 4\theta$, $y = 4 \sin \theta - \sin 4\theta$.

18 From $\mathbf{v} = |\mathbf{v}|\mathbf{T}$ and $\mathbf{a}$ in equation (8), derive $\kappa = |\mathbf{v} \times \mathbf{a}|/|\mathbf{v}|^3$.

19 From a point on the curve, go along the vector $\mathbf{N}/\kappa$ to find the ***center of curvature***. Locate this center for the point $(1, 0)$ on the circle $x = \cos t$, $y = \sin t$ and the ellipse $x = \cos t$, $y = 2 \sin t$ and the parabola $y = \frac{1}{2}(x^2 - 1)$. The path of the center of curvature is the "*evolute*" of the curve.

20 Which of these depend only on the shape of the curve, and which depend also on the speed? $\mathbf{v}$, $\mathbf{T}$, $|\mathbf{v}|$, s, κ, $\mathbf{a}$, $\mathbf{N}$, $\mathbf{B}$.

21 A plane curve through $(0, 0)$ and $(2, 0)$ with constant curvature κ is the circular arc ________. For which κ is there no such curve?

22 Sketch a smooth curve going through $(0, 0)$, $(1, -1)$, and $(2, 0)$. Somewhere d^2y/dx^2 is at least ________. Somewhere the curvature is at least ________. (Proof is for instructors only.)

23 For plane vectors, the ordinary product rule applied to $U_1V_1 + U_2V_2$ shows that $(\mathbf{U} \cdot \mathbf{V})' = \mathbf{U}' \cdot \mathbf{V} +$ ________.

24 If $\mathbf{v}$ is perpendicular to $\mathbf{a}$, prove that the speed is constant. True or false: The path is a circle.

Problems 25–30 work with the T-N-B system—along the curve, in the plane of the curve, perpendicular to that plane.

25 Compute $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ for the helix $\mathbf{R} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$ in Examples 3–4.

26 Using Problem 23, differentiate $\mathbf{B} \cdot \mathbf{T} = 0$ and $\mathbf{B} \cdot \mathbf{B} = 1$ to show that $\mathbf{B}'$ is perpendicular to $\mathbf{T}$ and $\mathbf{B}$. So $d\mathbf{B}/ds = -\tau\mathbf{N}$ for some number τ called the ***torsion***.

27 Compute the torsion $\tau = |d\mathbf{B}/ds|$ for the helix in Problem 25.

28 Find $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ for the curve $x = 1$, $y = t$, $z = t^2$.

29 A circle lies in the xy plane. Its normal $\mathbf{N}$ lies ________ and $\mathbf{B} =$ ________ and $\tau = |d\mathbf{B}/ds| =$ ________.

30 The Serret-Frenet formulas are $d\mathbf{T}/ds = \kappa\mathbf{N}$, $d\mathbf{N}/ds = -\kappa\mathbf{T} + \tau\mathbf{B}$, $d\mathbf{B}/ds = -\tau\mathbf{N}$. We know the first and third. Differentiate $\mathbf{N} = -\mathbf{T} \times \mathbf{B}$ to find the second.

31 The angle θ from the x axis to the tangent line is $\theta = \tan^{-1}(dy/dx)$, when dy/dx is the slope of the curve.

(a) Compute $d\theta/dx$.

(b) Divide by $ds/dx = (1 + (dy/dx)^2)^{1/2}$ to show that $|d\theta/ds|$ is κ in equation (5). Curvature is change in direction $|d\theta|$ divided by change in position $|ds|$.

32 If the tangent direction is at angle θ then $\mathbf{T} = \cos \theta\,\mathbf{i} + \sin \theta\,\mathbf{j}$. In Problem 31 $|d\theta/ds|$ agreed with $\kappa = |d\mathbf{T}/ds|$ because $|d\mathbf{T}/d\theta| =$ ________.

In 33–37 find the T and N components of acceleration.

