CHAPTER 15

Vector Calculus

Chapter 14 introduced double and triple integrals. We went from $\int dx$ to $\iint dx\,dy$ and $\iiint dx\,dy\,dz$. All those integrals add up small pieces, and the limit gives area or volume or mass. What could be more natural than that? I regret to say, after the success of those multiple integrals, that something is missing. It is even more regrettable that we didn't notice it. The missing piece is nothing less than the Fundamental Theorem of Calculus.

The double integral $\iint dx\,dy$ equals the area. To compute it, we did not use an antiderivative of 1. At least not consciously. The method was almost trial and error, and the hard part was to find the limits of integration. This chapter goes deeper, to show how the step from a double integral to a single integral is really a new form of the Fundamental Theorem—when it is done right.

Two new ideas are needed early, one pleasant and one not. You will like ***vector fields***. You may not think so highly of ***line integrals***. Those are ordinary single integrals like $\int v(x)dx$, but they go along curves instead of straight lines. The nice step dx becomes the confusing step ds. Where $\int dx$ equals the length of the interval, $\int ds$ is the length of the curve. The point is that regions are enclosed by curves, and we have to integrate along them. The Fundamental Theorem in its two-dimensional form (Green's Theorem) connects ***a double integral over the region*** to ***a single integral along its boundary curve***.

The great applications are in science and engineering, where vector fields are so natural. But there are changes in the language. Instead of an antiderivative, we speak about a ***potential function***. Instead of the derivative, we take the "*divergence*" and "*curl*." Instead of area, we compute ***flux*** and ***circulation*** and ***work***. Examples come first.

15.1 Vector Fields

For an ordinary scalar function, the input is a number x and the output is a number $f(x)$. For a vector field (or vector function), the input is a point (x, y) and the output is a two-dimensional vector $\mathbf{F}(x, y)$. There is a "field" of vectors, one at every point.

In three dimensions the input point is (x, y, z) and the output vector $\mathbf{F}$ has three components.

DEFINITION Let R be a region in the xy plane. A ***vector field*** $\mathbf{F}$ assigns to every point (x, y) in R a vector $\mathbf{F}(x, y)$ with two components:

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}. \tag{1}$$

This plane vector field involves *two functions of two variables*. They are the components M and N, which vary from point to point. A vector has fixed components, a vector field has varying components.

A three-dimensional vector field has components $M(x, y, z)$ and $N(x, y, z)$ and $P(x, y, z)$. Then the vectors are $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$.

EXAMPLE 1 The ***position vector*** at (x, y) is $\mathbf{R} = x\mathbf{i} + y\mathbf{j}$. Its components are $M = x$ and $N = y$. The vectors grow larger as we leave the origin (Figure 15.1a). Their direction is outward and their length is $|\mathbf{R}| = \sqrt{x^2 + y^2} = r$. The vector $\mathbf{R}$ is boldface, the number r is lightface.

EXAMPLE 2 The vector field $\mathbf{R}/r$ consists of ***unit vectors*** $\mathbf{u}_r$, pointing outward. We divide $\mathbf{R} = x\mathbf{i} + y\mathbf{j}$ by its length, at every point except the origin. The components of $\mathbf{R}/r$ are $M = x/r$ and $N = y/r$. Figure 15.1 shows a third field $\mathbf{R}/r^2$, whose length is $1/r$.

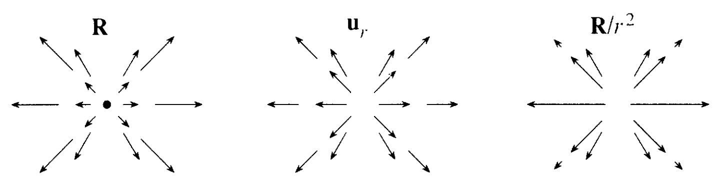

Fig. 15.1 The vector fields $\mathbf{R}$ and $\mathbf{R}/r$ and $\mathbf{R}/r^2$ are radial. Lengths r and 1 and $1/r$.

EXAMPLE 3 The ***spin field*** or rotation field or turning field goes around the origin instead of away from it. The field is $\mathbf{S}$. Its components are $M = -y$ and $N = x$:

$$\mathbf{S} = -y\mathbf{i} + x\mathbf{j} \text{ also has length } |\mathbf{S}| = \sqrt{(-y)^2 + x^2} = r. \tag{2}$$

$\mathbf{S}$ is perpendicular to $\mathbf{R}$—their dot product is zero: $\mathbf{S} \cdot \mathbf{R} = (-y)(x) + (x)(y) = 0$. The spin fields $\mathbf{S}/r$ and $\mathbf{S}/r^2$ have lengths 1 and $1/r$:

$$\frac{\mathbf{S}}{r} = -\frac{y}{r}\mathbf{i} + \frac{x}{r}\mathbf{j} \text{ has } \left|\frac{\mathbf{S}}{r}\right| = 1 \qquad \frac{\mathbf{S}}{r^2} = -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} \text{ has } \left|\frac{\mathbf{S}}{r^2}\right| = \frac{1}{r}.$$

The unit vector $\mathbf{S}/r$ is $\mathbf{u}_\theta$. Notice the blank at $(0, 0)$, where this field is not defined.

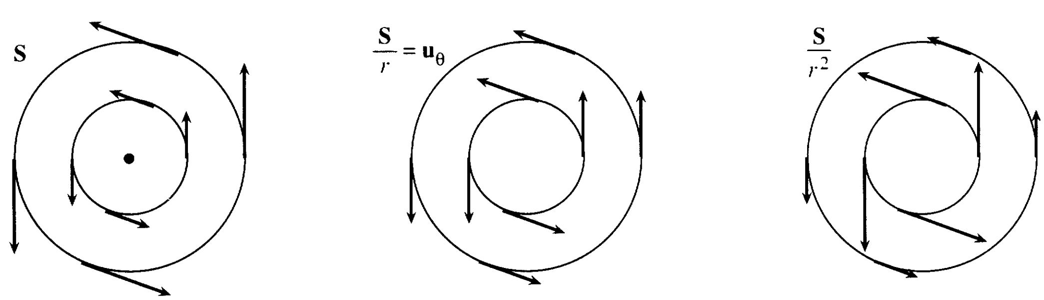

Fig. 15.2 The spin fields $\mathbf{S}$ and $\mathbf{S}/r$ and $\mathbf{S}/r^2$ go around the origin. Lengths r and 1 and $1/r$.

EXAMPLE 4 A ***gradient field*** starts with an ordinary function $f(x, y)$. The components M and N are the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$. Then the field $\mathbf{F}$ is the gradient of f:

$$\mathbf{F} = \operatorname{grad} f = \nabla f = \partial f/\partial x\, \mathbf{i} + \partial f/\partial y\, \mathbf{j}. \tag{3}$$

This vector field grad f ***is everywhere perpendicular to the level curves*** $f(x, y) = c$. The length $|\operatorname{grad} f|$ tells how fast f is changing (in the direction it changes fastest). Invent a function like $f = x^2 y$, and you immediately have its gradient field $\mathbf{F} = 2xy\mathbf{i} + x^2\mathbf{j}$. To repeat, M is $\partial f/\partial x$ and N is $\partial f/\partial y$.

