At this point in the book, I feel I can speak to you directly. You can guess what triple integrals are like. Instead of a small interval or a small rectangle, there is a small box. Instead of length $dx$ or area $dx \, dy$, the box has volume $dV = dx \, dy \, dz$. That is length times width times height. The goal is to put small boxes together (by integration). The main problem will be to discover the correct limits on $x, y, z$.

We could dream up more and more complicated regions in three-dimensional space. But I don’t think you can see the method clearly without seeing the region clearly. In practice six shapes are the most important:

- box
- prism
- cylinder
- cone
- tetrahedron
- sphere.

The box is easiest and the sphere may be the hardest (but no problem in spherical coordinates). Circular cylinders and cones fall in the middle, where $xyz$ coordinates are possible but $r \theta z$ are the best. I start with the box and prism and $xyz$.

**EXAMPLE 1** By triple integrals find the volume of a box and a prism (Figure 14.12).

\[
\iiint dV = \int_{z=0}^{1} \int_{y=0}^{3} \int_{x=0}^{2} dx \, dy \, dz \quad \text{and} \quad \iiint dV = \int_{z=0}^{1} \int_{y=0}^{3} \int_{x=0}^{2} dx \, dy \, dz
\]

The inner integral for both is $\int dx = 2$. Lines in the $x$ direction have length 2, cutting through the box and the prism. The middle integrals show the limits on $y$ (since $dy$ comes second):

\[
\int_{y=0}^{3} 2 \, dy = 6 \quad \text{and} \quad \int_{y=0}^{3} 2 \, dy = 6 - 6z.
\]

After two integrations these are areas. The first area 6 is for a plane section through the box. The second area $6 - 6z$ is cut through the prism. The shaded rectangle goes from $y = 0$ to $y = 3 - 3z$—we needed and used the equation $y + 3z = 3$ for the boundary of the prism. At this point $z$ is still constant! But the area depends on $z$, because the prism gets thinner going upwards. The base area is $6 - 6z = 6$, the top area is $6 - 6z = 0$.

The outer integral multiplies those areas by $dz$, to give the volume of slices. They are horizontal slices because $z$ came last. Integration adds up the slices to find the total volume:

\[
\text{box volume} = \int_{z=0}^{1} 6 \, dz = 6 \quad \text{prism volume} = \int_{z=0}^{1} (6 - 6z)dz = \left[ 6z - 3z^2 \right]_0^1 = 3.
\]

The box volume $2 \cdot 3 \cdot 1$ didn’t need calculus. The prism is half of the box, so its volume was sure to be 3—but it is satisfying to see how $6z - 3z^2$ gives the answer. Our purpose is to see how a triple integral works.

**Fig. 14.12** Box with sides 2, 3, 1. The prism is half of the box: volume $\int (6 - 6z)dz$ or $\int \frac{3}{2} \, dx$. 
Question  Find the prism volume in the order \( dz \, dy \, dx \) (six orders are possible).

Answer  \[ \int_0^2 \int_0^3 \int_0^{(3-y)/3} dz \, dy \, dx = \int_0^2 \int_0^3 \left( \frac{3-y}{3} \right) dy \, dx = \int_0^2 \frac{3}{2} dx = 3. \]

To find those limits on the \( z \) integral, follow a line in the \( z \) direction. It enters the prism at \( z = 0 \) and exits at the sloping face \( y + 3z = 3 \). That gives the upper limit \( z = (3 - y)/3 \). It is the height of a thin stick as in Section 14.1. This section writes out \( \int dz \) for the height, but a quicker solution starts at the double integral.

What is the number \( \frac{3}{2} \) in the last integral? It is the area of a vertical slice, cut by a plane \( x = \) constant. The outer integral adds up slices.

\[ \iiint f(x, y, z) \, dV \text{ is computed from three single integrals } \int \left[ \int \left[ \int f \, dx \right] dy \right] dz. \]

That step cannot be taken in silence—some basic calculus is involved. The triple integral is the limit of \( \sum f_i \Delta V \), a sum over small boxes of volume \( \Delta V \). Here \( f_i \) is any value of \( f(x, y, z) \) in the \( i \)th box. (In the limit, the boxes fit a curved region.) Now take those boxes in a certain order. Put them into lines in the \( x \) direction and put the lines of boxes into planes. The lines lead to the inner \( x \) integral, whose answer depends on \( y \) and \( z \). The \( y \) integral combines the lines into planes. Finally the outer integral accounts for all planes and all boxes.

Example 2 is important because it displays more possibilities than a box or prism.

**EXAMPLE 2**  Find the volume of a tetrahedron (4-sided pyramid). Locate \((\bar{x}, \bar{y}, \bar{z})\).

Solution  A tetrahedron has four flat faces, all triangles. The fourth face in Figure 14.13 is on the plane \( x + y + z = 1 \). A line in the \( x \) direction enters at \( x = 0 \) and exits at \( x = 1 - y - z \). (The length depends on \( y \) and \( z \). The equation of the boundary plane gives \( x \).) Then those lines are put into plane slices by the \( y \) integral:

\[ \int_{y=0}^{1-z} \int_{x=0}^{1-y-z} dx \, dy = \int_{y=0}^{1-z} (1 - y - z) dy = \left[ y - \frac{1}{2}y^2 - zy \right]_{y=0}^{1-z} = \frac{1}{2}(1 - z)^2. \]

What is this number \( \frac{1}{2}(1 - z)^2 \)? It is the area at height \( z \). The plane at that height slices out a right triangle, whose legs have length \( 1 - z \). The area is correct, but look at the limits of integration. If \( x \) goes to \( 1 - y - z \), why does \( y \) go to \( 1 - z \)? Reason: We are assembling lines, not points. The figure shows a line at every \( y \) up to \( 1 - z \).
Adding the slices gives the volume: \[ \int_0^1 \frac{1}{2}(1-z)^2 \, dz = \left[ \frac{1}{2}(z-1)^3 \right]_0^1 = \frac{1}{2}. \] This agrees with \( \frac{1}{2}(\text{base times height}) \), the volume of a pyramid.

