14.3 Triple Integrals

At this point in the book, I feel I can speak to you directly. You can guess what triple integrals are like. Instead of a small interval or a small rectangle, there is a small box. Instead of length dx or area $dx\,dy$, the box has volume $dV = dx\,dy\,dz$. That is length times width times height. The goal is to put small boxes together (by integration). The main problem will be to discover the correct limits on x, y, z.

We could dream up more and more complicated regions in three-dimensional space. But I don't think you can see the method clearly without seeing the region clearly. In practice six shapes are the most important:

box prism cylinder cone tetrahedron sphere.

The box is easiest and the sphere may be the hardest (but no problem in spherical coordinates). Circular cylinders and cones fall in the middle, where xyz coordinates are possible but $r\theta z$ are the best. I start with the box and prism and xyz.

EXAMPLE 1 By triple integrals find the volume of a box and a prism (Figure 14.12).

$$\iiint_{\text{box}} dV = \int_{z=0}^{1}\int_{y=0}^{3}\int_{x=0}^{2} dx\,dy\,dz \quad\text{and}\quad \iiint_{\text{prism}} dV = \int_{z=0}^{1}\int_{y=0}^{3-3z}\int_{x=0}^{2} dx\,dy\,dz$$

The inner integral for both is $\int dx = 2$. Lines in the x direction have length 2, cutting through the box and the prism. The middle integrals show the limits on y (since dy comes second):

$$\int_{y=0}^{3} 2\,dy = 6 \quad\text{and}\quad \int_{y=0}^{3-3z} 2\,dy = 6 - 6z.$$

After two integrations these are ***areas***. The first area 6 is for a plane section through the box. The second area $6 - 6z$ is cut through the prism. The shaded rectangle goes from $y = 0$ to $y = 3 - 3z$—we needed and used the equation $y + 3z = 3$ for the boundary of the prism. *At this point z is still constant*! But the area depends on z, because the prism gets thinner going upwards. The base area is $6 - 6z = 6$, the top area is $6 - 6z = 0$.

The outer integral multiplies those areas by dz, to give the volume of slices. They are horizontal slices because z came last. Integration adds up the slices to find the total volume:

$$\text{box volume} = \int_{z=0}^{1} 6\,dz = 6 \qquad \text{prism volume} = \int_{z=0}^{1} (6-6z)dz = \Big[6z - 3z^2\Big]_0^1 = 3.$$

The box volume $2 \cdot 3 \cdot 1$ didn't need calculus. The prism is half of the box, so its volume was sure to be 3—but it is satisfying to see how $6z - 3z^2$ gives the answer. Our purpose is to see how a triple integral works.

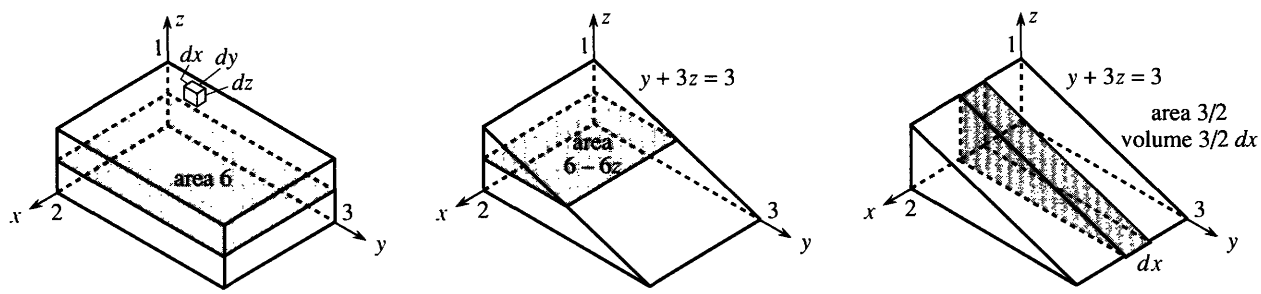

Fig. 14.12 Box with sides 2, 3, 1. The prism is half of the box: volume $\int(6-6z)dz$ or $\int \frac{3}{2}\,dx$.

Question Find the prism volume in the order $dz\,dy\,dx$ (***six orders are possible***).

Answer $\displaystyle\int_0^2\int_0^3\int_0^{(3-y)/3} dz\,dy\,dx=\int_0^2\int_0^3\left(\frac{3-y}{3}\right)dy\,dx=\int_0^2\frac{3}{2}\,dx=3.$

To find those limits on the z integral, follow a line in the z direction. It enters the prism at $z=0$ and exits at the sloping face $y+3z=3$. That gives the upper limit $z=(3-y)/3$. It is the height of a thin stick as in Section 14.1. This section writes out $\int dz$ for the height, but a quicker solution starts at the double integral.

What is the number $\frac{3}{2}$ in the last integral? It is the ***area of a vertical slice***, cut by a plane $x=$ constant. The outer integral adds up slices.

$$\iiint f(x,y,z)\,dV \textit{ is computed from three single integrals } \int\left[\int\left[\int f\,dx\right]dy\right]dz.$$

That step cannot be taken in silence—some basic calculus is involved. The triple integral is the limit of $\sum f_i\,\Delta V$, a sum over small boxes of volume ΔV. Here f_i is any value of $f(x,y,z)$ in the ith box. (In the limit, the boxes fit a curved region.) Now take those boxes *in a certain order*. Put them into lines in the x direction and put the lines of boxes into planes. The lines lead to the inner x integral, whose answer depends on y and z. The y integral combines the lines into planes. Finally the outer integral accounts for all planes and all boxes.

Example 2 is important because it displays more possibilities than a box or prism.

EXAMPLE 2 Find the volume of a tetrahedron (4-sided pyramid). Locate $(\bar{x},\bar{y},\bar{z})$.

Solution A tetrahedron has four flat faces, all triangles. The fourth face in Figure 14.13 is on the plane $x+y+z=1$. A line in the x direction enters at $x=0$ and exits at $x=1-y-z$. (The length depends on y and z. The equation of the boundary plane gives x.) Then those lines are put into plane slices by the y integral:

$$\int_{y=0}^{1-z}\int_{x=0}^{1-y-z} dx\,dy=\int_{y=0}^{1-z}(1-y-z)dy=\Big[y-\tfrac{1}{2}y^2-zy\Big]_0^{1-z}=\tfrac{1}{2}(1-z)^2.$$

What is this number $\frac{1}{2}(1-z)^2$? ***It is the area at height*** z. The plane at that height slices out a right triangle, whose legs have length $1-z$. The area is correct, but look at the limits of integration. ***If*** x ***goes to*** $1-y-z$, ***why does*** y ***go to*** $1-z$? Reason: We are assembling lines, not points. The figure shows a line at every y up to $1-z$.