For every vector field you should ask two questions: ***Is it a gradient field***? ***If so, what is*** f? Here are answers for the radial fields and spin fields:

> **15A** The radial fields $\mathbf{R}$ and $\mathbf{R}/r$ and $\mathbf{R}/r^2$ are all gradient fields.
> The spin fields $\mathbf{S}$ and $\mathbf{S}/r$ are not gradients of any $f(x, y)$.
> The spin field $\mathbf{S}/r^2$ is the gradient of the polar angle $\theta = \tan^{-1}(y/x)$.

The derivatives of $f = \frac{1}{2}(x^2 + y^2)$ are x and y. Thus $\mathbf{R}$ is a gradient field. The gradient of $f = r$ is the unit vector $\mathbf{R}/r$ pointing outwards. Both fields are perpendicular to circles around the origin. Those are the level curves of $f = \frac{1}{2}r^2$ and $f = r$.

Question Is every $\mathbf{R}/r^n$ a gradient field?
Answer *Yes.* But among the spin fields, the only gradient is $\mathbf{S}/r^2$.

A major goal of this chapter is to recognize gradient fields by a simple test. The rejection of $\mathbf{S}$ and $\mathbf{S}/r$ will be interesting. For some reason $-y\mathbf{i} + x\mathbf{j}$ is rejected and $y\mathbf{i} + x\mathbf{j}$ is accepted. (It is the gradient of _______.) The acceptance of $\mathbf{S}/r^2$ as the gradient of $f = \theta$ contains a surprise at the origin (Section 15.3).

Gradient fields are called ***conservative***. The function f is the ***potential function***. These words, and the next examples, come from physics and engineering.

EXAMPLE 5 The ***velocity field*** is $\mathbf{V}$ and the ***flow field*** is $\rho\mathbf{V}$.

Suppose fluid moves steadily down a pipe. Or a river flows smoothly (no waterfall). Or the air circulates in a fixed pattern. The velocity can be different at different points, but there is no change with time. The velocity vector $\mathbf{V}$ gives the *direction of flow* and *speed of flow* at every point.

In reality the velocity field is $\mathbf{V}(x, y, z)$, with three components M, N, P. Those are the velocities v_1, v_2, v_3 in the x, y, z directions. The speed $|\mathbf{V}|$ is the length: $|\mathbf{V}|^2 = v_1^2 + v_2^2 + v_3^2$. In a "plane flow" the $\mathbf{k}$ component is zero, and the velocity field is $v_1\mathbf{i} + v_2\mathbf{j} = M\mathbf{i} + N\mathbf{j}$.

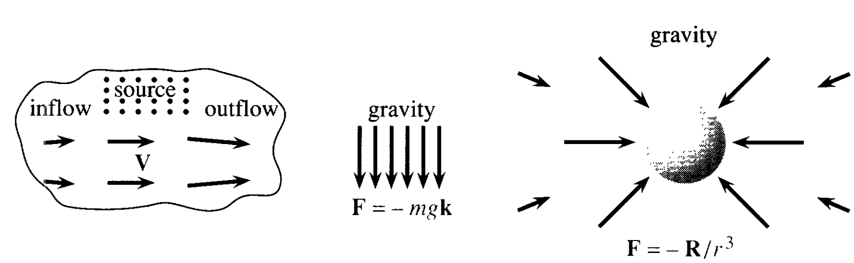

Fig. 15.3 A steady velocity field $\mathbf{V}$ and two force fields $\mathbf{F}$.

For a compact disc or a turning wheel, **V** is a spin field ($\mathbf{V} = \omega\mathbf{S}$, ω = angular velocity). A tornado might be closer to $\mathbf{V} = \mathbf{S}/r^2$ (except for a dead spot at the center). An explosion could have $\mathbf{V} = \mathbf{R}/r^2$. A quieter example is flow in and out of a lake with steady rain as a source term.

The *flow field* $\rho\mathbf{V}$ is the density ρ times the velocity field. While **V** gives the rate of movement, $\rho\mathbf{V}$ gives the ***rate of movement of mass***. A greater density means a greater rate $|\rho\mathbf{V}|$ of "mass transport." It is like the number of passengers on a bus times the speed of the bus.

EXAMPLE 6 Force fields from gravity: **F** is downward in the classroom, **F** is radial in space.

When gravity pulls downward, it has only one nonzero component: $\mathbf{F} = -mg\mathbf{k}$. This assumes that vectors to the center of the Earth are parallel—almost true in a classroom. Then **F** is the gradient of $-mgz$ (note $\partial f/\partial z = -mg$).
In physics the usual potential is not $-mgz$ ***but*** $+mgz$. The force field is *minus* grad f also in electrical engineering. Electrons flow from high potential to low potential. The mathematics is the same, but the sign is reversed.

In space, the force is radial inwards: $\mathbf{F} = -mMG\mathbf{R}/r^3$. Its magnitude is proportional to $1/r^2$ (Newton's inverse square law). The masses are m and M, and the gravitational constant is $G = 6.672 \times 10^{-11}$—with distance in meters, mass in kilograms, and time in seconds. The dimensions of G are (force)(distance)^2/(mass)^2. This is different from the acceleration $g = 9.8\text{m/sec}^2$, which already accounts for the mass and radius of the Earth.

Like all radial fields, ***gravity is a gradient field.*** It comes from a potential f:

$$f = \frac{mMG}{r} \text{ and } \frac{\partial f}{\partial x} = -\frac{mMGx}{r^3} \text{ and } \frac{\partial f}{\partial y} = -\frac{mMGy}{r^3} \text{ and } \frac{\partial f}{\partial z} = -\frac{mMGz}{r^3}. \quad (4)$$

EXAMPLE 7 (a short example) Current in a wire produces a ***magnetic field*** **B**. It is the spin field $\mathbf{S}/r^2$ around the wire, times the strength of the current.

STREAMLINES AND LINES OF FORCE

Drawing a vector field is not always easy. Even the spin field looks messy when the vectors are too long (they go in circles and fall across each other). *The circles give a clearer picture than the vectors.* In any field, the vectors are tangent to "*field lines*"—which in the spin case are circles.

DEFINITION C is a ***field line*** or ***integral curve*** if the vectors $\mathbf{F}(x, y)$ are tangent to C. The slope dy/dx of the curve C equals the slope N/M of the vector $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$:

$$\frac{dy}{dx} = \frac{N(x, y)}{M(x, y)} \quad \left(= -\frac{x}{y} \text{ for the spin field}\right). \quad (6)$$

We are still drawing the field of vectors, but now they are infinitesimally short. They are connected into curves! What is lost is their length, because **S** and $\mathbf{S}/r$ and $\mathbf{S}/r^2$ all have the same field lines (circles). For the position field **R** and gravity field $\mathbf{R}/r^3$, the field lines are rays from the origin. In this case the "curves" are actually straight.