The height \( \bar{z} \) of the centroid is “\( z_{\text{average}} \)” We compute \( \iiint z \, dV \) and divide by the volume. Each horizontal slice is multiplied by its height \( z \), and the limits of integration don’t change:

\[
\iiint z \, dV = \int_0^1 \int_0^{1-y} \int_0^{1-y-z} z \, dx \, dy \, dz = \int_0^1 \frac{z(1-z)^2}{2} \, dz = \frac{1}{24}.
\]

This is quick because \( z \) is constant in the \( x \) and \( y \) integrals. Each triangular slice contributes \( z \) times its area \( \frac{1}{2}(1-z)^2 \) times \( dz \). Then the \( z \) integral gives the moment \( 1/24 \). To find the average height, divide \( 1/24 \) by the volume:

\[
\bar{z} = \frac{\iiint z \, dV}{\iiint dV} = \frac{1/24}{1/6} = \frac{1}{4}.
\]

By symmetry \( \bar{x} = \frac{1}{4} \) and \( \bar{y} = \frac{1}{4} \). The centroid is the point \((\frac{1}{4}, \frac{1}{4}, \frac{1}{4})\). Compare that with \((\frac{1}{4}, \frac{1}{4}, \frac{1}{4})\), the centroid of the standard right triangle. Compare also with \( \frac{1}{2} \), the center of the unit interval. There must be a five-sided region in four dimensions centered at \((\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})\).

For area and volume we meet another pattern. Length of standard interval is 1, area of standard triangle is \( \frac{1}{2} \), volume of standard tetrahedron is \( \frac{1}{6} \), hypervolume in four dimensions must be \( \frac{1}{24} \). The interval reaches the point \( x = 1 \), the triangle reaches the line \( x + y = 1 \), the tetrahedron reaches the plane \( x + y + z = 1 \). The four-dimensional region stops at the hyperplane \( x + y + z + w = 1 \).

**EXAMPLE 3** Find the volume \( \iiint dx \, dy \, dz \) inside the unit sphere \( x^2 + y^2 + z^2 = 1 \).

First question: What are the limits on \( x \)? If a needle goes through the sphere in the \( x \) direction, where does it enter and leave? Moving in the \( x \) direction, the numbers \( y \) and \( z \) stay constant. The inner integral deals only with \( x \). The smallest and largest \( x \) are at the boundary where \( x^2 + y^2 + z^2 = 1 \). This equation does the work—we solve it for \( x \). Look at the limits on the \( x \) integral:

\[
\text{volume of sphere} = \iiint \sqrt{1-y^2-z^2} \, dx \, dy \, dz = \iiint 2\sqrt{1-y^2-z^2} \, dy \, dz.
\] (1)

The limits on \( y \) are \(-\sqrt{1-z^2}\) and \(+\sqrt{1-z^2}\). You can use algebra on the boundary equation \( x^2 + y^2 + z^2 = 1 \). But notice that \( x \) is gone! We want the smallest and largest \( y \), for each \( z \). It helps very much to draw the plane at height \( z \), slicing through the sphere in Figure 14.14. The slice is a circle of radius \( r = \sqrt{1-z^2} \). So the area is \( \pi r^2 \), which must come from the \( y \) integral:

\[
\int 2\sqrt{1-y^2-z^2} \, dy = \text{area of slice} = \pi(1-z^2).
\] (2)

I admit that I didn’t integrate. Is it cheating to use the formula \( \pi r^2 \)? I don’t think so. Mathematics is hard enough, and we don’t have to work blindfolded. The goal is understanding, and if you know the area then use it. Of course the integral of \( \sqrt{1-y^2-z^2} \) can be done if necessary—use Section 7.2.

The triple integral is down to a single integral. We went from one needle to a circle of needles and now to a sphere of needles. The volume is a sum of slices of area \( \pi(1-z^2) \). The South Pole is at \( z = -1 \), the North Pole is at \( z = +1 \), and the integral
14.3 Triple Integrals

is the volume $4\pi/3$ inside the unit sphere:

$$
\int_{-1}^{1} \pi(1-z^2)dz = \pi \left[ z - \frac{1}{3}z^3 \right]_{-1}^{1} = \frac{2}{3}\pi - \left( -\frac{2}{3}\pi \right) = \frac{4}{3}\pi.
$$

(3)

**Question 1** A cone also has circular slices. How is the last integral changed?

**Answer** The slices of a cone have radius $1-z$. Integrate $(1-z)^2$ not $\sqrt{1-z^2}$.

**Question 2** How does this compare with a circular cylinder (height 1, radius 1)?

**Answer** Now all slices have radius 1. Above $z=0$, a cylinder has volume $\pi$ and a half-sphere has volume $\frac{4}{3}\pi$ and a cone has volume $\frac{1}{3}\pi$.

For solids with equal surface area, the sphere has largest volume.

**Question 3** What is the average height $\bar{z}$ in the cone and half-sphere and cylinder?

**Answer**

$$
\bar{z} = \frac{\int z \text{(slice area)} dz}{\int \text{(slice area)} dz} = \frac{1}{4} \quad \text{and} \quad \frac{3}{8} \quad \text{and} \quad \frac{1}{2}.
$$

For solids with equal surface area, the sphere has largest volume.

**Example 4** Find the volume $\iiint dx \, dy \, dz$ inside the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

The limits on $x$ are now $\pm \sqrt{1-y^2/b^2 - z^2/c^2}$. The algebra looks terrible. The geometry is better—all slices are ellipses. A change of variable is absolutely the best.