33 $x = 5 \cos \omega t$, $y = 5 \sin \omega t$, $z = 0$ (circle)

34 $x = 1 + t$, $y = 1 + 2t$, $z = 1 + 3t$ (line)

35 $x = t \cos t$, $y = t \sin t$, $z = 0$ (spiral)

36 $x = e^t \cos t$, $y = e^t \sin t$, $z = 0$ (spiral)

37 $x = 1$, $y = t$, $z = t^2$.

38 For the spiral in 36, show that the angle between **R** and **a** (position and acceleration) is constant. Find the angle.

39 Find the curvature of a polar curve $r = F(\theta)$.

12.4 Polar Coordinates and Planetary Motion

This section has a general purpose—to do vector calculus in ***polar coordinates***. It also has a specific purpose—to study ***central forces*** and the ***motion of planets***. The main gravitational force on a planet is from the sun. It is a central force, because it comes from the sun at the center. Polar coordinates are natural, so the two purposes go together.

You may feel that the planets are too old for this course. But Kepler's laws are more than theorems, they are something special in the history of mankind—"the greatest scientific discovery of all time." If we can recapture that glory we should do it. Part of the greatness is in the difficulty—Kepler was working sixty years before Newton discovered calculus. From pages of observations, and some terrific guesses, a theory was born. We will try to preserve the greatness without the difficulty, and show how elliptic orbits come from calculus. The first conclusion is quick.

Motion in a central force field always stays in a plane.

F is a multiple of the vector **R** from the origin (central force). **F** also equals $m\mathbf{a}$ (Newton's Law). Therefore **R** and **a** are in the same direction and $\mathbf{R} \times \mathbf{a} = \mathbf{0}$. Then $\mathbf{R} \times \mathbf{v}$ has zero derivative and is constant:

$$\text{by the product rule:} \quad \frac{d}{dt}(\mathbf{R} \times \mathbf{v}) = \mathbf{v} \times \mathbf{v} + \mathbf{R} \times \mathbf{a} = \mathbf{0} + \mathbf{0}. \tag{1}$$

$\mathbf{R} \times \mathbf{v}$ is a constant vector **H**. So **R** stays in the plane perpendicular to **H**.

How does a planet move in that plane? We turn to polar coordinates. At each point except the origin (where the sun is), $\mathbf{u}_r$ ***is the unit vector pointing outward.*** It is the position vector **R** divided by its length r (which is $\sqrt{x^2 + y^2}$):

$$\mathbf{u}_r = \mathbf{R}/r = (x\mathbf{i} + y\mathbf{j})/r = \cos \theta\, \mathbf{i} + \sin \theta\, \mathbf{j}. \tag{2}$$

That is a unit vector because $\cos^2\theta + \sin^2\theta = 1$. It goes out from the center. Figure 12.9 shows $\mathbf{u}_r$ and the second unit vector $\mathbf{u}_\theta$ at a 90° angle:

$$\mathbf{u}_\theta = -\sin \theta\, \mathbf{i} + \cos \theta\, \mathbf{j}. \tag{3}$$

The dot product is $\mathbf{u}_r \cdot \mathbf{u}_\theta = 0$. The subscripts r and θ indicate direction (not derivative).

Question 1: How do $\mathbf{u}_r$ and $\mathbf{u}_\theta$ change as r changes (out a ray)? *They don't.*

Question 2: How do $\mathbf{u}_r$ and $\mathbf{u}_\theta$ change as θ changes? *Take the derivative*:

$$\begin{aligned} d\mathbf{u}_r/d\theta &= -\sin \theta\, \mathbf{i} + \cos \theta\, \mathbf{j} = \mathbf{u}_\theta \\ d\mathbf{u}_\theta/d\theta &= -\cos \theta\, \mathbf{i} - \sin \theta\, \mathbf{j} = -\mathbf{u}_r. \end{aligned} \tag{4}$$