EXAMPLE 8 Show that the field lines for the velocity field $\mathbf{V} = y\mathbf{i} + x\mathbf{j}$ are hyperbolas.

$$\frac{dy}{dx} = \frac{N}{M} = \frac{x}{y} \quad \Rightarrow \quad y\,dy = x\,dx \quad \Rightarrow \quad \tfrac{1}{2}y^2 - \tfrac{1}{2}x^2 = \text{constant}.$$

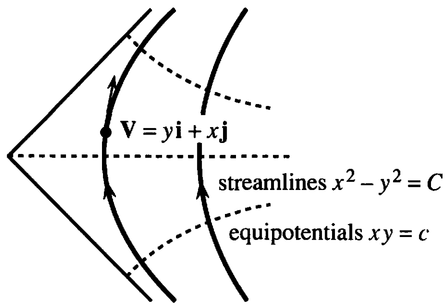

Fig. 15.4 Velocity fields are tangent to streamlines. Gradient fields also have equipotentials.

At every point these hyperbolas line up with the velocity **V**. ***Each particle of fluid travels on a field line.*** In fluid flow those hyperbolas are called ***streamlines***. Drop a leaf into a river, and it follows a streamline. Figure 15.4 shows the streamlines for a river going around a bend.

Don't forget the essential question about each vector field. Is it a gradient field? For $\mathbf{V} = y\mathbf{i} + x\mathbf{j}$ the answer is *yes*, and the potential is $f = xy$:

$$\text{the gradient of } xy \text{ is } (\partial f/\partial x)\mathbf{i} + (\partial f/\partial y)\mathbf{j} = y\mathbf{i} + x\mathbf{j}. \tag{7}$$

When there is a potential, it has level curves. They connect points of equal potential, so the curves $f(x, y) = c$ are called ***equipotentials***. Here they are the curves $xy = c$— also hyperbolas. Since gradients are perpendicular to level curves, ***the streamlines are perpendicular to the equipotentials***. Figure 15.4 is sliced one way by streamlines and the other way by equipotentials.

A gradient field $\mathbf{F} = \partial f/\partial x\, \mathbf{i} + \partial f/\partial y\, \mathbf{j}$ is tangent to the field lines (streamlines) and perpendicular to the equipotentials (level curves of f).

In the gradient direction f changes fastest. In the level direction f doesn't change at all. The chain rule along $f(x, y) = c$ proves these directions to be perpendicular:

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0 \quad \text{or} \quad (\text{grad}\, f)\cdot(\text{tangent to level curve}) = 0.$$

EXAMPLE 9 The streamlines of $\mathbf{S}/r^2$ are circles around (0, 0). The equipotentials are rays $\theta = c$. Add rays to Figure 15.2 for the gradient field $\mathbf{S}/r^2$.

For the gravity field those are reversed. A body is pulled in along the field lines (rays). The equipotentials are the circles where $f = 1/r$ is constant. The plane is crisscrossed by "orthogonal trajectories"—curves that meet everywhere at right angles.

If you bring a magnet near a pile of iron filings, a little shake will display the field lines. In a force field, they are "lines of force." *Here are the other new words.*

Vector field $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$	Plane field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$
Radial field: multiple of $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$	Spin field: multiple of $\mathbf{S} = -y\mathbf{i} + x\mathbf{j}$
Gradient field = conservative field: $M = \partial f/\partial x$, $N = \partial f/\partial y$, $P = \partial f/\partial z$	
Potential $f(x, y)$ (not a vector)	Equipotential curves $f(x, y) = c$
Streamline = field line = integral curve: a curve that has $\mathbf{F}(x, y)$ as its tangent vectors.	

15.1 EXERCISES

Read-through questions

A vector field assigns a __a__ to each point (x, y) or (x, y, z). In two dimensions $\mathbf{F}(x,y) =$ __b__ $\mathbf{i} +$ __c__ $\mathbf{j}$. An example is the position field $\mathbf{R} =$ __d__. Its magnitude is $|\mathbf{R}| =$ __e__ and its direction is __f__. It is the gradient field for $f =$ __g__. The level curves are __h__, and they are __i__ to the vectors $\mathbf{R}$.

Reversing this picture, the spin field is $\mathbf{S} =$ __j__. Its magnitude is $|\mathbf{S}| =$ __k__ and its direction is __l__. It is not a gradient field, because no function has $\partial f/\partial x =$ __m__ and $\partial f/\partial y =$ __n__. $\mathbf{S}$ is the velocity field for flow going __o__. The streamlines or __p__ lines or integral __q__ are __r__. The flow field $\rho\mathbf{V}$ gives the rate at which __s__ is moved by the flow.

A gravity field from the origin is proportional to $\mathbf{F} =$ __t__ which has $|\mathbf{F}| =$ __u__. This is Newton's __v__ square law. It is a gradient field, with potential $f =$ __w__. The equipotential curves $f(x, y) = c$ are __x__. They are __y__ to the field lines which are __z__. This illustrates that the __A__ of a function $f(x, y)$ is __B__ to its level curves.

The velocity field $y\mathbf{i} + x\mathbf{j}$ is the gradient of $f =$ __C__. Its streamlines are __D__. The slope dy/dx of a streamline equals the ratio __E__ of velocity components. The field is __F__ to the streamlines. Drop a leaf onto the flow, and it goes along __G__.

Find a potential $f(x, y)$ for the gradient fields 1–8. Draw the streamlines perpendicular to the equipotentials $f(x, y) = c$.

1 $\mathbf{F} = \mathbf{i} + 2\mathbf{j}$ (constant field)

2 $\mathbf{F} = x\mathbf{i} + \mathbf{j}$

3 $\mathbf{F} = \cos(x+y)\mathbf{i} + \cos(x+y)\mathbf{j}$

4 $\mathbf{F} = (1/y)\mathbf{i} - (x/y^2)\mathbf{j}$

5 $\mathbf{F} = (2x\mathbf{i} + 2y\mathbf{j})/(x^2+y^2)$

6 $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j}$

7 $\mathbf{F} = xy\mathbf{i} +$ ____ $\mathbf{j}$

8 $\mathbf{F} = \sqrt{y}\,\mathbf{i} +$ ____ $\mathbf{j}$

9 Draw the shear field $\mathbf{F} = x\mathbf{j}$. Check that it is not a gradient field: If $\partial f/\partial x = 0$ then $\partial f/\partial y = x$ is impossible. What are the streamlines (field lines) in the direction of $\mathbf{F}$?

10 Find all functions that satisfy $\partial f/\partial x = -y$ and show that none of them satisfy $\partial f/\partial y = x$. Then the spin field $\mathbf{S} = -y\mathbf{i} + x\mathbf{j}$ is not a gradient field.

Compute $\partial f/\partial x$ and $\partial f/\partial y$ in 11–18. Draw the gradient field $\mathbf{F} = \mathbf{grad}\, f$ and the equipotentials $f(x, y) = c$:

11 $f = 3x + y$

12 $f = x - 3y$

13 $f = x + y^2$

14 $f = (x-1)^2 + y^2$

15 $f = x^2 - y^2$

16 $f = e^x \cos y$

17 $f = e^{x-y}$

18 $f = y/x$

Find equations for the streamlines in 19–24 by solving $dy/dx = N/M$ (including a constant C). Draw the streamlines.