Introduce $u = x/a$ and $v = y/b$ and $w = z/c$. Then the outer boundary becomes $u^2 + v^2 + w^2 = 1$. In these new variables the shape is a sphere. The triple integral for a sphere is $\iint du \, dv \, dw = 4\pi/3$. But what volume $dV$ in $xyz$ space corresponds to a small box with sides $du$ and $dv$ and $dw$?

Every $uvw$ box comes from an $xyz$ box. The box is stretched with no bending or twisting. Since $u$ is $x/a$, the length $dx$ is $a \, du$. Similarly $dy = b \, dv$ and $dz = c \, dw$. The volume of the $xyz$ box (Figure 14.14) is $dx \, dy \, dz = (abc) \, du \, dv \, dw$. The stretching factor $J = abc$ is a constant, and the volume of the ellipsoid is

$$
\iiint_{\text{ellipsoid}} dx \, dy \, dz = \iiint_{\text{sphere}} (abc) \, du \, dv \, dw = \frac{4\pi}{3} abc.
$$

(4)

You realize that this is special—other volumes are much more complicated. The sphere and ellipsoid are curved, but the small $xyz$ boxes are straight. The next section introduces spherical coordinates, and we can finally write "good limits." But then we need a different $J$. 

**Fig. 14.14** $\int dx =$ length of needle, $\iiint dx \, dy =$ area of slice. Ellipsoid is a stretched sphere.
14.3 EXERCISES

Read-through questions

Six important solid shapes are ____. The integral \( \iiint \) adds the volume ____ of small _____. For computation it becomes ____ single integrals. The inner integral \( \int \) is the ____ of a line through the solid. The variables ____ and ____ are held constant. The double integral \( \iint \) is the ____ of a slice, with ____ held constant. Then the z integral adds up the volumes of _____.

If the solid region \( V \) is bounded by the planes \( x = 0, \ y = 0, \ z = 0, \) and \( x + 2y + 3z = 1, \) the limits on the inner \( x \) integral are _____. The limits on \( y \) are ____. The limits on \( z \) are _____. In the new variables \( u = x, \ v = 2y, \ w = 3z, \) the equation of the outer boundary is ____. The volume of the tetrahedron in \( uvw \) space is ___. From \( dx = du \) and \( dy = dv/2 \) and \( dz = dw, \) the volume of an \( xyz \) box is \( dx \ dy \ dz = \) _____ \( du \ dv \ dw. \) So the volume of \( V \) is ___.

To find the average height \( \bar{z} \) in \( V \) we compute ____. To find the total mass in \( V \) if the density is \( \rho = e^x \) we compute the integral ____. To find the average density we compute ____. In the order \( \iiint \) the limits on the inner integral can depend on ___. The limits on the middle integral can depend on ___. The outer limits for the ellipsoid \( x^2 + 2y^2 + 3z^2 \leq 8 \) are _____.

1. For the solid region \( 0 \leq x \leq y \leq z \leq 1, \) find the limits in \( \iiint \) and compute the volume.
2. Reverse the order in Problem 1 to \( \iiint \) and find the limits of integration. The four faces of this tetrahedron are the planes \( x = 0 \) and \( y = x \) and ________.
3. This tetrahedron and five others like it fill the unit cube. Change the inequalities in Problem 1 to describe the other five.

4. Find the centroid \((\bar{x}, \bar{y}, \bar{z})\) in Problem 1.

Find the limits of integration in \( \iiint \) and the volume of solids 5–16. Draw a very rough picture.

5. A cube with sides of length 2, centered at \((0, 0, 0)\).
6. Half of that cube, the box above the \( xy \) plane.
7. Part of the same cube, the prism above the plane \( z = y.\)
8. Part of the same cube, above \( z = y \) and \( z = 0.\)
9. Part of the same cube, above \( z = x \) and below \( z = y.\)
10. Part of the same cube, where \( x \leq y \leq z. \) What shape is this?
11. The tetrahedron bounded by planes \( x = 0, \ y = 0, \ z = 0, \) and \( x + y + 2z = 2.\)
12. The tetrahedron with corners \((0, 0, 0), (2, 0, 0), (0, 4, 0), (0, 0, 4).\) First find the plane through the last three corners.
13. The part of the tetrahedron in Problem 11 below \( z = \frac{1}{2}.\)
14. The tetrahedron in Problem 12 with its top sliced off by the plane \( z = 1.\)
15. The volume above \( z = 0 \) below the cone \( \sqrt{x^2 + y^2} = 1 - z.\)
16. The tetrahedron in Problem 12, after it falls across the \( x \) axis onto the \( xy \) plane.

17–20 find the limits in \( \iiint \) or \( \iiint \) and compute the volume.

17. A circular cylinder with height 6 and base \( x^2 + y^2 \leq 1.\)
18. The part of that cylinder below the plane \( z = x. \) Watch the base. Draw a picture.
19. The volume shared by the cube (Problem 5) and cylinder.
20. The same cylinder lying along the \( x \) axis.

21. A cube is inscribed in a sphere: radius 1, both centers at \((0, 0, 0)\). What is the volume of the cube?
22. Find the volume and the centroid of the region bounded by \( x = 0, \ y = 0, \ z = 0, \) and \( x/a + y/b + z/c = 1.\)
23. Find the volume and centroid of the solid \( 0 \leq z \leq 4 - x^2 - y^2.\)
24. Based on the text, what is the volume inside \( x^2 + 4y^2 + 9z^2 = 16? \) What is the "hypervolume" of the 4-dimensional pyramid that stops at \( x + y + z + w = 1?\)
25. Find the partial derivatives \( \partial I/\partial x, \partial I/\partial y, \partial I/\partial z \) of

\[
I = \iiint f(x, y, z) \, dx \, dy \, dz.
\]