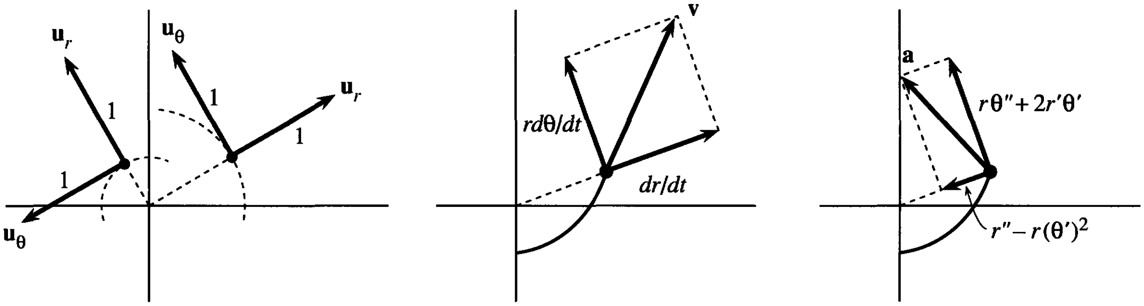

Fig. 12.9 $\mathbf{u}_r$ is outward, $\mathbf{u}_\theta$ is around the center. Components of $\mathbf{v}$ and $\mathbf{a}$ in those directions.

Since $\mathbf{u}_r = \mathbf{R}/r$, one formula is simple: ***The position vector is*** $\mathbf{R} = r\mathbf{u}_r$. For its derivative $\mathbf{v} = d\mathbf{R}/dt$, use the chain rule $d\mathbf{u}_r/dt = (d\mathbf{u}_r/d\theta)(d\theta/dt) = (d\theta/dt)\mathbf{u}_\theta$:

$$\textit{The velocity is } \mathbf{v} = \frac{d}{dt}(r\mathbf{u}_r) = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta. \tag{5}$$

The outward speed is dr/dt. The circular speed is $r\,d\theta/dt$. The sum of squares is $|\mathbf{v}|^2$.

Return one more time to steady motion around a circle, say $r = 3$ and $\theta = 2t$. The velocity is $\mathbf{v} = 6\mathbf{u}_\theta$, all circular. The acceleration is $-12\mathbf{u}_r$, all inward. For circles $\mathbf{u}_\theta$ is the tangent vector $\mathbf{T}$. But the unit vector $\mathbf{u}_r$ points outward and $\mathbf{N}$ points inward—the way the curve turns.

Now we tackle acceleration for any motion in polar coordinates. There can be speedup in r and speedup in θ (also change of direction). Differentiate $\mathbf{v}$ in (5) by the product rule:

$$\frac{d\mathbf{v}}{dt} = \frac{d^2r}{dt^2}\mathbf{u}_r + \frac{dr}{dt}\frac{d\mathbf{u}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta + r\frac{d\theta}{dt}\frac{d\mathbf{u}_\theta}{dt}.$$

For $d\mathbf{u}_r/dt$ and $d\mathbf{u}_\theta/dt$, multiply equation (4) by $d\theta/dt$. Then all terms contain $\mathbf{u}_r$ or $\mathbf{u}_\theta$. The formula for $\mathbf{a}$ is famous but not popular (except it got us to the moon):

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{u}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{u}_\theta. \tag{6}$$

In the steady motion with $r = 3$ and $\theta = 2t$, only one acceleration term is nonzero: $\mathbf{a} = -12\mathbf{u}_r$. Formula (6) can be memorized (maybe). Problem 14 gives a new way to reach it, using $re^{i\theta}$.

EXAMPLE 1 Find $\mathbf{R}$ and $\mathbf{v}$ and $\mathbf{a}$ for speedup $\theta = t^2$ around the circle $r = 1$.

Solution The position vector is $\mathbf{R} = \mathbf{u}_r$. Then $\mathbf{v}$ and $\mathbf{a}$ come from (5–6):

$$\mathbf{v} = (r\,d\theta/dt)\mathbf{u}_\theta = 2t\mathbf{u}_\theta \qquad \mathbf{a} = -(2t)^2\mathbf{u}_r + 2\mathbf{u}_\theta.$$

This question and answer were also in Example 6 of the previous section. The acceleration was $2\mathbf{T} + 4t^2\mathbf{N}$. Notice again that $\mathbf{T} = \mathbf{u}_\theta$ and $\mathbf{N} = -\mathbf{u}_r$, going round the circle.