19 $\mathbf{F} = \mathbf{i} - \mathbf{j}$

20 $\mathbf{F} = \mathbf{i} + x\mathbf{j}$

21 $\mathbf{F} = \mathbf{S}$ (spin field)

22 $\mathbf{F} = \mathbf{S}/r$ (spin field)

23 $\mathbf{F} = \text{grad}\,(x/y)$

24 $\mathbf{F} = \text{grad}\,(2x + y)$.

25 The Earth's gravity field is radial, but in a room the field lines seem to go straight down into the floor. This is because nearby field lines always look ________.

26 A line of charges produces the electrostatic force field $\mathbf{F} = \mathbf{R}/r^2 = (x\mathbf{i} + y\mathbf{j})/(x^2+y^2)$. Find the potential $f(x, y)$. ($\mathbf{F}$ is also the gravity field for a line of masses.)

In 27–32 write down the vector fields $M\mathbf{i} + N\mathbf{j}$.

27 $\mathbf{F}$ points radially away from the origin with magnitude 5.

28 The velocity is perpendicular to the curves $x^3 + y^3 = c$ and the speed is 1.

29 The gravitational force $\mathbf{F}$ comes from two unit masses at $(0, 0)$ and $(1, 0)$.

30 The streamlines are in the 45° direction and the speed is 4.

31 The streamlines are circles clockwise around the origin and the speed is 1.

32 The equipotentials are the parabolas $y = x^2 + c$ and $\mathbf{F}$ is a gradient field.

33 Show directly that the hyperbolas $xy = 2$ and $x^2 - y^2 = 3$ are perpendicular at the point $(2, 1)$, by computing both slopes dy/dx and multiplying to get -1.

34 The derivative of $f(x, y) = c$ is $f_x + f_y(dy/dx) = 0$. Show that the slope of this level curve is $dy/dx = -M/N$. It is perpendicular to streamlines because $(-M/N)(N/M) =$ ________.

35 The x and y derivatives of $f(r)$ are $\partial f/\partial x =$ ________ and $\partial f/\partial y =$ ________ by the chain rule. (Test $f = r^2$.) The equipotentials are ________.

36 $\mathbf{F} = (ax + by)\mathbf{i} + (bx + cy)\mathbf{j}$ is a gradient field. Find the potential f and describe the equipotentials.

37 *True or false*:

1. The constant field $\mathbf{i} + 2\mathbf{k}$ is a gradient field.
2. For non-gradient fields, equipotentials meet streamlines at non-right angles.
3. In three dimensions the equipotentials are surfaces instead of curves.
4. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ points outward from $(0, 0, 0)$—a radial field.

38 Create and draw f and $\mathbf{F}$ and your own equipotentials and streamlines.

39 How can different vector fields have the same streamlines? Can they have the same equipotentials? Can they have the same f?

40 Draw arrows at six or eight points to show the direction and magnitude of each field:

(a) $\mathbf{R}+\mathbf{S}$ (b) $\mathbf{R}/r-\mathbf{S}/r$ (c) $x^2\mathbf{i}+x^2\mathbf{j}$ (d) $y\mathbf{i}$.

15.2 Line Integrals

A line integral is ***an integral along a curve***. It can equal an area, but that is a special case and not typical. Instead of area, here are two important line integrals in physics and engineering:

$$\textit{Work along a curve} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds \qquad \textit{Flow across a curve} = \int_C \mathbf{F}\cdot\mathbf{n}\,ds.$$

In the first integral, $\mathbf{F}$ is a ***force field.*** In the second integral, $\mathbf{F}$ is a ***flow field.*** Work is done in the direction of movement, so we integrate $\mathbf{F}\cdot\mathbf{T}$. Flow is measured through the curve C, so we integrate $\mathbf{F}\cdot\mathbf{n}$. Here $\mathbf{T}$ is the unit *tangent* vector, and $\mathbf{F}\cdot\mathbf{T}$ is the force component along the curve. Similarly $\mathbf{n}$ is the unit *normal* vector, at right angles with $\mathbf{T}$. Then $\mathbf{F}\cdot\mathbf{n}$ is the component of flow perpendicular to the curve.

We will write those integrals in several forms. They may never be as comfortable as $\int y(x)\,dx$, but eventually we get them under control. I mention these applications early, so you can see where we are going. This section concentrates on work, and flow comes later. (It is also called ***flux***—the Latin word for flow.) You recognize ds as the step along the curve, corresponding to dx on the x axis. Where $\int dx$ gives the length of an interval (it equals $b-a$), $\int ds$ is the length of the curve.

EXAMPLE 1 Flight from Atlanta to Los Angeles on a straight line and a semicircle.

According to Delta Airlines, the distance straight west is 2000 miles. Atlanta is at (1000, 0) and Los Angeles is at (−1000, 0), with the origin halfway between. The semicircle route C has radius 1000. *This is not a great circle route.* It is more of a "flat circle," which goes north past Chicago. No plane could fly it (it probably goes into space).

The equation for the semicircle is $x^2+y^2=1000^2$. Parametrically this path is $x = 1000\cos t$, $y=1000\sin t$. For a line integral the parameter is better. The plane leaves Atlanta at $t=0$ and reaches L.A. at $t=\pi$, more than three hours later. On the straight 2000-mile path, Delta could almost do it. Around the semicircle C, the distance is 1000π miles and the speed has to be 1000 miles per hour. Remember that speed is distance ds divided by time dt:

$$ds/dt = \sqrt{(dx/dt)^2+(dy/dt)^2} = 1000\sqrt{(-\sin t)^2+(\cos t)^2} = 1000. \qquad (1)$$

The tangent vector to C is proportional to $(dx/dt, dy/dt) = (-1000\sin t, 1000\cos t)$. But $\mathbf{T}$ is a unit vector, so we divide by 1000—which is the speed.

Suppose the wind blows due east with force $\mathbf{F}=M\mathbf{i}$. The components are M and zero. For $M=$ constant, compute the dot product $\mathbf{F}\cdot\mathbf{T}$ and the work $-2000\,M$:

$$\mathbf{F}\cdot\mathbf{T} = M\mathbf{i}\cdot(-\sin t\,\mathbf{i}+\cos t\,\mathbf{j}) = M(-\sin t)+0(\cos t) = -M\sin t$$

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_{t=0}^{\pi}(-M\sin t)\left(\frac{ds}{dt}dt\right) = \int_0^{\pi} -1000M\sin t\,dt = -2000M.$$

Work is force times distance moved. It is negative, because the wind acts *against* the movement. You may point out that the work could have been found more simply—go 2000 miles and multiply by $-M$. I would object that ***this straight route is a different path***. But you claim that ***the path doesn't matter***—the work of the wind is $-2000M$ on every path. I concede that this time you are right (but not always).

Most line integrals depend on the path. Those that don't are crucially important. For a *gradient field*, we only need to know the starting point P and the finish Q.