26. Define the average value of \( f(x, y, z) \) in a solid \( V.\)
27. Find the moment of inertia \( \iiint I \, dV \) of the cube \( |x| \leq 1, |y| \leq 1, |z| \leq 1 \) when \( I \) is the distance to (a) the \( x \) axis (b) the edge \( y = z = 1 \) (c) the diagonal \( x = y = z.\)
28. Add upper limits to produce the volume of a unit cube from small cubes: \( V = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 (\Delta x)^3 = 1.\)
29. Find the limit as \( \Delta x \to 0 \) of \( \sum_{i=1}^{3/\Delta x} \sum_{j=1}^{2/\Delta x} \sum_{k=1}^{1} (\Delta x)^3.\)
30. The midpoint rule for an integral over the unit cube chooses the center value \( f(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}). \) Which functions \( f = x^n y^m z^p \) are integrated correctly?
31. The trapezoidal rule estimates \( \int_0^1 \int_0^1 \int_0^1 f(x, y, z) \, dx \, dy \, dz \) as \( \frac{1}{4} \) times the sum of \( f(x, y, z) \) at 8 corners. This correctly integrates \( x^n y^m z^p \) for which \( m, n, p?\)
32. Propose a 27-point "Simpson's Rule" for integration over a cube. If many small cubes fill a large box, why are there only 8 new points per cube?
Cylindrical and Spherical Coordinates

Cylindrical coordinates are good for describing solids that are symmetric around an axis. The solid is three-dimensional, so there are three coordinates $r, \theta, z$:

$r$: out from the axis  \quad \theta$: around the axis  \quad z$: along the axis.

This is a mixture of polar coordinates $r\theta$ in a plane, plus $z$ upward. You will not find $r\theta z$ difficult to work with. Start with a cylinder centered on the $z$ axis:

**solid cylinder:** $0 \leq r \leq 1 \quad$ flat bottom and top: $0 \leq z \leq 3 \quad$ half-cylinder: $0 \leq \theta \leq \pi$

Integration over this half-cylinder is \[ \int_0^3 \int_0^1 \int_0^\pi r \, dr \, d\theta \, dz. \] These limits on $r, \theta, z$ are especially simple. Two other axially symmetric solids are almost as convenient:

cone: integrate to $r + z = 1$ \quad sphere: integrate to $r^2 + z^2 = R^2$

I would not use cylindrical coordinates for a box. Or a tetrahedron.

The integral needs one thing more—the volume $dV$. The movements $dr$ and $d\theta$ and $dz$ give a “curved box” in $xyz$ space, drawn in Figure 14.15c. The base is a polar rectangle, with area $r \, dr \, d\theta$. The new part is the height $dz$. The volume of the curved box is $r \, dr \, d\theta \, dz$. Then $r$ goes into the blank space in the triple integral—the stretching factor is $J = r$. There are six orders of integration (we give two):

\[
\text{volume} = \int_0^3 \int_0^1 \int_0^\pi r \, dr \, d\theta \, dz = \int_0^3 \int_0^\pi r \, dr \, dz \, d\theta. \quad (1)
\]

**EXAMPLE 1** (Volume of the half-cylinder). The integral of $r \, dr$ from 0 to 1 is \( \frac{1}{2} \). The $\theta$ integral is $\pi$ and the $z$ integral is 3. The volume is $3\pi/2$.

**EXAMPLE 2** The surface $r = 1 - z$ encloses the cone in Figure 14.16. Find its volume.

First solution Since $r$ goes out to $1 - z$, the integral of $r \, dr$ is $\frac{1}{2}(1 - z)^2$. The $\theta$ integral is $2\pi$ (a full rotation). Stop there for a moment.

We have reached $\int \int r \, dr \, d\theta = \frac{1}{2}(1 - z)^2 \cdot 2\pi$. This is the area of a slice at height $z$. The slice is a circle, its radius is $1 - z$, its area is $\pi(1 - z)^2$. The $z$ integral adds those slices to give $\pi/3$. That is correct, but it is not the only way to compute the volume.
Second solution  Do the \( z \) and \( \theta \) integrals first. Since \( z \) goes up to \( 1 - r \), and \( \theta \) goes around to \( 2\pi \), those integrals produce \( \iint r \, dz \, d\theta = r(1-\theta)2\pi \). Stop again—this must be the area of something.

After the \( z \) and \( \theta \) integrals we have a **shell at radius** \( r \). The height is \( 1-r \) (the outer shells are shorter). This height times \( 2\pi r \) gives the area around the shell. The choice between shells and slices is exactly as in Chapter 8. **Different orders of integration give different ways to cut up the solid.**

The volume of the shell is area times thickness \( dr \). The volume of the complete cone is the integral of shell volumes: \( \int_0^r r(1-\theta)2\pi \, dr = \pi/3 \).

**Third solution**  Do the \( r \) and \( z \) integrals first: \( \iint r \, dr \, dz = \frac{1}{2} \). Then the \( \theta \) integral is \( \int \frac{1}{2} \, d\theta \), which gives \( \frac{1}{6} \) times \( 2\pi \). This is the volume \( \pi/3 \)—but what is \( \frac{1}{6} \, d\theta \)?

The third cone is cut into wedges. The volume of a wedge is \( \frac{1}{6} \, d\theta \). It is quite common to do the \( \theta \) integral last, especially when it just multiplies by \( 2\pi \). It is not so common to think of wedges.

**Question**  Is the volume \( \frac{1}{6} \, d\theta \) equal to an area \( \frac{1}{6} \) times a thickness \( d\theta \)?

**Answer**  No! The triangle in the third cone has area \( \frac{1}{2} \) not \( \frac{1}{6} \). **Thickness is never \( d\theta \).**

![Fig. 14.16](image)

A cone cut three ways: slice at height \( z \), shell at radius \( r \), wedge at angle \( \theta \).