EXAMPLE 2 Find $\mathbf{R}$ and $\mathbf{v}$ and $|\mathbf{v}|$ and $\mathbf{a}$ for the spiral motion $r = 3t$, $\theta = 2t$.

Solution The position vector is $\mathbf{R} = 3t\,\mathbf{u}_r$. Equation (5) gives velocity and speed:

$$\mathbf{v} = 3\mathbf{u}_r + 6t\mathbf{u}_\theta \qquad \text{and} \qquad |\mathbf{v}| = \sqrt{(3)^2 + (6t)^2}.$$

The motion goes *out* and also *around*. From (6) the acceleration is $-12t\,\mathbf{u}_r + 12\mathbf{u}_\theta$. The same answers would come more slowly from $\mathbf{R} = 3t\cos 2t\,\mathbf{i} + 3t\sin 2t\,\mathbf{j}$.

This example uses polar coordinates, but *the motion is not circular*. One of Kepler's inspirations, after many struggles, was to get away from circles.

KEPLER'S LAWS

You may know that before Newton and Leibniz and calculus and polar coordinates, Johannes Kepler discovered three laws of planetary motion. He was the court mathematician to the Holy Roman Emperor, who mostly wanted predictions of wars. Kepler also determined the date of every Easter—no small problem. His triumph was to discover patterns in the observations made by astronomers (especially by Tycho Brahe). Galileo and Copernicus expected circles, but Kepler found ellipses.

Law 1: Each planet travels in an ellipse with one focus at the sun.

Law 2: The vector from sun to planet sweeps out area at a steady rate: $dA/dt =$ constant.

Law 3: The length of the planet's year is $T = ka^{3/2}$, where $a =$ maximum distance from the center (not the sun) and $k = 2\pi/\sqrt{GM}$ is the same for all planets.

With calculus the proof of these laws is a thousand times quicker. But Law 2 is the only easy one. The sun exerts a central force. Equation (1) gave $\mathbf{R}\times\mathbf{v} = \mathbf{H} =$ constant for central forces. Replace $\mathbf{R}$ by $r\mathbf{u}_r$ and replace $\mathbf{v}$ by equation (5):

$$\mathbf{H} = r\mathbf{u}_r \times \left(\frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta\right) = r^2\frac{d\theta}{dt}(\mathbf{u}_r \times \mathbf{u}_\theta). \tag{7}$$

This vector $\mathbf{H}$ is constant, so ***its length*** $h = r^2 d\theta/dt$ ***is constant.*** In polar coordinates, the area is $dA = \frac{1}{2}r^2 d\theta$. This area dA is swept out by the planet (Figure 12.10), and we have proved Law 2:

$$dA/dt = \tfrac{1}{2}r^2\,d\theta/dt = \tfrac{1}{2}h = \textbf{\textit{constant}}. \tag{8}$$

Near the sun r is small. So $d\theta/dt$ is big and planets go around faster.

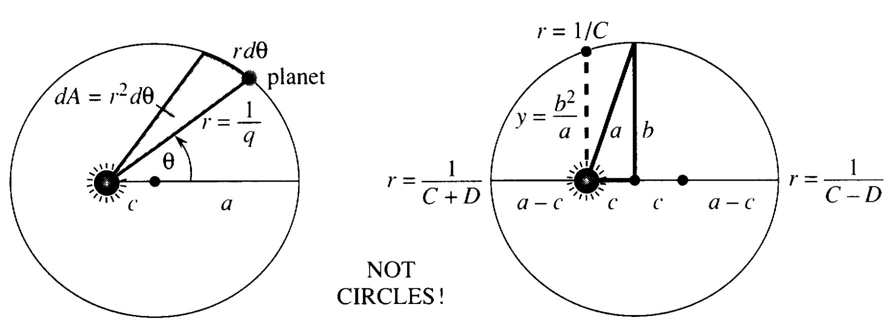

Fig. 12.10 The planet is on an ellipse with the sun at a focus. Note a, b, c, q.