15B When $\mathbf{F}$ is the gradient of a potential function $f(x, y)$, the work $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ depends only on the endpoints P and Q. ***The work is the change in*** f:

$$\text{if} \quad \mathbf{F} = \partial f/\partial x\,\mathbf{i} + \partial f/\partial y\,\mathbf{j} \quad \text{then} \quad \int_C \mathbf{F}\cdot\mathbf{T}\,ds = f(Q) - f(P). \tag{2}$$

When $\mathbf{F} = M\mathbf{i}$, its components M and zero are the partial derivatives of $f = Mx$. To compute the line integral, just evaluate f at the endpoints. Atlanta has $x = 1000$, Los Angeles has $x = -1000$, and ***the potential function*** $f = Mx$ ***is like an antiderivative***:

$$\text{work} = f(Q) - f(P) = M(-1000) - M(1000) = -2000M. \tag{3}$$

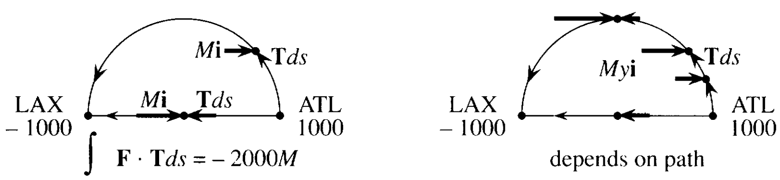

Fig. 15.5 Force $M\mathbf{i}$, work $-2000M$ on all paths. Force $My\mathbf{i}$, no work on straight path.

May I give a rough explanation of the work integral $\int \mathbf{F}\cdot\mathbf{T}\,ds$? It becomes clearer when the small movement $\mathbf{T}\,ds$ is written as $dx\,\mathbf{i} + dy\,\mathbf{j}$. The work is the dot product with $\mathbf{F}$:

$$\mathbf{F}\cdot\mathbf{T}\,ds = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)\cdot(dx\,\mathbf{i} + dy\,\mathbf{j}) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df. \tag{4}$$

The infinitesimal work is df. The total work is $\int df = f(Q) - f(P)$. This is the ***Fundamental Theorem for a line integral***. Only one warning: When $\mathbf{F}$ is not the gradient of any f (Example 2), the Theorem does not apply.

EXAMPLE 2 Fly these paths against the non-constant force field $\mathbf{F} = My\mathbf{i}$. Compute the work.

There is no force on the straight path where $y = 0$. Along the x axis the wind does no work. But the semicircle goes up where $y = 1000 \sin t$ and the wind is strong:

$$\mathbf{F}\cdot\mathbf{T} = (My\mathbf{i})\cdot(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j}) = -My \sin t = -1000M \sin^2 t$$

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^\pi (-1000M \sin^2 t)\frac{ds}{dt}dt = \int_0^\pi -10^6 M \sin^2 t\,dt = -\frac{\pi}{2}10^6 M.$$

This work is enormous (and unrealistic). But the calculations make an important point—everything is converted to the parameter t. The second point is that $\mathbf{F} = My\mathbf{i}$ is not a gradient field. ***First reason***: The work was zero on the straight path and nonzero on the semicircle. ***Second reason***: No function has $\partial f/\partial x = My$ and $\partial f/\partial y = 0$. (The first makes f depend on y and the second forbids it. This $\mathbf{F}$ is called a *shear force*.) Without a potential we cannot substitute P and Q—and the work depends on the path.

THE DEFINITION OF LINE INTEGRALS

We go back to the start, to define $\int \mathbf{F}\cdot\mathbf{T}\,ds$. We can think of $\mathbf{F}\cdot\mathbf{T}$ as a function $g(x, y)$ along the path, and define its integral as a limit of sums:

$$\int_C g(x, y)\,ds = \text{limit of} \sum_{i=1}^{N} g(x_i, y_i)\Delta s_i \quad \text{as} \quad (\Delta s)_{\max} \to 0. \tag{5}$$

The points (x_i, y_i) lie on the curve C. The last point Q is (x_N, y_N); the first point P is (x_0, y_0). The step Δs_i is the distance to (x_i, y_i) from the previous point. As the steps get small ($\Delta s \to 0$) the straight pieces follow the curve. Exactly as in Section 8.2, the special case $g = 1$ gives the arc length. As long as $g(x, y)$ is piecewise continuous (jumps allowed) and the path is piecewise smooth (corners allowed), the limit exists and defines the line integral.

When g is the density of a wire, the line integral is the total mass. When g is $\mathbf{F}\cdot\mathbf{T}$, the integral is the work. But nobody does the calculation by formula (5). We now introduce a parameter t—which could be the time, or the arc length s, or the distance x along the base.

The differential ds ***becomes*** $(ds/dt)dt$. Everything changes over to t:

$$\int g(x, y)ds = \int_{t=a}^{t=b} g(x(t), y(t))\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt. \tag{6}$$

The curve starts when $t = a$, runs through the points $(x(t), y(t))$, and ends when $t = b$. The square root in the integral is the speed ds/dt. In three dimensions the points on C are $(x(t), y(t), z(t))$ and $(dz/dt)^2$ is in the square root.

EXAMPLE 3 The points on a coil spring are $(x, y, z) = (\cos t, \sin t, t)$. Find the mass of two complete turns (from $t = 0$ to $t = 4\pi$) if the density is $\rho = 4$.

Solution The key is $(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 = \sin^2 t + \cos^2 t + 1 = 2$. Thus $ds/dt = \sqrt{2}$. To find the mass, integrate the mass per unit length which is $g = \rho = 4$:

$$\text{mass} = \int_0^{4\pi} \rho\frac{ds}{dt}dt = \int_0^{4\pi} 4\sqrt{2}\,dt = 16\sqrt{2}\pi.$$

That is a line integral in three-dimensional space. It shows how to introduce t. But it misses the main point of this section, because it contains no vector field $\mathbf{F}$. This section is about *work*, not just mass.

DIFFERENT FORMS OF THE WORK INTEGRAL

The work integral $\int \mathbf{F}\cdot\mathbf{T}\,ds$ can be written in a better way. The force is $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$. A small step along the curve is $dx\,\mathbf{i} + dy\,\mathbf{j}$. Work is force times distance, but it is only the force component *along the path* that counts. The dot product $\mathbf{F}\cdot\mathbf{T}\,ds$ finds that component automatically.

> **15C** The vector to a point on C is $\mathbf{R} = x\mathbf{i} + y\mathbf{j}$. Then $d\mathbf{R} = \mathbf{T}\,ds = dx\,\mathbf{i} + dy\,\mathbf{j}$:
>
> $$\textit{work} = \int_C \mathbf{F} \cdot d\mathbf{R} = \int_C M\,dx + N\,dy. \tag{7}$$
>
> Along a space curve the work is $\int \mathbf{F} \cdot \mathbf{T}\,ds = \int \mathbf{F} \cdot d\mathbf{R} = \int M\,dx + N\,dy + P\,dz$.

The product $M\,dx$ is (force in x direction)(movement in x direction). This is zero if either factor is zero. When the only force is gravity, pushing a piano takes no work. It is friction that hurts. Carrying the piano up the stairs brings in Pdz, and the total work is the piano weight P times the change in z.