This cone is typical of a **solid of revolution**. The axis is in the \( z \) direction. The \( \theta \) integral yields \( 2\pi \), whether it comes first, second, or third. The \( r \) integral goes out to a radius \( f(z) \), which is 1 for the cylinder and \( 1-z \) for the cone. The integral \( \iint r \, dr \, d\theta \) is \( \pi(f(z))^2 \) = area of circular slice. This leaves the \( z \) integral \( \int \pi(f(z))^2 \, dz \). That is our old volume formula \( \int \pi(f(x))^2 \, dx \) from Chapter 8, where the slices were cut through the \( x \) axis.

**Example 3**  The **moment of inertia** around the \( z \) axis is \( \iiint r^3 \, dr \, d\theta \, dz \). The extra \( r^2 \) is (distance to axis)\(^2\). For the cone this triple integral is \( \pi/10 \).

**Example 4**  The **moment** around the \( z \) axis is \( \iiint r^2 \, dr \, d\theta \, dz \). For the cone this is \( \pi/6 \).

The **average distance** \( \bar{r} \) is (moment)/(volume) = \( (\pi/6)/(\pi/3) = \frac{1}{2} \).

**Example 5**  A sphere of radius \( R \) has the boundary \( r^2 + z^2 = R^2 \), in cylindrical coordinates. The outer limit on the \( r \) integral is \( \sqrt{R^2 - z^2} \). That is not acceptable in difficult problems. To avoid it we now change to coordinates that are natural for a sphere.
SPHERICAL COORDINATES

The Earth is a solid sphere (or near enough). On its surface we use two coordinates—latitude and longitude. To dig inward or fly outward, there is a third coordinate—\( \rho \) from the center. This Greek letter \( \rho \) replaces \( r \) to avoid confusion with cylindrical coordinates. Where \( r \) is measured from the \( z \) axis, \( \rho \) is measured from the origin. Thus \( r^2 = x^2 + y^2 \) and \( \rho^2 = x^2 + y^2 + z^2 \).

The angle \( \theta \) is the same as before. It goes from 0 to \( 2\pi \). It is the longitude, which increases as you travel east around the Equator.

The angle \( \phi \) is new. It equals 0 at the North Pole and \( \pi \) (not \( 2\pi \)) at the South Pole. It is the polar angle, measured down from the \( z \) axis. The Equator has a latitude of 0 but a polar angle of \( \pi/2 \) (halfway down). Here are some typical shapes:

- solid sphere (or ball): \( 0 \leq \rho \leq R \)
- surface of sphere: \( \rho = R \)
- upper half-sphere: \( 0 \leq \phi \leq \pi/2 \)
- eastern half-sphere: \( 0 \leq \theta \leq \pi \)

\( \text{Fig. 14.17} \) Spherical coordinates \( \rho \phi \theta \). The volume \( dV = \rho^2 \sin \phi \ d\rho \ d\phi \ d\theta \) of a spherical box.

The angle \( \phi \) is constant on a cone from the origin. It cuts the surface in a circle (Figure 14.17b), but not a great circle. The angle \( \theta \) is constant along a half-circle from pole to pole. The distance \( \rho \) is constant on each inner sphere, starting at the center \( \rho = 0 \) and moving out to \( \rho = R \).

**In spherical coordinates the volume integral is** \[ \iiint \rho^2 \sin \phi \ d\rho \ d\phi \ d\theta \]. To explain that surprising factor \( J = \rho^2 \sin \phi \), start with \( x = r \cos \theta \) and \( y = r \sin \theta \). In spherical coordinates \( r = \rho \sin \phi \) and \( z = \rho \cos \phi \)—see the triangle in the figure. So substitute \( \rho \sin \phi \) for \( r \):

\[
x = \rho \sin \phi \cos \theta, \ y = \rho \sin \phi \sin \theta, \ z = \rho \cos \phi. \tag{1}
\]

Remember those two steps, \( \rho \phi \theta \) to \( \rho \theta z \) to \( xyz \). We check that \( x^2 + y^2 + z^2 = \rho^2 \):

\[
\rho^2(\sin^2\phi \cos^2\theta + \sin^2\phi \sin^2\theta + \cos^2\phi) = \rho^2(\sin^2\phi + \cos^2\phi) = \rho^2.
\]

The volume integral is explained by Figure 14.17c. That shows a "spherical box" with right angles and curved edges. Two edges are \( d\phi \) and \( \rho d\phi \). The third edge is horizontal. The usual \( rd\theta \) becomes \( \rho \sin \phi \ d\theta \). Multiplying those lengths gives \( dV \).

**The volume of the box is** \( dV = \rho^2 \sin \phi \ d\rho \ d\phi \ d\theta \). This is a distance cubed, from \( \rho^2 d\rho \).
EXAMPLE 6 A solid ball of radius $R$ has known volume $V = \frac{4}{3} \pi R^3$. Notice the limits:

$$\int_0^{2\pi} \int_0^\pi \int_0^R \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \left[ \frac{1}{3} \rho^3 \right]^R_0 \left[ -\cos \phi \right]^\pi_0 \left[ \theta \right]^{2\pi}_0 = (\frac{1}{3} R^3)(2)(2\pi).$$

**Question** What is the volume above the cone in Figure 14.17?

**Answer** The $\phi$ integral stops at $[-\cos \phi]^\pi_3 = \frac{1}{3}$. The volume is $(\frac{1}{3} R^3)(\frac{1}{3})(2\pi)$.

EXAMPLE 7 The surface area of a sphere is $A = 4\pi R^2$. Forget the $\rho$ integral:

$$A = \int_0^{2\pi} \int_0^\pi R^2 \sin \phi \, d\phi \, d\theta = R^2 \left[ -\cos \phi \right]^\pi_0 \left[ \theta \right]^{2\pi}_0 = R^2(2)(2\pi).$$

*After those examples from geometry, here is the real thing from science.* I want to compute one of the most important triple integrals in physics—“the gravitational attraction of a solid sphere.” For some reason Isaac Newton had trouble with this integral. He refused to publish his masterpiece on astronomy until he had solved it. I think he didn’t use spherical coordinates—and the integral is not easy even now.