Now for Law 1, about ellipses. We are aiming for $1/r = C - D\cos\theta$, which is *the polar coordinate equation of an ellipse*. It is easier to write q than $1/r$, and find an equation for q. The equation we will reach is $d^2q/d\theta^2 + q = C$. The desired $q = C - D\cos\theta$ solves that equation (check this), and gives us Kepler's ellipse.

The first step is to connect dr/dt to $dq/d\theta$ by the chain rule:

$$\frac{dr}{dt}=\frac{d}{dt}\left(\frac{1}{q}\right)=\frac{-1}{q^2}\frac{dq}{dt}=\frac{-1}{q^2}\frac{dq}{d\theta}\frac{d\theta}{dt}=-h\frac{dq}{d\theta}. \tag{9}$$

Notice especially $d\theta/dt=h/r^2=hq^2$. What we really want are second derivatives:

$$\frac{d^2r}{dt^2}=-h\frac{d}{dt}\left(\frac{dq}{d\theta}\right)=-h\frac{d}{d\theta}\left(\frac{dq}{d\theta}\right)\frac{d\theta}{dt}=-h^2q^2\frac{d^2q}{d\theta^2}. \tag{10}$$

After this trick of introducing q, we are ready for physics. The planet obeys Newton's Law $\mathbf{F}=m\mathbf{a}$, and the central force $\mathbf{F}$ is the sun's gravity:

$$\frac{\mathbf{F}}{m}=\mathbf{a} \quad \text{is} \quad -\frac{GM}{r^2}=\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2. \tag{11}$$

That right side is the $\mathbf{u}_r$ component of $\mathbf{a}$ in (6). Change r to $1/q$ and change $d\theta/dt$ to hq^2. The preparation in (10) allows us to rewrite d^2r/dt^2 in equation (11). That equation becomes

$$-GM\ q^2=-h^2q^2\frac{d^2q}{d\theta^2}-\frac{1}{q}(hq^2)^2.$$

Dividing by $-h^2q^2$ gives what we hoped for—the simple equation for q:

$$d^2q/d\theta^2+q=GM/h^2=C\ (\textbf{\textit{a constant}}). \tag{12}$$

The solution is $q=C-D\cos\theta$. Section 9.3 gave this polar equation for an ellipse or parabola or hyperbola. To be sure it is an ellipse, an astronomer computes C and D from the sun's mass M and the constant G and the earth's position and velocity. *The main point is that* $C>D$. Then q is never zero and r is never infinite. Hyperbolas and parabolas are ruled out, and the orbit in Figure 12.10 must be an ellipse.†

Astronomy is really impressive. You should visit the Greenwich Observatory in London, to see how Halley watched his comet. He amazed the world by predicting the day it would return. Also the discovery of Neptune was pure mathematics—the path of Uranus was not accounted for by the sun and known planets. LeVerrier computed a point in the sky and asked a Berlin astronomer to look. Sure enough Neptune was there.

Recently one more problem was solved—to explain the gap in the asteroids around Jupiter. The reason is "*chaos*"—the three-body problem goes unstable and an asteroid won't stay in that orbit. We have come a long way from circles.

Department of Royal Mistakes The last pound note issued by the Royal Mint showed Newton looking up from his great book *Principia Mathematica.* He is not smiling and we can see why. The artist put the sun at the center! Newton has just proved it is at the focus. True, the focus is marked S and the planet is P. But those rays at the center brought untold headaches to the Mint—the note is out of circulation. I gave an antique dealer three pounds for it (in coins).

Kepler's third law gives the time T to go around the ellipse—the planet's year. What is special in the formula is $a^{3/2}$—and for Kepler himself, the 15th of May 1618 was unforgettable: "the right ratio outfought the darkness of my mind, by the great proof afforded by my labor of seventeen years on Brahe's observations." The second

†An amateur sees the planet come around again, and votes for an ellipse.

law $dA/dt = \frac{1}{2}h$ is the key, plus two facts about an ellipse—its area πab and the height b^2/a above the sun:

1. The area $A = \int_0^T \frac{dA}{dt}\,dt = \frac{1}{2}hT$ must equal πab, so $T = \frac{2\pi ab}{h}$

2. The distance $r = 1/C$ at $\theta = \pi/2$ must equal b^2/a, so $b = \sqrt{a/C}$.

The height b^2/a is in Figure 12.10 and Problems 25–26. The constant $C = GM/h^2$ is in equation (12). Put them together to find the period:

$$T = \frac{2\pi ab}{h} = \frac{2\pi a}{h}\sqrt{\frac{a}{C}} = \frac{2\pi}{\sqrt{GM}}a^{3/2}. \tag{13}$$

To think of Kepler guessing $a^{3/2}$ is amazing. To think of Newton proving Kepler's laws by calculus is also wonderful—because we can do it too.