To connect the new $\int \mathbf{F} \cdot d\mathbf{R}$ with the old $\int \mathbf{F} \cdot \mathbf{T}\,ds$, remember the tangent vector $\mathbf{T}$. It is $d\mathbf{R}/ds$. ***Therefore*** $\mathbf{T}\,ds$ ***is*** $d\mathbf{R}$. The best for computations is $d\mathbf{R}$, because the unit vector $\mathbf{T}$ has a division by $ds/dt = \sqrt{(dx/dt)^2 + (dy/dt)^2}$. Later we multiply by this square root, in converting ds to $(ds/dt)dt$. It makes no sense to compute the square root, divide by it, and then multiply by it. That is avoided in the improved form $\int M\,dx + N\,dy$.

EXAMPLE 4 Vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$, path from $(1, 0)$ to $(0, 1)$: Find the work.

Note 1 This $\mathbf{F}$ is the spin field $\mathbf{S}$. It goes *around* the origin, while $\mathbf{R} = x\mathbf{i} + y\mathbf{j}$ goes outward. Their dot product is $\mathbf{F} \cdot \mathbf{R} = -yx + xy = 0$. This does not mean that $\mathbf{F} \cdot d\mathbf{R} = 0$. The force is perpendicular to $\mathbf{R}$, but not to the *change* in $\mathbf{R}$. The work to move from $(1, 0)$ to $(0, 1)$, x axis to y axis, is not zero.

Note 2 We have not described the path C. That must be done. The spin field is not a gradient field, and the work along a straight line does not equal the work on a quarter-circle:

$$\textit{straight line } x = 1 - t,\ y = t \qquad \textit{quarter-circle } x = \cos t,\ y = \sin t.$$

Calculation of work Change $\mathbf{F} \cdot d\mathbf{R} = M\,dx + N\,dy$ to the parameter t:

$$\text{Straight line: } \int -y\,dx + x\,dy = \int_0^1 -t(-dt) + (1 - t)dt = 1$$

$$\text{Quarter-circle: } \int -y\,dx + x\,dy = \int_0^{\pi/2} -\sin t(-\sin t\,dt) + \cos t(\cos t\,dt) = \frac{\pi}{2}.$$

General method The path is given by $x(t)$ and $y(t)$. Substitute those into $M(x, y)$ and $N(x, y)$—then $\mathbf{F}$ is a function of t. Also find dx/dt and dy/dt. Integrate $M\,dx/dt + N\,dy/dt$ from the starting time t to the finish.

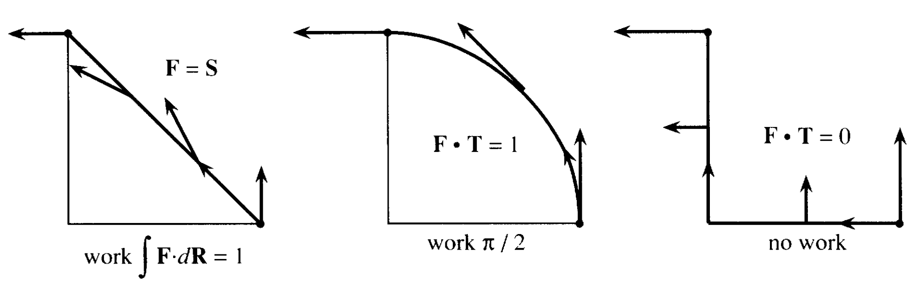

Fig. 15.6 Three paths for $\int \mathbf{F} \cdot d\mathbf{R} = \int -y\,dx + x\,dy = 1,\ \pi/2,\ 0$.

For practice, take the path down the x axis to the origin ($x=1-t$, $y=0$). Then go up the y axis ($x=0$, $y=t-1$). The starting time at (1, 0) is $t=0$. The turning time at the origin is $t=1$. The finishing time at (0, 1) is $t=2$. The integral has two parts because this new path has two parts:

$$\text{Bent path:}\quad \int -y\,dx + x\,dy = 0 + 0 \quad (y=0 \text{ on one part, then } x=0).$$

Note 3 The answer depended on the path, for this spin field $\mathbf{F}=\mathbf{S}$. The answer did *not* depend on the choice of parameter. If we follow the same path at a different speed, the work is the same. We can choose another parameter τ, since $(ds/dt)dt$ and $(ds/d\tau)d\tau$ both equal ds. Traveling twice as fast on the straight path ($x=1-2\tau$, $y=2\tau$) we finish at $\tau=\frac{1}{2}$ instead of $t=1$. The work is still 1:

$$\int -y\,dx + x\,dy = \int_0^{1/2} (-2\tau)(-2d\tau) + (1-2\tau)(2d\tau) = \int_0^{1/2} 2\,d\tau = 1.$$

CONSERVATION OF TOTAL ENERGY (KINETIC + POTENTIAL)

When a force field does work on a mass m, it normally gives that mass a new velocity. Newton's Law is $\mathbf{F}=m\mathbf{a}=m\,d\mathbf{v}/dt$. (It is a vector law. Why write out three components?) The work $\int \mathbf{F}\cdot d\mathbf{R}$ is

$$\int (m\,d\mathbf{v}/dt)\cdot(\mathbf{v}\,dt) = \tfrac{1}{2}m\mathbf{v}\cdot\mathbf{v}\Big]_P^Q = \tfrac{1}{2}m|\mathbf{v}(Q)|^2 - \tfrac{1}{2}m|\mathbf{v}(P)|^2. \tag{8}$$

The work equals the change in the kinetic energy $\frac{1}{2}m|\mathbf{v}|^2$. But for a gradient field the work is also the *change in potential*—with a minus sign from physics:

$$\textit{work} = \int \mathbf{F}\cdot d\mathbf{R} = -\int df = f(P) - f(Q). \tag{9}$$

Comparing (8) with (9), the combination $\frac{1}{2}m|\mathbf{v}|^2 + f$ is the same at P and Q. ***The total energy, kinetic plus potential, is conserved.***

INDEPENDENCE OF PATH: GRADIENT FIELDS

The work of the spin field $\mathbf{S}$ depends on the path. Example 4 took three paths—straight line, quarter-circle, bent line. The work was 1, $\pi/2$, and 0, different on each path. This happens for more than 99.99% of all vector fields. It does not happen for the most important fields. Mathematics and physics concentrate on very special fields—for which the work depends only on the endpoints. We now explain what happens, *when the integral is independent of the path.*

Suppose you integrate from P to Q on one path, and back to P on another path. Combined, that is a ***closed path*** from P to P (Figure 15.7). But a backward integral is the negative of a forward integral, since $d\mathbf{R}$ switches sign. ***If the integrals from P to Q are equal, the integral around the closed path is zero***:

$$\underset{\text{closed}}{\oint_P^P \mathbf{F}\cdot d\mathbf{R}} = \underset{\text{path 1}}{\int_P^Q \mathbf{F}\cdot d\mathbf{R}} + \underset{\text{back path 2}}{\int_Q^P \mathbf{F}\cdot d\mathbf{R}} = \underset{\text{path 1}}{\int_P^Q \mathbf{F}\cdot d\mathbf{R}} - \underset{\text{path 2}}{\int_P^Q \mathbf{F}\cdot d\mathbf{R}} = 0. \tag{10}$$

The circle on the first integral indicates a closed path. Later we will drop the P's.