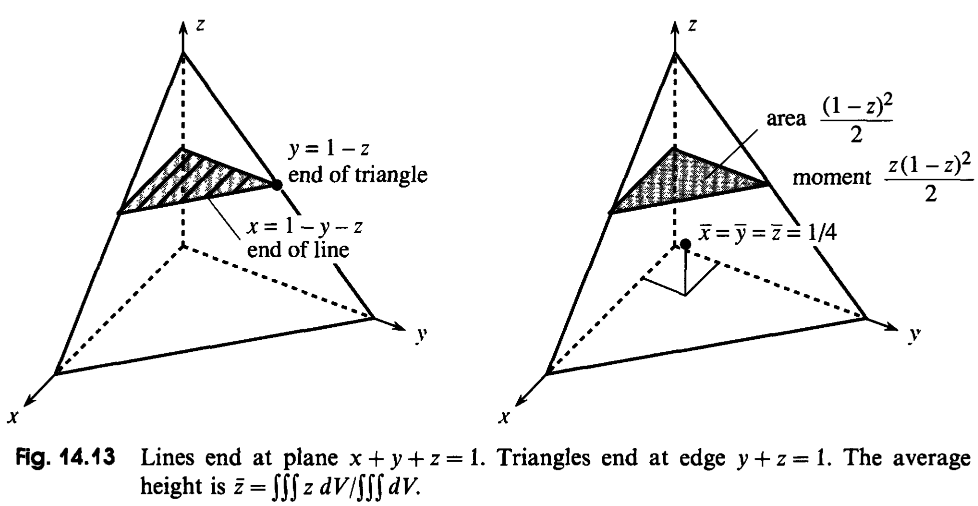

Fig. 14.13 Lines end at plane $x+y+z=1$. Triangles end at edge $y+z=1$. The average height is $\bar{z}=\iiint z\,dV/\iiint dV$.

Adding the slices gives the volume: $\int_0^1 \frac{1}{2}(1-z)^2 dz = [\frac{1}{6}(z-1)^3]_0^1 = \frac{1}{6}$. This agrees with $\frac{1}{3}$(base times height), the volume of a pyramid.

The height $\bar{z}$ of the centroid is "z_{average}." We compute $\iiint z\,dV$ and divide by the volume. Each horizontal slice is multiplied by its height z, and the limits of integration don't change:

$$\iiint z\,dV = \int_0^1 \int_0^{1-y} \int_0^{1-y-z} z\,dx\,dy\,dz = \int_0^1 \frac{z(1-z)^2}{2}\,dz = \frac{1}{24}.$$

This is quick because z is constant in the x and y integrals. Each triangular slice contributes z times its area $\frac{1}{2}(1-z)^2$ times dz. Then the z integral gives the moment $1/24$. To find the *average* height, divide $1/24$ by the volume:

$$\bar{z} = \text{height of centroid} = \frac{\iiint z\,dV}{\iiint dV} = \frac{1/24}{1/6} = \frac{1}{4}.$$

By symmetry $\bar{x} = \frac{1}{4}$ and $\bar{y} = \frac{1}{4}$. The centroid is the point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Compare that with $(\frac{1}{3}, \frac{1}{3})$, the centroid of the standard right triangle. Compare also with $\frac{1}{2}$, the center of the unit interval. There must be a five-sided region in four dimensions centered at $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$.

For area and volume we meet another pattern. Length of standard interval is 1, area of standard triangle is $\frac{1}{2}$, volume of standard tetrahedron is $\frac{1}{6}$, hypervolume in four dimensions must be _______. The interval reaches the point $x = 1$, the triangle reaches the line $x + y = 1$, the tetrahedron reaches the plane $x + y + z = 1$. The four-dimensional region stops at the hyperplane _______ $= 1$.

EXAMPLE 3 Find the volume $\iiint dx\,dy\,dz$ inside the unit sphere $x^2 + y^2 + z^2 = 1$.

First question: What are the limits on x? If a needle goes through the sphere in the x direction, where does it enter and leave? Moving in the x direction, the numbers y and z stay constant. The inner integral deals only with x. The smallest and largest x are at the boundary where $x^2 + y^2 + z^2 = 1$. This equation does the work—we solve it for x. Look at the limits on the x integral:

$$\text{volume of sphere} = \int_{?}^{?}\int_{?}^{?}\int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} dx\,dy\,dz = \int_{?}^{?}\int_{?}^{?} 2\sqrt{1-y^2-z^2}\,dy\,dz. \tag{1}$$

The limits on y are $-\sqrt{1-z^2}$ and $+\sqrt{1-z^2}$. You can use algebra on the boundary equation $x^2 + y^2 + z^2 = 1$. But notice that x is gone! We want the smallest and largest y, for each z. It helps very much to draw the plane at height z, slicing through the sphere in Figure 14.14. The slice is a circle of radius $r = \sqrt{1-z^2}$. So the area is πr^2, which must come from the y integral:

$$\int 2\sqrt{1-y^2-z^2}\,dy = \text{area of slice} = \pi(1-z^2). \tag{2}$$

I admit that I didn't integrate. Is it cheating to use the formula πr^2? I don't think so. Mathematics is hard enough, and we don't have to work blindfolded. The goal is understanding, and if you know the area then use it. Of course the integral of $\sqrt{1-y^2-z^2}$ can be done if necessary—use Section 7.2.

The triple integral is down to a single integral. We went from one needle to a circle of needles and now to a sphere of needles. The volume is a sum of slices of area $\pi(1-z^2)$. The South Pole is at $z = -1$, the North Pole is at $z = +1$, and the integral is the volume $4\pi/3$ inside the unit sphere:

$$\int_{-1}^{1} \pi(1-z^2)dz = \pi\left(z - \frac{1}{3}z^3\right)\Big]_{-1}^{1} = \frac{2}{3}\pi - \left(-\frac{2}{3}\pi\right) = \frac{4}{3}\pi. \tag{3}$$

Question 1 A cone also has circular slices. How is the last integral changed?
Answer The slices of a cone have radius $1 - z$. Integrate $(1-z)^2$ not $\sqrt{1-z^2}$.

Question 2 How does this compare with a circular cylinder (height 1, radius 1)?
Answer Now all slices have radius 1. Above $z = 0$, a cylinder has volume π and a half-sphere has volume $\frac{2}{3}\pi$ and a cone has volume $\frac{1}{3}\pi$.

For solids with equal surface area, the sphere has largest volume.

Question 3 What is the average height $\bar{z}$ in the cone and half-sphere and cylinder?