The answer that Newton finally found is beautiful. **The sphere acts as if all its mass were concentrated at the center.** At an outside point $(0, 0, D)$, the force of gravity is proportional to $1/D^2$. The force from a uniform solid sphere equals the force from a point mass, at every outside point $P$. That is exactly what Newton wanted and needed, to explain the solar system and to prove Kepler’s laws.

Here is the difficulty. Some parts of the sphere are closer than $D$, some parts are farther away. The actual distance $q$, from the outside point $P$ to a typical inside point, is shown in Figure 14.18. The average distance $\bar{q}$ to all points in the sphere is not $D$. But what Newton needed was a different average, and by good luck or some divine calculus it works perfectly: **The average of $1/\bar{q}$ is $1/D$.** This gives the potential energy:

$$\text{potential at point } P = \iiint_{\text{sphere}} \frac{1}{q} \, dV = \frac{\text{volume of sphere}}{D}. \quad (2)$$

A small volume $dV$ at the distance $q$ contributes $dV/q$ to the potential (Section 8.6, with density 1). The integral adds the contributions from the whole sphere. Equation (2) says that the potential at $r = D$ is not changed when the sphere is squeezed to the center. The potential equals the whole volume divided by the single distance $D$.

Important point: The average of $1/\bar{q}$ is $1/D$ and not $1/\bar{q}$. The average of $\frac{1}{3}$ and $\frac{1}{4}$ is not $\frac{1}{3}$. Smaller point: I wrote “sphere” where I should have written “ball.” The sphere is solid: $0 \leq \rho \leq R$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$.

What about the force? For the small volume it is proportional to $dV/q^2$ (this is the inverse square law). But **force is a vector**, pulling the outside point toward $dV$—not toward the center of the sphere. The figure shows the geometry and the symmetry. **We want the $z$ component of the force.** (By symmetry the overall $x$ and $y$ components are zero.) The angle between the force vector and the $z$ axis is $\alpha$, so for the $z$ component we multiply by $\cos \alpha$. The total force comes from the integral that Newton discovered:

$$\text{force at point } P = \iiint_{\text{sphere}} \frac{\cos \alpha}{q^2} \, dV = \frac{\text{volume of sphere}}{D^2}. \quad (3)$$

I will compute the integral (2) and leave you the privilege of solving (3). I mean that word seriously. If you have come this far, you deserve the pleasure of doing what at
14.4 Cylindrical and Spherical Coordinates

one time only Isaac Newton could do. Problem 26 offers a suggestion (just the law of cosines) but the integral is yours.

The law of cosines also helps with (2). For the triangle in the figure it gives $q^2 = D^2 - 2\rho D \cos \phi + \rho^2$. Call this whole quantity $u$. We do the surface integral first ($d\phi$ and $d\theta$ with $\rho$ fixed). Then $q^2 = u$ and $q = \sqrt{u}$ and $du = 2\rho D \sin \phi \, d\phi$:

$$
\int_0^{2\pi} \int_0^\pi \frac{\rho^2 \sin \phi \, d\phi \, d\theta}{q} = \int_0^{2\pi} \int_0^\pi \frac{2\pi \rho^2 \, du}{2\rho D \sqrt{u}} = \left[ \frac{2\pi \rho}{D} \sqrt{u} \right]_0^{\phi = \pi}.
$$

(4)

$2\pi$ came from the $\theta$ integral. The integral of $du/\sqrt{u}$ is $2\sqrt{u}$. Since $\cos \phi = -1$ at the upper limit, $u$ is $D^2 + 2\rho D + \rho^2$. The square root of $u$ is $D + \rho$. At the lower limit $\cos \phi = +1$ and $u = D^2 - 2\rho D + \rho^2$. This is another perfect square—its square root is $D - \rho$. The surface integral (4) with fixed $\rho$ is

$$
\int \int \frac{dA}{q} = \frac{2\pi \rho}{D} \left[ (D + \rho) - (D - \rho) \right] = \frac{4\pi \rho^2}{D}.
$$

(5)

Last comes the $\rho$ integral: $\int_0^R 4\pi \rho^2 \, d\rho/D = \frac{4}{3}\pi R^3/D$. This proves formula (2): potential equals volume of the sphere divided by $D$.

Note 1 Physicists are also happy about equation (5). The average of $1/q$ is $1/D$ not only over the solid sphere but over each spherical shell of area $4\pi \rho^2$. The shells can have different densities, as they do in the Earth, and still Newton is correct. This also applies to the force integral (3)—each separate shell acts as if its mass were concentrated at the center. Then the final $\rho$ integral yields this property for the solid sphere.

Note 2 Physicists also know that force is minus the derivative of potential. The derivative of (2) with respect to $D$ produces the force integral (3). Problem 27 explains this shortcut to equation (3).

EXAMPLE 8 Everywhere inside a hollow sphere the force of gravity is zero.

When $D$ is smaller than $\rho$, the lower limit $\sqrt{u}$ in the integral (4) changes from $D - \rho$ to $\rho - D$. That way the square root stays positive. This changes the answer in (5) to $4\pi \rho^2/\rho$, so the potential no longer depends on $D$. The potential is constant inside the hollow shell. Since the force comes from its derivative, the force is zero.

A more intuitive proof is in the second figure. The infinitesimal areas on the surface are proportional to $q^2$ and $Q^2$. But the distances to those areas are $q$ and $Q$, so the
forces involve $1/q^2$ and $1/Q^2$ (the inverse square law). Therefore the two areas exert equal and opposite forces on the inside point, and they cancel each other. The total force from the shell is zero.

I believe this zero integral is the reason that the inside of a car is safe from lightning. Of course a car is not a sphere. But electric charge distributes itself to keep the surface at constant potential. The potential stays constant inside—therefore no force. The tires help to prevent conduction of current (and electrocution of driver).