Not all closed path integrals are zero! For most fields $\mathbf{F}$, different paths yield different work. For "*conservative*" fields, all paths yield the same work. Then zero work around a closed path conserves energy. The big question is: ***How to decide which fields are conservative, without trying all paths***? Here is the crucial information about conservative fields, in a plane region R *with no holes*:

15D $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is a conservative field if it has these properties:

A. The work $\int \mathbf{F} \cdot d\mathbf{R}$ around every closed path is zero.

B. The work $\int_P^Q \mathbf{F} \cdot d\mathbf{R}$ depends only on P and Q, not on the path.

C. $\mathbf{F}$ is a ***gradient field***: $M = \partial f/\partial x$ and $N = \partial f/\partial y$ for some potential $f(x, y)$.

D. The components satisfy $\partial M/\partial y = \partial N/\partial x$.

A field with one of these properties has them all. **D** is the quick test.

These statements **A**–**D** bring everything together for conservative fields (alias gradient fields). A closed path goes one way to Q and back the other way to P. The work cancels, and statements **A** and **B** are equivalent. We now connect them to **C**. *Note*: Test **D** says that the "***curl***" of **F** is zero. That can wait for Green's Theorem in the next section—the full discussion of the curl comes in 15.6.

First, ***a gradient field*** $\mathbf{F} = \operatorname{grad} f$ ***is conservative***. The work is $f(Q) - f(P)$, by the fundamental theorem for line integrals. It depends only on the endpoints and not the path. Therefore statement **C** leads back to **B**.

Our job is in the other direction, to show that conservative fields $M\mathbf{i} + N\mathbf{j}$ are gradients. Assume that the work integral depends only on the endpoints. We must construct a potential f, so that **F** is its gradient. In other words, $\partial f/\partial x$ must be M and $\partial f/\partial y$ must be N.

Fix the point P. Define f(Q) as the work to reach Q. Then* F *equals grad f.

Check the reasoning. At the starting point P, f is zero. At every other point Q, f is the work $\int M\,dx + N\,dy$ to reach that point. ***All paths from P to Q give the same f(Q)***, because the field is assumed conservative. After two examples we prove that grad f agrees with **F**—the construction succeeds.

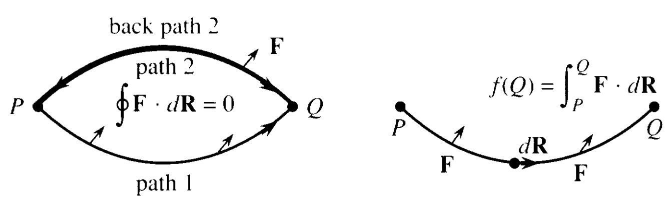

Fig. 15.7 Conservative fields: $\oint \mathbf{F} \cdot d\mathbf{R} = 0$ and $\int_P^Q \mathbf{F} \cdot d\mathbf{R} = f(Q) - f(P)$. Here $f(P) = 0$.

EXAMPLE 5 Find $f(x, y)$ when $\mathbf{F} = M\mathbf{i} + N\mathbf{j} = 2xy\mathbf{i} + x^2\mathbf{j}$. We want $\partial f/\partial x = 2xy$ and $\partial f/\partial y = x^2$.

Solution 1 Choose $P = (0, 0)$. Integrate $M\,dx + N\,dy$ along to $(x, 0)$ and up to (x, y):

$$\int_{(0,0)}^{(x,0)} 2xy\,dx = 0 \text{ (since } y = 0) \qquad \int_{(x,0)}^{(x,y)} x^2 dy = x^2 y \text{ (which is } f).$$

Certainly $f = x^2 y$ meets the requirements: $f_x = 2xy$ and $f_y = x^2$. Thus $\mathbf{F} = \operatorname{grad} f$. Note that $dy = 0$ in the first integral (on the x axis). Then $dx = 0$ in the second integral (x is fixed). The integrals add to $f = x^2 y$.

Solution 2 Integrate $2xy\,dx + x^2dy$ on the straight line (xt, yt) from $t=0$ to $t=1$:

$$\int_0^1 2(xt)(yt)(x\,dt) + (xt)^2(y\,dt) = \int_0^1 3x^2yt^2dt = x^2yt^3]_0^1 = x^2y.$$

Most authors use Solution 1. I use Solution 2. ***Most students use Solution 3***:

Solution 3 Directly solve $\partial f/\partial x = M = 2xy$ and then fix up $\partial f/\partial y = N = x^2$:

$$\partial f/\partial x = 2xy \quad \text{gives} \quad f = x^2y \quad (\textit{plus any function of } y).$$

In this example x^2y already has the correct derivative $\partial f/\partial y = x^2$. No additional function of y is necessary. When we integrate with respect to x, *the constant of integration* (usually C) *becomes a function* $C(y)$.

You will get practice in finding f. This is only possible for conservative fields! I tested $M = 2xy$ and $N = x^2$ in advance (using **D**) to be sure that $\partial M/\partial y = \partial N/\partial x$.

EXAMPLE 6 Look for $f(x, y)$ when $M\mathbf{i} + N\mathbf{j}$ is the spin field $-y\mathbf{i} + x\mathbf{j}$.

Attempted solution 1 Integrate $-y\,dx + x\,dy$ from $(0, 0)$ to $(x, 0)$ to (x, y):

$$\int_{(0,0)}^{(x,0)} -y\,dx = 0 \quad \text{and} \quad \int_{(x,0)}^{(x,y)} x\,dy = xy \quad (\text{which seems like } f).$$

Attempted solution 2 Integrate $-y\,dx + x\,dy$ on the line (xt, yt) from $t = 0$ to 1:

$$\int_0^1 -(yt)(x\,dt) + (xt)(y\,dt) = 0 \quad (\text{a different } f, \text{ also wrong}).$$

Attempted solution 3 Directly solve $\partial f/\partial x = -y$ and try to fix up $\partial f/\partial y = x$:

$$\partial f/\partial x = -y \quad \text{gives} \quad f = -xy \quad (\text{plus any function } C(y)).$$

The y derivative of this f is $-x + dC/dy$. That does not agree with the required $\partial f/\partial y = x$. ***Conclusion***: ***The spin field*** $-y\mathbf{i} + x\mathbf{j}$ ***is not conservative***. There is no f. Test **D** gives $\partial M/\partial y = -1$ and $\partial N/\partial x = +1$.

To finish this section, we move from examples to a proof. The potential $f(Q)$ is defined as the work to reach Q. We must show that its partial derivatives are M and N. This seems reasonable from the formula $f(Q) = \int M\,dx + N\,dy$, but we have to think it through.

Remember statement **A**, that all paths give the same $f(Q)$. Take a path that goes from P to the left of Q. It comes in to Q on a line $y = \text{constant}$ (so $dy = 0$). As the path reaches Q, we are only integrating $M\,dx$. The derivative of this integral, at Q, is $\partial f/\partial x = M$. That is the Fundamental Theorem of Calculus.