Answer $$\bar{z} = \frac{\int z(\text{slice area})dz}{\int(\text{slice area})dz} = \frac{1}{4} \text{ and } \frac{3}{8} \text{ and } \frac{1}{2}.$$

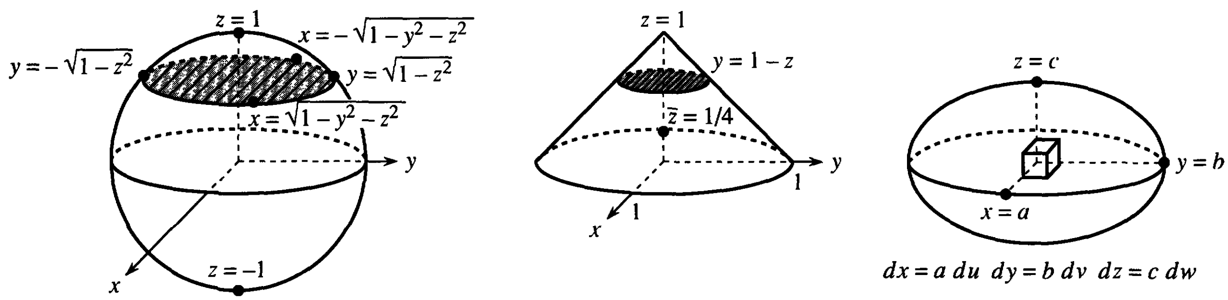

Fig. 14.14 $\int dx$ = length of needle, $\iint dx\,dy$ = area of slice. Ellipsoid is a stretched sphere.

EXAMPLE 4 Find the volume $\iiint dx\,dy\,dz$ inside the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

The limits on x are now $\pm\sqrt{1 - y^2/b^2 - z^2/c^2}$. The algebra looks terrible. The geometry is better—all slices are ellipses. A ***change of variable*** is absolutely the best.

Introduce $u = x/a$ and $v = y/b$ and $w = z/c$. Then the outer boundary becomes $u^2 + v^2 + w^2 = 1$. In these new variables the shape is a sphere. The triple integral for a sphere is $\iiint du\,dv\,dw = 4\pi/3$. But what volume dV in xyz space corresponds to a small box with sides du and dv and dw?

Every uvw box comes from an xyz box. The box is stretched with no bending or twisting. Since u is x/a, the length dx is $a\,du$. Similarly $dy = b\,dv$ and $dz = c\,dw$. The volume of the xyz box (Figure 14.14) is $dx\,dy\,dz = (abc)\,du\,dv\,dw$. The ***stretching factor*** $J = abc$ is a constant, and the volume of the ellipsoid is

$$\overset{\text{bad limits}}{\underset{\text{ellipsoid}}{\iiint}} dx\,dy\,dz = \overset{\text{better limits}}{\underset{\text{sphere}}{\iiint}} (abc)\,du\,dv\,dw = \frac{4\pi}{3}abc. \tag{4}$$

You realize that this is special—other volumes are much more complicated. The sphere and ellipsoid are curved, but the small xyz boxes are straight. The next section introduces spherical coordinates, and we can finally write "*good limits.*" But then we need a different J.

14.3 EXERCISES

Read-through questions

Six important solid shapes are __a__. The integral $\iiint dx\,dy\,dz$ adds the volume __b__ of small __c__. For computation it becomes __d__ single integrals. The inner integral $\int dx$ is the __e__ of a line through the solid. The variables __f__ and __g__ are held constant. The double integral $\iint dx\,dy$ is the __h__ of a slice, with __i__ held constant. Then the z integral adds up the volumes of __j__.

If the solid region V is bounded by the planes $x=0$, $y=0$, $z=0$, and $x+2y+3z=1$, the limits on the inner x integral are __k__. The limits on y are __l__. The limits on z are __m__. In the new variables $u=x$, $v=2y$, $w=3z$, the equation of the outer boundary is __n__. The volume of the tetrahedron in uvw space is __o__. From $dx=du$ and $dy=dv/2$ and $dz=$ __p__, the volume of an xyz box is $dx\,dy\,dz=$ __q__ $du\,dv\,dw$. So the volume of V is __r__.

To find the average height $\bar{z}$ in V we compute __s__/__t__. To find the total mass in V if the density is $\rho=e^z$ we compute the integral __u__. To find the average density we compute __v__/__w__. In the order $\iiint dz\,dx\,dy$ the limits on the inner integral can depend on __x__. The limits on the middle integral can depend on __y__. The outer limits for the ellipsoid $x^2+2y^2+3z^2\leqslant 8$ are __z__.

1 For the solid region $0\leqslant x\leqslant y\leqslant z\leqslant 1$, find the limits in $\iiint dx\,dy\,dz$ and compute the volume.

2 Reverse the order in Problem 1 to $\iiint dz\,dy\,dx$ and find the limits of integration. The four faces of this tetrahedron are the planes $x=0$ and $y=x$ and _______.

3 This tetrahedron and five others like it fill the unit cube. Change the inequalities in Problem 1 to describe the other five.

4 Find the centroid $(\bar{x},\bar{y},\bar{z})$ in Problem 1.

Find the limits of integration in $\iiint dx\,dy\,dz$ and the volume of solids 5–16. Draw a very rough picture.

5 A cube with sides of length 2, centered at (0, 0, 0).

6 Half of that cube, the box above the xy plane.

7 Part of the same cube, the prism above the plane $z=y$.

8 Part of the same cube, above $z=y$ and $z=0$.

9 Part of the same cube, above $z=x$ and below $z=y$.

10 Part of the same cube, where $x\leqslant y\leqslant z$. What shape is this?

11 The tetrahedron bounded by planes $x=0$, $y=0$, $z=0$, and $x+y+2z=2$.

12 The tetrahedron with corners (0, 0, 0), (2, 0, 0), (0, 4, 0), (0, 0, 4). First find the plane through the last three corners.

13 The part of the tetrahedron in Problem 11 below $z=\frac{1}{2}$.

14 The tetrahedron in Problem 12 with its top sliced off by the plane $z=1$.

15 The volume above $z=0$ below the cone $\sqrt{x^2+y^2}=1-z$.

***16** The tetrahedron in Problem 12, after it falls across the x axis onto the xy plane.

In 17–20 find the limits in $\iiint dx\,dy\,dz$ or $\iiint dz\,dy\,dx$. Compute the volume.