EXAMPLE 3 When a satellite goes around in a circle, find the time T.

Let r be the radius and ω be the angular velocity. The time for a complete circle (angle 2π) is $T = 2\pi/\omega$. The acceleration is GM/r^2 from gravity, and it is also $r\omega^2$ for circular motion. Therefore Kepler is proved right:

$$r\omega^2 = GM/r^2 \quad \Rightarrow \quad \omega = \sqrt{GM/r^3} \quad \Rightarrow \quad T = 2\pi/\omega = 2\pi r^{3/2}/\sqrt{GM}.$$

12.4 EXERCISES

Read-through questions

A central force points toward ___a___. Then $\mathbf{R} \times d^2\mathbf{R}/dt^2 = \mathbf{0}$ because ___b___. Therefore $\mathbf{R} \times d\mathbf{R}/dt$ is a ___c___ (called $\mathbf{H}$).

In polar coordinates, the outward unit vector is $\mathbf{u}_r = \cos\theta\,\mathbf{i} +$ ___d___. Rotated by 90° this becomes $\mathbf{u}_\theta =$ ___e___. The position vector $\mathbf{R}$ is the distance r times ___f___. The velocity $\mathbf{v} = d\mathbf{R}/dt$ is ___g___ $\mathbf{u}_r +$ ___h___ $\mathbf{u}_\theta$. For steady motion around the circle $r = 5$ with $\theta = 4t$, $\mathbf{v}$ is ___i___ and $|\mathbf{v}|$ is ___j___ and $\mathbf{a}$ is ___k___.

For motion under a circular force, r^2 times ___l___ is constant. Dividing by 2 gives Kepler's second law $dA/dt =$ ___m___. The first law says that the orbit is an ___n___ with the sun at ___o___. The polar equation for a conic section is ___p___ $= C - D\cos\theta$. Using $\mathbf{F} = m\mathbf{a}$ we found $q_{\theta\theta} +$ ___q___ $= C$. So the path is a conic section; it must be an ellipse because ___r___. The properties of an ellipse lead to the period $T =$ ___s___, which is Kepler's third law.

1 Find the unit vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$ at the point $(0, 2)$. The $\mathbf{u}_r$ and $\mathbf{u}_\theta$ components of $\mathbf{v} = \mathbf{i} + \mathbf{j}$ at that point are ________.

2 Find $\mathbf{u}_r$ and $\mathbf{u}_\theta$ at $(3, 3)$. If $\mathbf{v} = \mathbf{i} + \mathbf{j}$ then $\mathbf{v} =$ ________$\mathbf{u}_r$. Equation (5) gives $dr/dt =$ ________ and $d\theta/dt =$ ________.

3 At the point $(1, 2)$, velocities in the direction ________ will give $dr/dt = 0$. Velocities in the direction ________ will give $d\theta/dt = 0$.

4 Traveling on the cardioid $r = 1 - \cos\theta$ with $d\theta/dt = 2$, what is $\mathbf{v}$? How long to go around the cardioid (no integration involved)?

5 If $r = e^\theta$ and $\theta = 3t$, find $\mathbf{v}$ and $\mathbf{a}$ when $t = 1$.

6 If $r = 1$ and $\theta = \sin t$, describe the path and find $\mathbf{v}$ and $\mathbf{a}$ from equations (5–6). Where is the velocity zero?