P.S. Don’t just step out of the car. Let a metal chain conduct the charge to the ground. Otherwise you could be the conductor.

**CHANGE OF COORDINATES—STRETCHING FACTOR $J$**

Once more we look to calculus for a formula. We need the volume of a small curved box in any $uvw$ coordinate system. The $r\theta z$ box and the $\rho\phi\theta$ box have right angles, and their volumes were read off from the geometry (stretching factors $J = r$ and $J = \rho^2 \sin \phi$ in Figures 14.15 and 14.17). Now we change from $xyz$ to other coordinates $uvw$—which are chosen to fit the problem.

Going from $xy$ to $uv$, the area $dA = J \ du \ dv$ was a 2 by 2 determinant. In three dimensions the determinant is 3 by 3. The matrix is always the “Jacobian matrix,” containing first derivatives. There were four derivatives from $xy$ to $uv$, now there are nine from $xyz$ to $uvw$.

**EXAMPLE 9** Find the factor $J$ for $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $z = \rho \cos \phi$.

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix}.$$  

The determinant has six terms, but two are zero—because of the zero in the corner. The other four terms are $\rho^2 \sin \phi \cos^2 \theta \sin^2 \phi$ and $\rho^2 \sin \phi \cos^2 \phi \cos^2 \theta$ and $\rho^2 \sin^3 \phi \sin^2 \theta$ and $\rho^2 \sin^3 \phi \cos^2 \theta$. Add the first two (note $\sin^2 \theta + \cos^2 \theta$) and separately add the second two. Then add the sums to reach $J = \rho^2 \sin \phi$.

Geometry already gave this answer. For most $uvw$ variables, use the determinant.
14.4 Cylindrical and Spherical Coordinates

EXERCISES

14.4 Read-through questions

The three _α_ coordinates are _rθz_. The point at _x = y = z = 1_ has _r = b_, _θ = c_, _z = d_. The volume integral is _∫∫_ _e_. The solid region _1 ≤ r ≤ 2, 0 ≤ θ ≤ 2π, 0 ≤ z ≤ 4_ is a _f_. Its volume is _g_. From the _r_ and _θ_ integrals the area of a _h_ equals _i_. From the _z_ and _θ_ integrals the area of a _j_ equals _k_. In _rθz_ coordinates the shapes of _l_ are convenient, while _m_ are not.

14.4 The three _n_ coordinates are _pθφ_. The point at _x = y = z = 1_ has _ρ = o_, _φ = p_, _θ = q_. The angle _φ_ is measured from _r_. _θ_ is measured from _s_. _ρ_ is the distance to _t_, where _r_ was the distance to _u_. If _ρφθ_ are known then _x = v_, _y = w_, _z = x_. The stretching factor _J_ is a 3 by 3 _Y_, and volume is _∫∫∫_ _z_.

The solid region _1 ≤ ρ ≤ 2, 0 ≤ Φ ≤ π, 0 ≤ θ ≤ 2π_ is a _A_. Its volume is _B_. From the _φ_ and _θ_ integrals the area of a _C_ at radius _ρ_ equals _D_. Newton discovered that the outside gravitational attraction of a _E_ is the same as for an equal mass located at _F_.

14.4 Convert the _xyz_ coordinates in 1–4 to _rθz_ and _ρφθ_.

1 (D, 0, 0) 2 (0, -D, 0)
3 (0, 0, D)(watch θ) 4 (3, 4, 5)

14.4 Convert the spherical coordinates in 5–7 to _xyz_ and _rθz_.

5 _ρ = 4, φ = π/4, θ = π/4_
6 _ρ = 2, φ = π/3, θ = π/6_
7 _ρ = 1, φ = π, θ = anything._
8 Where does _x = r_ and _y = θ_?
9 Find the polar angle _φ_ for the point with cylindrical coordinates _rθz_.
10 What are _x(t), y(t), z(t)_ on the great circle from _ρ = 1, φ = π/2, θ = 0_ with speed 1 to _ρ = 1, φ = π/4, θ = π/2_

14.4 From the limits of integration describe each region in 11–20 and find its volume. The inner integral has the inner limits.

11 _∫_ _0_ ^ _2π_ _∫_ _0_ ^ _1/2_ _∫_ _r_ _dz_ _dr_ _dθ_
12 _∫_ _0_ ^ _π_ _∫_ _0_ ^ _1_ _∫_ _r_ _dz_ _dr_ _dθ_
13 _∫_ _0_ ^ _2π_ _∫_ _0_ ^ _1_ _∫_ _r_ _dr_ _dz_ _dθ_
14 _∫_ _0_ ^ _2π_ _∫_ _0_ ^ _1_ _∫_ _r_ _dθ_ _dr_ _dz_
15 _∫_ _0_ ^ _π/2_ _∫_ _0_ ^ _π/2_ _∫_ _r_ _2π sin φ_ _dφ_ _dr_ _dθ_ _dz_
16 _∫_ _0_ ^ _2π_ _∫_ _0_ ^ _π/3_ _∫_ _0_ _π/2_ _ρ_ _sin θ_ _dθ_ _dφ_ _dr_
17 _∫_ _0_ ^ _π_ _∫_ _0_ _π/2_ _ρ_ _2 sin θ_ _dθ_ _dφ_ _dr_
18 _∫_ _0_ ^ _2π_ _∫_ _0_ ^ _π/4_ _∫_ _0_ _π/4_ _ρ_ _2 sin θ_ _dθ_ _dφ_ _dr_
19 _∫_ _0_ ^ _π_ _∫_ _0_ _π/3_ _∫_ _r_ _ρ_ _2 sin θ_ _dθ_ _dφ_ _dr_
20 _∫_ _0_ ^ _1_ _∫_ _0_ ^ _1_ _∫_ _ρ_ _2 sin θ_ _dθ_ _dφ_ _dr_

21 Example 5 gave the volume integral for a sphere in _rθz_ coordinates. What is the area of the circular slice at height _z_?