17 A circular cylinder with height 6 and base $x^2+y^2\leqslant 1$.

18 The part of that cylinder below the plane $z=x$. Watch the base. Draw a picture.

19 The volume shared by the cube (Problem 5) and cylinder.

20 The same cylinder lying along the x axis.

21 A cube is inscribed in a sphere: radius 1, both centers at (0, 0, 0). What is the volume of the cube?

22 Find the volume and the centroid of the region bounded by $x=0$, $y=0$, $z=0$, and $x/a+y/b+z/c=1$.

23 Find the volume and centroid of the solid $0\leqslant z\leqslant 4-x^2-y^2$.

24 Based on the text, what is the volume inside $x^2+4y^2+9z^2=16$? What is the "hypervolume" of the 4-dimensional pyramid that stops at $x+y+z+w=1$?

25 Find the partial derivatives $\partial I/\partial x$, $\partial I/\partial y$, $\partial^2 I/\partial y\,\partial z$ of

$$I=\int_0^z\int_0^y dx\,dy \text{ and } I=\int_0^z\int_0^y\int_0^x f(x,y,z)\,dx\,dy\,dz.$$

26 Define the average value of $f(x,y,z)$ in a solid V.

27 Find the moment of inertia $\iiint l^2\,dV$ of the cube $|x|\leqslant 1$, $|y|\leqslant 1$, $|z|\leqslant 1$ when l is the distance to
(a) the x axis (b) the edge $y=z=1$ (c) the diagonal $x=y=z$.

28 Add upper limits to produce the volume of a unit cube from small cubes: $V=\sum_{i=1}\sum_{j=1}\sum_{k=1}(\Delta x)^3=1$.

***29** Find the limit as $\Delta x\to 0$ of $\sum_{i=1}^{3/\Delta x}\sum_{j=1}^{2/\Delta x}\sum_{k=1}^{j}(\Delta x)^3$.

30 The midpoint rule for an integral over the unit cube chooses the center value $f(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Which functions $f=x^m y^n z^p$ are integrated correctly?

31 The trapezoidal rule estimates $\int_0^1\int_0^1\int_0^1 f(x,y,z)\,dx\,dy\,dz$ as $\frac{1}{8}$ times the sum of $f(x,y,z)$ at 8 corners. This correctly integrates $x^m y^n z^p$ for which m, n, p?

32 Propose a 27-point "Simpson's Rule" for integration over a cube. If many small cubes fill a large box, why are there only 8 new points per cube?

14.4 Cylindrical and Spherical Coordinates

Cylindrical coordinates are good for describing solids that are ***symmetric around an axis***. The solid is three-dimensional, so there are three coordinates r, θ, z:

r: *out* from the axis $\quad\theta$: *around* the axis $\quad z$: *along* the axis.

This is a mixture of polar coordinates $r\theta$ in a plane, plus z upward. You will not find $r\theta z$ difficult to work with. Start with a cylinder centered on the z axis:

solid cylinder: $0 \leqslant r \leqslant 1$ *flat bottom and top*: $0 \leqslant z \leqslant 3$ *half-cylinder*: $0 \leqslant \theta \leqslant \pi$

Integration over this half-cylinder is $\int_0^3 \int_0^\pi \int_0^1 \underline{\quad ? \quad} \, dr\, d\theta\, dz$. These limits on r, θ, z are especially simple. Two other axially symmetric solids are almost as convenient:

cone: *integrate to* $r + z = 1 \qquad$ ***sphere***: *integrate to* $r^2 + z^2 = R^2$

I would not use cylindrical coordinates for a box. Or a tetrahedron.

The integral needs one thing more—the volume dV. The movements dr and $d\theta$ and dz give a "curved box" in xyz space, drawn in Figure 14.15c. The base is a polar rectangle, with area $r\,dr\,d\theta$. The new part is the height dz. ***The volume of the curved box is*** $r\,dr\,d\theta\,dz$. Then r goes in the blank space in the triple integral—the stretching factor is $J = r$. There are six orders of integration (we give two):

$$\text{volume} = \int_z \int_\theta \int_r r\,dr\,d\theta\,dz = \int_\theta \int_z \int_r r\,dr\,dz\,d\theta. \tag{1}$$

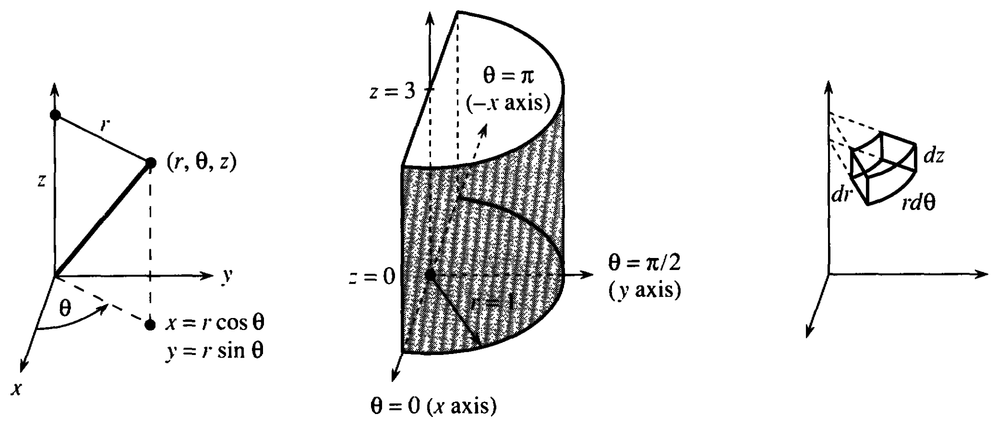

Fig. 14.15 Cylindrical coordinates for a point and a half-cylinder. Small volume $r\,dr\,d\theta\,dz$.

EXAMPLE 1 (Volume of the half-cylinder). The integral of $r\,dr$ from 0 to 1 is $\frac{1}{2}$. The θ integral is π and the z integral is 3. The volume is $3\pi/2$.

EXAMPLE 2 The surface $r = 1 - z$ encloses the cone in Figure 14.16. Find its volume.

First solution Since r goes out to $1 - z$, the integral of $r\,dr$ is $\frac{1}{2}(1 - z)^2$. The θ integral is 2π (a full rotation). Stop there for a moment.

We have reached $\iint r\,dr\,d\theta = \frac{1}{2}(1 - z)^2\, 2\pi$. This is the ***area of a slice at height*** z. The slice is a circle, its radius is $1 - z$, its area is $\pi(1 - z)^2$. The z integral adds those slices to give $\pi/3$. That is correct, but it is not the only way to compute the volume.

Second solution Do the z and θ integrals first. Since z goes up to $1-r$, and θ goes around to 2π, those integrals produce $\iint r\,dz\,d\theta = r(1-r)2\pi$. Stop again—this must be the area of something.