7 (important) $\mathbf{R} = 4\cos 5t\,\mathbf{i} + 4\sin 5t\,\mathbf{j} = 4\mathbf{u}_r$ travels on a circle of radius 4 with $\theta = 5t$ and speed 20. Find the components of $\mathbf{v}$ and $\mathbf{a}$ in three systems: $\mathbf{i}$ and $\mathbf{j}$, $\mathbf{T}$ and $\mathbf{N}$, u_r and $\mathbf{u}_\theta$.

8 When is the circle $r = 4$ completed, if the speed is $8t$? Find $\mathbf{v}$ and $\mathbf{a}$ at the return to the starting point $(4, 0)$.

9 The $\mathbf{u}_\theta$ component of acceleration is ________ $= 0$ for a central force, which is in the direction of ________. Then $r^2 d\theta/dt$ is constant (new proof) because its derivative is r times ________.

10 If $r^2 d\theta/dt = 2$ for travel up the line $x = 1$, draw a triangle to show that $r = \sec\theta$ and integrate to find the time to reach $(1, 1)$.

11 A satellite is $r = 10{,}000$ km from the center of the Earth, traveling perpendicular to the radius vector at 4 km/sec. Find $d\theta/dt$ and h.

12 From $|\mathbf{u}_r| = 1$, it follows that $d\mathbf{u}_r/dr$ and $d\mathbf{u}_r/d\theta$ are ________ to $\mathbf{u}_r$ (Section 12.3). In fact $d\mathbf{u}_r/dr$ is ________ and $d\mathbf{u}_r/d\theta$ is ________.

13 Momentum is $m\mathbf{v}$ and its derivative is $m\mathbf{a}$ = force. Angular momentum is $m\mathbf{H} = m\mathbf{R} \times \mathbf{v}$ and its derivative is ________ = torque. Angular momentum is constant under a central force because the ________ is zero.

14 To find (and remember) $\mathbf{v}$ and $\mathbf{a}$ in polar coordinates, start with the complex number $re^{i\theta}$ and take its derivatives:

$$\mathbf{R} = re^{i\theta} \qquad \frac{d\mathbf{R}}{dt} = \frac{dr}{dt}e^{i\theta} + ir\frac{d\theta}{dt}e^{i\theta}$$

$$\frac{d^2\mathbf{R}}{dt^2} = \_\_\_\_ + \_\_\_\_ + \_\_\_\_ + \_\_\_\_ + \_\_\_\_.$$

Key idea: The coefficients of $e^{i\theta}$ and $ie^{i\theta}$ are the $\mathbf{u}_r$ and $\mathbf{u}_\theta$ components of $\mathbf{R}$, $\mathbf{v}$, $\mathbf{a}$:

$$\mathbf{R} = r\mathbf{u}_r + 0\mathbf{u}_\theta \qquad \mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta \qquad \mathbf{a} = \_\_\_\_\_\_\_\_.$$

(a) Fill in the five terms from the derivative of $d\mathbf{R}/dt$

(b) Convert $e^{i\theta}$ to $\mathbf{u}_r$ and $ie^{i\theta}$ to $\mathbf{u}_\theta$ to find $\mathbf{a}$

(c) Compare $\mathbf{R}$, $\mathbf{v}$, $\mathbf{a}$ with formulas (5–6)

(d) (for instructors only) Why does this method work?

Note how $e^{i\theta} = \cos\theta + i\sin\theta$ corresponds to $\mathbf{u}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$. This is one place where electrical engineers are allowed to write j instead of i for $\sqrt{-1}$.

15 If the period is T find from (13) a formula for the distance a.

16 To stay above New York what should be the period of a satellite? What should be its distance a from the center of the Earth?

17 From T and a find a formula for the mass M.

18 If the moon has a period of 28 days at an average distance of $a = 380{,}000$ km, estimate the mass of the ________.

19 The Earth takes $365\frac{1}{4}$ days to go around the sun at a distance $a \approx 93$ million miles ≈ 150 million kilometers. Find the mass of the sun.

20 True or false:

(a) The paths of all comets are ellipses.

(b) A planet in a circular orbit has constant speed.

(c) Orbits in central force fields are conic sections.