22 Describe the solid with _0 ≤ ρ ≤ 1 - cos θ_. What is the area of the cylindrical shell at radius _r_? Integrate over slices (_dz_) and over shells (_dr_) to reach _4πR_3/3.

23 A cylindrical tree has radius _a_. A saw cuts horizontally, ending halfway in at the _x_ axis. Then it cuts on a sloping plane (angle _r_ with the horizontal), also ending at the _x_ axis. What is the volume of the wedge that falls out?

24 Find the mass of a planet of radius _R_, if its density at each radius _ρ_ is _ρ_ = _ρ_3 + _1_)/_ρ_. Notice the infinite density at the center, but finite mass _M = ∫∫∫_ _δ dV_. Here _ρ_ is radius, not density.

25 For the cone out to _r = 1 - z_, the average distance from the _z_ axis is _q_ = _3_. For the triangle out to _r = 1 - z_ the average is _r_ = _3_. How can they be different when rotating the triangle produces the cone?

Problems 26–32, on the attraction of a sphere, use Figure 14.18 and the law of cosines _q_^2 = _D_^2 - 2_ρ_ _D_ cos _φ_ + _ρ_^2 = _u_.

26 Newton's achievement Show that _∫∫∫_ _cos x_ _dV_/_q_^2 _equals_ volume/D^2_. One hint only: Find _cos x_ from a second law of cosines _ρ_^2 = _D_^2 - 2_ρ_ _D_ _cos x_ + _x_^2. The _φ_ integral should involve _1/_ _q_ and _1/_ _q_^3. Equation (2) integrates _1/_ _q_ leaving _∫∫∫_ _dV_/_q_^3 still to do.

27 Compute _∂q/∂D_ in the first cosine law and show from Figure 14.18 that it equals _cos x_. Then the derivative of equation (2) with respect to _D_ is a shortcut to Newton's equation (3).

28 The lines of length _D_ and _q_ meet at the angle _x_. Move the meeting point up by _ΔD_. Explain why the other line stretches by _Δq_ ≈ _ΔD_ _cos x_. So _∂q/∂D_ = _cos x_ as before.
29 Show that the average distance is \( \bar{q} = 4R/3 \), from the North Pole \((D = R)\) to points on the Earth's surface \((\rho = R)\). To compute: \( \bar{q} = \int_{s_{D}} qR^2 \sin \phi \, d\phi \, d\theta / \text{(area } 4\pi R^2) \). Use the same substitution \( u \).

30 Show as in Problem 29 that the average distance is \( \bar{q} = D + \frac{1}{2} \rho^2 / D \), from the outside point \((0, 0, D)\) to points on the shell of radius \( \rho \). Then integrate \( \iiint q \, dV \) and divide by \( 4\pi R^3 / 3 \) to find \( \bar{q} \) for the solid sphere.

31 In Figure 14.18b, it is not true that the areas on the surface are exactly proportional to \( q^2 \) and \( Q^2 \). Why not? What happens to the second proof in Example 8?

32 For two solid spheres attracting each other (sun and planet), can we concentrate both spheres into point masses at their centers?

*33 Compute \( \iiint \cos \alpha \, dV / q^3 \) to find the force of gravity at \((0, 0, D)\) from a cylinder \( x^2 + y^2 \leq a^2 \), \( 0 \leq z \leq h \). Show from a figure why \( q^2 = r^2 + (D - z)^2 \) and \( \cos \alpha = (D - z) / q \).

34 A linear change of variables has \( x = au + bv + cw \), \( y = du + ev + fw \), and \( z = gu + hv + iw \). Write down the six terms in the determinant \( J \). Three terms have minus signs.

35 A pure stretching has \( x = au \), \( y = bv \), and \( z = cw \). Find the 3 by 3 matrix and its determinant \( J \). What is special about the xyz box in this case?

36 (a) The matrix in Example 9 has three columns. Find the lengths of those three vectors (sum of squares, then square root). Compare with the edges of the box in Figure 14.17.

(b) Take the dot product of every column in \( J \) with every other column. Zero dot products mean right angles in the box. So \( J \) is the product of the column lengths.

37 Find the stretching factor \( J \) for cylindrical coordinates from the matrix of first derivatives.

38 Follow Problem 36 for cylindrical coordinates—find the length of each column in \( J \) and compare with the box in Figure 14.15.

39 Find the moment of inertia around the z axis of a spherical shell (radius \( \rho \), density \( 1 \)). The distance from the axis to a point on the shell is \( r = \ldots \). Substitute for \( r \) to find \( I(\rho) = \int_0^{2\pi} \int_0^{\pi} r^2 \rho^2 \sin \phi \, d\phi \, d\theta \).

Divide by \( m\rho^2 \) (which is \( 4\pi \rho^4 \)) to compute the number \( J \) for a hollow ball in the rolling experiment of Section 8.5.

40 The moment of inertia of a solid sphere (radius \( R \), density \( 1 \)) adds up the hollow spheres of Problem 39: \( I = \int_0^R I(\rho) \, d\rho = \ldots \). Divide by \( mR^2 \) (which is \( \frac{2}{3} \pi R^5 \)) to find \( J \) in the rolling experiment. A solid ball rolls faster than a hollow ball because \( \ldots \).

41 Inside the Earth, the force of gravity is proportional to the distance \( \rho \) from the center. Reason: The inner ball of radius \( \rho \) has mass proportional to \( \ldots \) (assume constant density). The force is proportional to that mass divided by \( \ldots \). The rest of the Earth (sphere with hole) exerts no force because \( \ldots \).

42 Dig a tunnel through the center to Australia. Drop a ball in the tunnel at \( y = R \); Australia is \( y = -R \). The force of gravity is \( -cy \) by Problem 41. Newton's law is \( my'' = -cy \). What does the ball do when it reaches Australia?