After the z and θ integrals we have a ***shell at radius r***. The height is $1-r$ (the outer shells are shorter). This height times $2\pi r$ gives the area around the shell. The choice betweeen shells and slices is exactly as in Chapter 8. ***Different orders of integration give different ways to cut up the solid.***

The volume of the shell is area times thickness dr. The volume of the complete cone is the integral of shell volumes: $\int_0^1 r(1-r)2\pi\,dr = \pi/3$.

Third solution Do the r and z integrals first: $\iint r\,dr\,dz = \frac{1}{6}$. Then the θ integral is $\int \frac{1}{6}\,d\theta$, which gives $\frac{1}{6}$ times 2π. This is the volume $\pi/3$—*but what is* $\frac{1}{6}\,d\theta$?

The third cone is cut into wedges. The volume of a wedge is $\frac{1}{6}\,d\theta$. It is quite common to do the θ integral last, especially when it just multiplies by 2π. It is not so common to think of wedges.

Question Is the volume $\frac{1}{6}\,d\theta$ equal to an area $\frac{1}{6}$ times a thickness $d\theta$?
Answer No! The triangle in the third cone has area $\frac{1}{2}$ not $\frac{1}{6}$. *Thickness is never* $d\theta$.

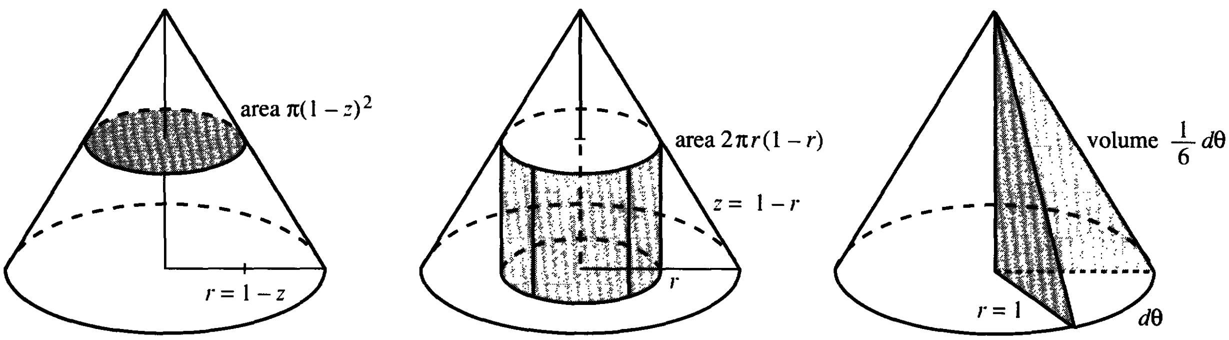

Fig. 14.16 A cone cut three ways: slice at height z, shell at radius r, wedge at angle θ.

This cone is typical of a ***solid of revolution***. The axis is in the z direction. The θ integral yields 2π, whether it comes first, second, or third. The r integral goes out to a radius $f(z)$, which is 1 for the cylinder and $1-z$ for the cone. The integral $\iint r\,dr\,d\theta$ is $\pi(f(z))^2$ = area of circular slice. This leaves the z integral $\int \pi(f(z))^2 dz$. That is our old volume formula $\int \pi(f(x))^2 dx$ from Chapter 8, where the slices were cut through the x axis.

EXAMPLE 3 The ***moment of inertia*** around the z axis is $\iiint r^3 dr\,d\theta\,dz$. The extra r^2 is (distance to axis)^2. For the cone this triple integral is $\pi/10$.

EXAMPLE 4 The ***moment*** around the z axis is $\iiint r^2\,dr\,d\theta\,dz$. For the cone this is $\pi/6$. The *average distance* $\bar{r}$ is (moment)/(volume) = $(\pi/6)/(\pi/3) = \frac{1}{2}$.

EXAMPLE 5 A sphere of radius R has the boundary $r^2 + z^2 = R^2$, in cylindrical coordinates. The outer limit on the r integral is $\sqrt{R^2 - z^2}$. That is not acceptable in difficult problems. To avoid it we now change to coordinates that are natural for a sphere.

SPHERICAL COORDINATES

The Earth is a solid sphere (or near enough). On its surface we use two coordinates—latitude and longitude. To dig inward or fly outward, there is a third coordinate—*the distance ρ from the center*. This Greek letter *rho* replaces r to avoid confusion with cylindrical coordinates. Where r is measured from the z axis, ρ is measured from the origin. Thus $r^2 = x^2 + y^2$ and $\rho^2 = x^2 + y^2 + z^2$.

The angle θ is the same as before. It goes from 0 to 2π. It is the longitude, which increases as you travel east around the Equator.

The angle ϕ is new. It equals 0 at the North Pole and π (not 2π) at the South Pole. It is the ***polar angle***, measured down from the z axis. The Equator has a latitude of 0 but a polar angle of $\pi/2$ (halfway down). Here are some typical shapes:

solid sphere (or ball): $0 \leqslant \rho \leqslant R$ surface of sphere: $\rho = R$

upper half-sphere: $0 \leqslant \phi \leqslant \pi/2$ eastern half-sphere: $0 \leqslant \theta \leqslant \pi$

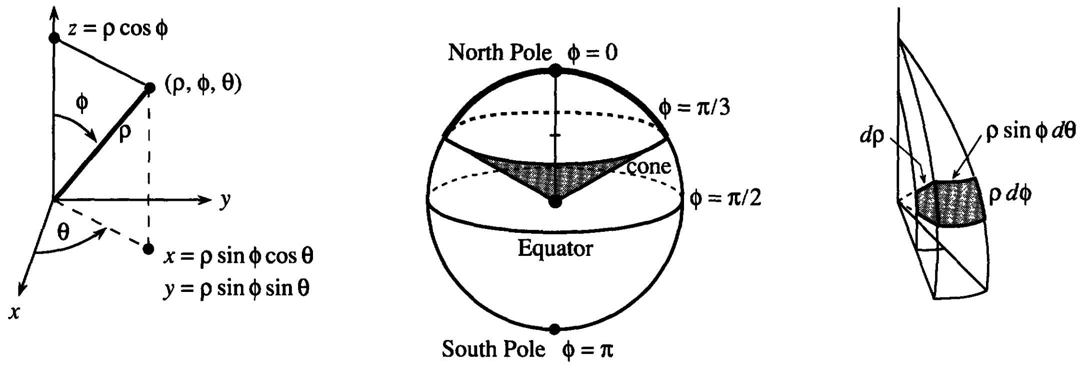

Fig. 14.17 Spherical coordinates $\rho\phi\theta$. The volume $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ of a spherical box.

The angle ϕ is constant on a cone from the origin. It cuts the surface in a circle (Figure 14.17b), but not a great circle. The angle θ is constant along a half-circle from pole to pole. The distance ρ is constant on each inner sphere, starting at the center $\rho = 0$ and moving out to $\rho = R$.