21 $\sqrt{GM} \approx 2 \cdot 10^7$ in what units, based on the Earth's mass $M = 6 \cdot 10^{24}$ kg and the constant $G = 6.67 \cdot 10^{-11}$ Nm^2/kg^2? A force of one kg$\cdot$meter/sec^2 is a Newton N.

22 If a satellite circles the Earth at 9000 km from the center, estimate its period T in seconds.

23 The Viking 2 orbiter around Mars had a period of about 10,000 seconds. If the mass of Mars is $M = 6.4 \cdot 10^{23}$ kg, what was the value of a?

24 Convert $1/r = C - D\cos\theta$, or $1 = Cr - Dx$, into the xy equation of an ellipse.

25 The distances a and c on the ellipse give the constants in $r = 1/(C - D\cos\theta)$. Substitute $\theta = 0$ and $\theta = \pi$ as in Figure 12.10 to find $D = c/(a^2 - c^2)$ and $C = a/(a^2 - c^2) = a/b^2$.

26 Show that $x = -c$, $y = b^2/a$ lies on the ellipse $x^2/a^2 + y^2/b^2 = 1$. Thus y is the height $1/C$ above the sun in Figure 12.10. The distance from the sun to the center has $c^2 = a^2 - b^2$.

27 The point $x = a\cos 2\pi t/T$, $y = b\sin 2\pi t/T$ travels around an ellipse centered at $(0, 0)$ and returns at time T. By symmetry it sweeps out area at the same rate at both ends of the major axis. Why does this *break* Kepler's second law?

28 If a central force is $\mathbf{F} = -ma(r)\mathbf{u}_r$, explain why $d^2r/dt^2 - r(d\theta/dt)^2 = -a(r)$. What is $a(r)$ for gravity? Equation (12) for $q = 1/r$ leads to $q_{\theta\theta} + q = r^2a(r)$.

29 When $F = 0$ the body should travel in a straight ________.

The equation $q_{\theta\theta} + q = 0$ allows $q = \cos\theta$, in which case the path $1/r = \cos\theta$ is ________. Extra credit: Mark off equal distances on a line, connect them to the sun, and explain why the triangles have equal area. So dA/dt is still constant.

30 The strong nuclear force increases with distance, $a(r) = r$. It binds quarks so tightly that up to now no top quarks have been seen (reliably). Problem 28 gives $q_{\theta\theta} + q = 1/q^3$.

(a) Multiply by q_θ and integrate to find $\frac{1}{2}q_\theta^2 + \frac{1}{2}q^2 =$ ________ $+ C$.

*(b) Integrate again (with tables) after setting $u = q^2$, $u_\theta = 2qq_\theta$.

31 The path of a quark in 30(b) can be written as $r^2(A + B\cos 2\theta) = 1$. Show that this is the same as the ellipse $(A + B)x^2 + (A - B)y^2 = 1$ with the origin at the *center*. The nucleus is not at a focus, and the pound note is correct for Newton watching quarks. (Quantum mechanics not accounted for.)

32 When will Halley's comet appear again? It disappeared in 1986 and its mean distance to the sun (average of $a + c$ and $a - c$) is $a = 1.6 \cdot 10^9$ kilometers.

33 You are walking at 2 feet/second toward the center of a merry-go-round that turns once every ten seconds. Starting from $r = 20$, $\theta = 0$ find $r(t)$, $\theta(t)$, $\mathbf{v}(t)$, $\mathbf{a}(t)$ and the length of your path to the center.

34 From Kepler's laws $r = 1/(C - D\cos\theta)$ and $r^2 d\theta/dt = h$, show that

1. $dr/dt = -Dh\sin\theta$ **2.** $d^2r/dt^2 = \left(\frac{1}{r} - C\right)h^2/r^2$

3. $d^2r/dt^2 - r(d\theta/dt)^2 = -Ch^2/r^2$.

When Newton reached **3**, he knew that Kepler's laws required a central force of Ch^2/r^2. This is his *inverse square law*. Then he went backwards, in our equations (8–12), to show that this force yields Kepler's laws.

35 How long is our year? The Earth's orbit has $a = 149.57 \cdot 10^6$ kilometers.