In spherical coordinates the volume integral is $\iiint \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$. To explain that surprising factor $J = \rho^2 \sin\phi$, start with $x = r\cos\theta$ and $y = r\sin\theta$. In spherical coordinates r is $\rho\sin\phi$ and z is $\rho\cos\phi$—see the triangle in the figure. So substitute $\rho\sin\phi$ for r:

$$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi. \tag{1}$$

Remember those two steps, $\rho\phi\theta$ to $r\theta z$ to xyz. We check that $x^2 + y^2 + z^2 = \rho^2$:

$$\rho^2(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi) = \rho^2(\sin^2\phi + \cos^2\phi) = \rho^2.$$

The volume integral is explained by Figure 14.17c. That shows a "*spherical box*" with right angles and curved edges. Two edges are $d\rho$ and $\rho d\phi$. The third edge is horizontal. The usual $r d\theta$ becomes $\rho\sin\phi\, d\theta$. Multiplying those lengths gives dV.

The volume of the box is $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$. This is a *distance cubed*, from $\rho^2 d\rho$.

EXAMPLE 6 A solid ball of radius R has known volume $V = \frac{4}{3}\pi R^3$. Notice the limits:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \left[\tfrac{1}{3}\rho^3\right]_0^R \left[-\cos\phi\right]_0^{\pi} \left[\theta\right]_0^{2\pi} = (\tfrac{1}{3}R^3)(2)(2\pi).$$

Question What is the volume above the cone in Figure 14.17?
Answer The ϕ integral stops at $[-\cos\phi]_0^{\pi/3} = \frac{1}{2}$. The volume is $(\frac{1}{3}R^3)(\frac{1}{2})(2\pi)$.

EXAMPLE 7 The *surface area* of a sphere is $A = 4\pi R^2$. Forget the ρ integral:

$$A = \int_0^{2\pi}\int_0^{\pi} R^2 \sin\phi \, d\phi \, d\theta = R^2\left[-\cos\phi\right]_0^{\pi} \left[\theta\right]_0^{2\pi} = R^2(2)(2\pi).$$

After those examples from geometry, here is the real thing from science. I want to compute one of the most important triple integrals in physics—"the gravitational attraction of a solid sphere." For some reason Isaac Newton had trouble with this integral. He refused to publish his masterpiece on astronomy until he had solved it. I think he didn't use spherical coordinates—and the integral is not easy even now.

The answer that Newton finally found is beautiful. ***The sphere acts as if all its mass were concentrated at the center.*** At an outside point $(0, 0, D)$, the force of gravity is proportional to $1/D^2$. The force from a uniform solid sphere equals the force from a point mass, at every outside point P. That is exactly what Newton wanted and needed, to explain the solar system and to prove Kepler's laws.

Here is the difficulty. Some parts of the sphere are closer than D, some parts are farther away. The actual distance q, from the outside point P to a typical inside point, is shown in Figure 14.18. The *average* distance $\bar{q}$ to all points in the sphere is not D. But what Newton needed was a different average, and by good luck or some divine calculus it works perfectly: ***The average of*** $1/q$ ***is*** $1/D$. This gives the potential energy:

$$\textbf{\textit{potential at point}}\ P = \iiint_{\text{sphere}} \frac{1}{q}\, dV = \frac{\text{volume of sphere}}{D}. \tag{2}$$

A small volume dV at the distance q contributes dV/q to the potential (Section 8.6, with density 1). The integral adds the contributions from the whole sphere. Equation (2) says that the potential at $r = D$ is not changed when the sphere is squeezed to the center. The potential equals the whole volume divided by the single distance D.

Important point: The average of $1/q$ is $1/D$ and not $1/\bar{q}$. The average of $\frac{1}{2}$ and $\frac{1}{4}$ is not $\frac{1}{3}$. Smaller point: I wrote "sphere" where I should have written "ball." The sphere is solid: $0 \leqslant \rho \leqslant R$, $0 \leqslant \phi \leqslant \pi$, $0 \leqslant \theta \leqslant 2\pi$.

What about the force? For the small volume it is proportional to dV/q^2 (this is the inverse square law). But *force is a vector*, pulling the outside point toward dV—not toward the center of the sphere. The figure shows the geometry and the symmetry. ***We want the z component of the force.*** (By symmetry the overall x and y components are zero.) The angle between the force vector and the z axis is α, so for the z component we multiply by $\cos\alpha$. The total force comes from the integral that Newton discovered:

$$\textbf{\textit{force at point}}\ P = \iiint_{\text{sphere}} \frac{\cos\alpha}{q^2}\, dV = \frac{\text{volume of sphere}}{D^2}. \tag{3}$$

I will compute the integral (2) and leave you the privilege of solving (3). I mean that word seriously. If you have come this far, you deserve the pleasure of doing what at

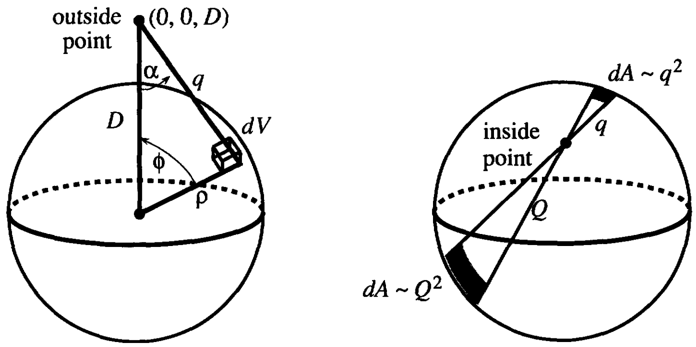

Fig. 14.18 Distance q from outside point to inside point. Distances q and Q to surface.

one time only Isaac Newton could do. Problem 26 offers a suggestion (just the law of cosines) but the integral is yours.

The law of cosines also helps with (2). For the triangle in the figure it gives $q^2 = D^2 - 2\rho D\cos\phi + \rho^2$. Call this whole quantity u. We do the surface integral first ($d\phi$ and $d\theta$ with ρ fixed). Then $q^2 = u$ and $q = \sqrt{u}$ and $du = 2\rho D \sin\phi\, d\phi$:

$$\int_0^{2\pi}\int_0^{\pi}\frac{\rho^2 \sin\phi\, d\phi\, d\theta}{q} = \int \frac{2\pi\rho^2}{2\rho D}\frac{du}{\sqrt{u}} = \left[\frac{2\pi\rho}{D}\sqrt{u}\right]_{\phi=0}^{\phi=\pi}. \tag{4}$$

2π came from the θ integral. The integral of $du/\sqrt{u}$ is $2\sqrt{u}$. Since $\cos\phi = -1$ at the upper limit, u is $D^2 + 2\rho D + \rho^2$. *The square root of u is $D + \rho$.* At the lower limit $\cos\phi = +1$ and $u = D^2 - 2\rho D + \rho^2$. This is another perfect square—its square root is $D - \rho$. The surface integral (4) with fixed ρ is

$$\iint \frac{dA}{q} = \frac{2\pi\rho}{D}\Big[(D+\rho) - (D-\rho)\Big] = \frac{4\pi\rho^2}{D}. \tag{5}$$

Last comes the ρ integral: $\int_0^R 4\pi\rho^2 d\rho/D = \frac{4}{3}\pi R^3/D$. This proves formula (2): ***potential equals volume of the sphere divided by D.***

Note* 1** Physicists are also happy about equation (5). The average of $1/q$ is $1/D$ not only over the solid sphere but over each spherical shell of area $4\pi\rho^2$. The shells can have different densities, as they do in the Earth, and still Newton is correct. This also applies to the force integral (3)—each separate shell acts as if its mass were concentrated at the center***. Then the final ρ integral yields this property for the solid sphere.

***Note* 2** Physicists also know that force is minus the derivative of potential. The derivative of (2) with respect to D produces the force integral (3). Problem 27 explains this shortcut to equation (3).

EXAMPLE 8 ***Everywhere inside a hollow sphere the force of gravity is zero.***

When D is smaller than ρ, the lower limit $\sqrt{u}$ in the integral (4) changes from $D - \rho$ to $\rho - D$. That way the square root stays positive. This changes the answer in (5) to $4\pi\rho^2/\rho$, so the potential no longer depends on D. *The potential is constant inside the hollow shell.* Since the force comes from its derivative, the force is zero.

A more intuitive proof is in the second figure. The infinitesimal areas on the surface are proportional to q^2 and Q^2. But the distances to those areas are q and Q, so the forces involve $1/q^2$ and $1/Q^2$ (the inverse square law). Therefore the two areas exert equal and opposite forces on the inside point, and they cancel each other. The total force from the shell is zero.

I believe this zero integral is the reason that the inside of a car is safe from lightning. Of course a car is not a sphere. But electric charge distributes itself to keep the surface at constant potential. The potential stays constant inside—therefore no force. The tires help to prevent conduction of current (and electrocution of driver).

P.S. Don't just step out of the car. Let a metal chain conduct the charge to the ground. Otherwise you could be the conductor.

CHANGE OF COORDINATES—STRETCHING FACTOR J

Once more we look to calculus for a formula. We need the volume of a small curved box in any uvw coordinate system. The $r\theta z$ box and the $\rho\phi\theta$ box have right angles, and their volumes were read off from the geometry (stretching factors $J = r$ and $J = \rho^2 \sin\phi$ in Figures 14.15 and 14.17). Now we change from xyz to other coordinates uvw—which are chosen to fit the problem.

Going from xy to uv, the area $dA = J\,du\,dv$ was a 2 by 2 determinant. In three dimensions the determinant is 3 by 3. The matrix is always the "Jacobian matrix," containing first derivatives. There were four derivatives from xy to uv, now there are nine from xyz to uvw.

14C Suppose x, y, z are given in terms of u, v, w. Then a small box in uvw space (sides du, dv, dw) comes from a volume $dV = J\,du\,dv\,dw$ in xyz space:

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \text{stretching factor } \frac{\partial(x, y, z)}{\partial(u, v, w)}. \tag{6}$$

The volume integral $\iiint dx\,dy\,dz$ becomes $\iiint |J|\,du\,dv\,dw$, with limits on uvw.

Remember that a 3 by 3 determinant is the sum of six terms (Section 11.5). One term in J is $(\partial x/\partial u)(\partial y/\partial v)(\partial z/\partial w)$, along the main diagonal. This comes from pure stretching, and the other five terms allow for rotation. The best way to exhibit the formula is for spherical coordinates—where the nine derivatives are easy but the determinant is not:

EXAMPLE 9 Find the factor J for $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho\cos\phi$.

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}.$$

The determinant has six terms, but two are zero—because of the zero in the corner. The other four terms are $\rho^2\sin\phi\cos^2\phi\sin^2\theta$ and $\rho^2\sin\phi\cos^2\phi\cos^2\theta$ and $\rho^2\sin^3\phi\sin^2\theta$ and $\rho^2\sin^3\phi\cos^2\theta$. Add the first two (note $\sin^2\theta + \cos^2\theta$) and separately add the second two. Then add the sums to reach $J = \rho^2\sin\phi$.

Geometry already gave this answer. For most uvw variables, use the determinant.

14.4 EXERCISES

Read-through questions

The three __a__ coordinates are $r\theta z$. The point at $x=y=z=1$ has $r=$ __b__, $\theta=$ __c__, $z=$ __d__. The volume integral is $\iiint$ __e__. The solid region $1\leqslant r\leqslant 2$, $0\leqslant\theta\leqslant 2\pi$, $0\leqslant z\leqslant 4$ is a __f__. Its volume is __g__. From the r and θ integrals the area of a __h__ equals __i__. From the z and θ integrals the area of a __j__ equals __k__. In $r\theta z$ coordinates the shapes of __l__ are convenient, while __m__ are not.

The three __n__ coordinates are $\rho\phi\theta$. The point at $x=y=z=1$ has $\rho=$ __o__, $\phi=$ __p__, $\theta=$ __q__. The angle ϕ is measured from __r__. θ is measured from __s__. ρ is the distance to __t__, where r was the distance to __u__. If $\rho\phi\theta$ are known then $x=$ __v__, $y=$ __w__, $z=$ __x__. The stretching factor J is a 3 by 3 __y__, and volume is $\iiint$ __z__.

The solid region $1\leqslant\rho\leqslant 2$, $0\leqslant\phi\leqslant\pi$, $0\leqslant\theta\leqslant 2\pi$ is a __A__. Its volume is __B__. From the ϕ and θ integrals the area of a __C__ at radius ρ equals __D__. Newton discovered that the outside gravitational attraction of a __E__ is the same as for an equal mass located at __F__.

Convert the xyz coordinates in 1–4 to $r\theta z$ and $\rho\phi\theta$.

1 $(D, 0, 0)$

2 $(0, -D, 0)$

3 $(0, 0, D)$(watch θ)

4 $(3, 4, 5)$

Convert the spherical coordinates in 5–7 to xyz and $r\theta z$.

5 $\rho=4$, $\phi=\pi/4$, $\theta=-\pi/4$

6 $\rho=2$, $\phi=\pi/3$, $\theta=\pi/6$

7 $\rho=1$, $\phi=\pi$, $\theta=$ anything.

8 Where does $x=r$ and $y=\theta$?

9 Find the polar angle ϕ for the point with cylindrical coordinates $r\theta z$.

10 What are $x(t)$, $y(t)$, $z(t)$ on the great circle from $\rho=1$, $\phi=\pi/2$, $\theta=0$ with speed 1 to $\rho=1$, $\phi=\pi/4$, $\theta=\pi/2$?

From the limits of integration describe each region in 11–20 and find its volume. The inner integral has the inner limits.

11 $\displaystyle\int_{\theta=0}^{2\pi}\int_{r=0}^{1/\sqrt{2}}\int_{z=r}^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta$

12 $\displaystyle\int_0^{\pi}\int_0^{1}\int_0^{1+r^2} r\,dz\,dr\,d\theta$

13 $\displaystyle\int_{\theta=0}^{2\pi}\int_{z=0}^{1}\int_{r=0}^{2-z} r\,dr\,dz\,d\theta$

14 $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} r\,d\theta\,dr\,dz$

15 $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{1} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

16 $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

17 $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{\sin\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

18 $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_1^{3} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

19 $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

20 $\displaystyle\int_0^{1}\int_0^{1}\int_0^{1} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

21 Example 5 gave the volume integral for a sphere in $r\theta z$ coordinates. What is the area of the circular slice at height z? What is the area of the cylindrical shell at radius r? Integrate over slices (dz) and over shells (dr) to reach $4\pi R^3/3$.

22 Describe the solid with $0\leqslant\rho\leqslant 1-\cos\phi$ and find its volume.

23 A cylindrical tree has radius a. A saw cuts horizontally, ending halfway in at the x axis. Then it cuts on a sloping plane (angle α with the horizontal), also ending at the x axis. What is the volume of the wedge that falls out?

24 Find the mass of a planet of radius R, if its density at each radius ρ is $\delta=(\rho+1)/\rho$. Notice the infinite density at the center, but finite mass $M=\iiint\delta\,dV$. Here ρ is radius, not density.

25 For the cone out to $r=1-z$, the average distance from the z axis is $\bar{r}=\frac{1}{2}$. For the triangle out to $r=1-z$ the average is $\bar{r}=\frac{1}{3}$. How can they be different when rotating the triangle produces the cone?

Problems 26–32, on the attraction of a sphere, use Figure 14.18 and the law of cosines $q^2=D^2-2\rho D\cos\phi+\rho^2=u$.

26 *Newton's achievement* Show that $\iiint(\cos\alpha)dV/q^2$ equals volume$/D^2$. *One hint only*: Find $\cos\alpha$ from a second law of cosines $\rho^2=D^2-2qD\cos\alpha+q^2$. The ϕ integral should involve $1/q$ and $1/q^3$. Equation (2) integrates $1/q$, leaving $\iiint dV/q^3$ still to do.

27 Compute $\partial q/\partial D$ in the first cosine law and show from Figure 14.18 that it equals $\cos\alpha$. Then the derivative of equation (2) with respect to D is a shortcut to Newton's equation (3).

28 The lines of length D and q meet at the angle α. Move the meeting point up by ΔD. Explain why the other line stretches by $\Delta q\approx\Delta D\cos\alpha$. So $\partial q/\partial D=\cos\alpha$ as before.

29 Show that the average distance is $\bar{q} = 4R/3$, from the North Pole ($D = R$) to points on the Earth's surface ($\rho = R$). To compute: $\bar{q} = \iint qR^2 \sin\phi \, d\phi \, d\theta/(\text{area } 4\pi R^2)$. Use the same substitution u.

30 Show as in Problem 29 that the average distance is $\bar{q} = D + \frac{1}{3}\rho^2/D$, from the outside point $(0, 0, D)$ to points on the shell of radius ρ. Then integrate $\iiint q \, dV$ and divide by $4\pi R^3/3$ to find $\bar{q}$ for the solid sphere.

31 In Figure 14.18b, it is not true that the areas on the surface are exactly proportional to q^2 and Q^2. Why not? What happens to the second proof in Example 8?

32 For two solid spheres attracting each other (sun and planet), can we concentrate *both* spheres into point masses at their centers?

*33 Compute $\iiint \cos\alpha \, dV/q^3$ to find the force of gravity at $(0, 0, D)$ from a *cylinder* $x^2 + y^2 \leqslant a^2$, $0 \leqslant z \leqslant h$. Show from a figure why $q^2 = r^2 + (D - z)^2$ and $\cos\alpha = (D - z)/q$.

34 A *linear* change of variables has $x = au + bv + cw$, $y = du + ev + fw$, and $z = gu + hv + iw$. Write down the six terms in the determinant J. Three terms have minus signs.

35 A pure stretching has $x = au$, $y = bv$, and $z = cw$. Find the 3 by 3 matrix and its determinant J. What is special about the xyz box in this case?

36 (a) The matrix in Example 9 has three columns. Find the lengths of those three vectors (sum of squares, then square root). Compare with the edges of the box in Figure 14.17.

(b) Take the dot product of every column in J with every other column. Zero dot products mean right angles in the box. So J is the product of the column lengths.

37 Find the stretching factor J for cylindrical coordinates from the matrix of first derivatives.

38 Follow Problem 36 for cylindrical coordinates—find the length of each column in J and compare with the box in Figure 14.15.

39 Find the moment of inertia around the z axis of a spherical shell (radius ρ, density 1). The distance from the axis to a point on the shell is $r =$ ________. Substitute for r to find

$$I(\rho) = \int_0^{2\pi} \int_0^{\pi} r^2 \rho^2 \sin\phi \, d\phi \, d\theta.$$

Divide by mr^2 (which is $4\pi\rho^4$) to compute the number J for a hollow ball in the rolling experiment of Section 8.5.

40 The moment of inertia of a solid sphere (radius R, density 1) adds up the hollow spheres of Problem 39: $I = \int_0^R I(\rho) d\rho =$ ________. Divide by mR^2 (which is $\frac{4}{3}\pi R^5$) to find J in the rolling experiment. A solid ball rolls faster than a hollow ball because ________.

41 Inside the Earth, the force of gravity is proportional to the distance ρ from the center. Reason: The inner ball of radius ρ has mass proportional to ________ (assume constant density). The force is proportional to that mass divided by ________. The rest of the Earth (sphere with hole) exerts no force because ________.

42 Dig a tunnel through the center to Australia. Drop a ball in the tunnel at $y = R$; Australia is $y = -R$. The force of gravity is $-cy$ by Problem 41. Newton's law is $my'' = -cy$. What does the ball do when it reaches Australia?