To show that $\partial f/\partial y = N$, take a different path. Go from P to a point below Q. The path comes up to Q on a vertical line (so $dx = 0$). Near Q we are only integrating $N\,dy$, so $\partial f/\partial y = N$.

The requirement that the region must have no holes will be critical for test **D**.

EXAMPLE 7 Find $f(x, y) = \int_{(0,0)}^{(x,y)} x\,dx + y\,dy$. Test **D** is passed: $\partial N/\partial x = 0 = \partial M/\partial y$.

Solution 1 $\int_{(0,0)}^{(x,0)} x\,dx = \frac{1}{2}x^2$ is added to $\int_{(x,0)}^{(x,y)} y\,dy = \frac{1}{2}y^2$.

Solution 2 $\int_0^1 (xt)(x\,dt) + (yt)(y\,dt) = \int_0^1 (x^2 + y^2)t\,dt = \frac{1}{2}(x^2 + y^2)$.

Solution 3 $\partial f/\partial x = x$ gives $f = \frac{1}{2}x^2 + C(y)$. Then $\partial f/\partial y = y$ needs $C(y) = \frac{1}{2}y^2$.

15.2 EXERCISES

Read-through questions

Work is the __a__ of $\mathbf{F}\cdot d\mathbf{R}$. Here $\mathbf{F}$ is the __b__ and $\mathbf{R}$ is the __c__. The __d__ product finds the component of __e__ in the direction of movement $d\mathbf{R}=dx\,\mathbf{i}+dy\,\mathbf{j}$. The straight path $(x, y)=(t, 2t)$ goes from __f__ at $t=0$ to __g__ at $t=1$ with $d\mathbf{R}=dt\,\mathbf{i}+$ __h__. The work of $\mathbf{F}=3\mathbf{i}+\mathbf{j}$ is $\int \mathbf{F}\cdot d\mathbf{R}=\int$ __i__ $dt=$ __j__.

Another form of $d\mathbf{R}$ is $\mathbf{T}\,ds$, where $\mathbf{T}$ is the __k__ vector to the path and $ds=\sqrt{\ \ \text{l}\ \ }$. For the path $(t, 2t)$, the unit vector $\mathbf{T}$ is __m__ and $ds=$ __n__ dt. For $\mathbf{F}=3\mathbf{i}+\mathbf{j}$, $\mathbf{F}\cdot\mathbf{T}\,ds$ is still __o__ dt. This $\mathbf{F}$ is the gradient of $f=$ __p__. The change in $f=3x+y$ from $(0, 0)$ to $(1, 2)$ is __q__.

When $\mathbf{F}=\text{grad} f$, the dot product $\mathbf{F}\cdot d\mathbf{R}$ is $(\partial f/\partial x)dx+$ __r__ $=df$. The work integral from P to Q is $\int df=$ __s__. In this case the work depends on the __t__ but not on the __u__. Around a closed path the work is __v__. The field is called __w__. $\mathbf{F}=(1+y)\mathbf{i}+x\mathbf{j}$ is the gradient of $f=$ __x__. The work from $(0, 0)$ to $(1, 2)$ is __y__, the change in potential.

For the spin field $\mathbf{S}=$ __z__, the work (does)(does not) depend on the path. The path $(x, y)=(3\cos t,\ 3\sin t)$ is a circle with $\mathbf{S}\cdot d\mathbf{R}=$ __A__. The work is __B__ around the complete circle. Formally $\int g(x, y)ds$ is the limit of the sum __C__.

The four equivalent properties of a conservative field $\mathbf{F}=M\mathbf{i}+N\mathbf{j}$ are **A**: __D__, **B**: __E__, **C**: __F__, and **D**: __G__. Test **D** is (passed)(not passed) by $\mathbf{F}=(y+1)\mathbf{i}+x\mathbf{j}$. The work $\int \mathbf{F}\cdot d\mathbf{R}$ around the circle $(\cos t, \sin t)$ is __H__. The work on the upper semicircle equals the work on __I__. This field is the gradient of $f=$ __J__, so the work to $(-1, 0)$ is __K__.

Compute the line integrals in 1–6.

1 $\int_C ds$ and $\int_C dy$: $x=t$, $y=2t$, $0\leqslant t\leqslant 1$.

2 $\int_C x\,ds$ and $\int_C xy\,ds$: $x=\cos t$, $y=\sin t$, $0\leqslant t\leqslant \pi/2$.

3 $\int_C xy\,ds$: bent line from $(0, 0)$ to $(1, 1)$ to $(1, 0)$.

4 $\int_C y\,dx-x\,dy$: any square path, sides of length 3.

5 $\int_C dx$ and $\int_C y\,dx$: any closed circle of radius 3.

6 $\int_C (ds/dt)\,dt$: any path of length 5.

7 Does $\int_P^Q xy\,dy$ equal $\frac{1}{2}xy^2]_P^Q$?

8 Does $\int_P^Q x\,dx$ equal $\frac{1}{2}x^2]_P^Q$?

9 Does $(\int_C ds)^2=(\int_C dx)^2+(\int_C dy)^2$?

10 Does $\int_C (ds)^2$ make sense?

In 11–16 find the work in moving from (1, 0) to (0, 1). When F is conservative, construct *f*. Choose your own path when F is not conservative.

11 $\mathbf{F}=\mathbf{i}+y\mathbf{j}$

12 $\mathbf{F}=y\mathbf{i}+\mathbf{j}$

13 $\mathbf{F}=xy^2\mathbf{i}+yx^2\mathbf{j}$

14 $\mathbf{F}=e^y\mathbf{i}+xe^y\mathbf{j}$

15 $\mathbf{F}=(x/r)\mathbf{i}+(y/r)\mathbf{j}$

16 $\mathbf{F}=-y^2\mathbf{i}+x^2\mathbf{j}$

17 For which powers n is $\mathbf{S}/r^n$ a gradient by test **D**?

18 For which powers n is $\mathbf{R}/r^n$ a gradient by test **D**?

19 A wire hoop around a vertical circle $x^2+z^2=a^2$ has density $\rho=a+z$. Find its mass $M=\int \rho\,ds$.

20 A wire of constant density ρ lies on the semicircle $x^2+y^2=a^2$, $y\geqslant 0$. Find its mass M and also its moment $M_x=\int \rho y\,ds$. Where is its center of mass $\bar{x}=M_y/M$, $\bar{y}=M_x/M$?

21 If the density around the circle $x^2+y^2=a^2$ is $\rho=x^2$, what is the mass and where is the center of mass?

22 Find $\int \mathbf{F}\cdot d\mathbf{R}$ along the space curve $x=t$, $y=t^2$, $z=t^3$, $0\leqslant t\leqslant 1$.

(a) $\mathbf{F}=\text{grad}\,(xy+xz)$ (b) $\mathbf{F}=y\mathbf{i}-x\mathbf{j}+z\mathbf{k}$

23 (a) Find the unit tangent vector $\mathbf{T}$ and the speed ds/dt along the path $\mathbf{R}=2t\,\mathbf{i}+t^2\,\mathbf{j}$.

(b) For $\mathbf{F}=3x\mathbf{i}+4\mathbf{j}$, find $\mathbf{F}\cdot\mathbf{T}\,ds$ using (a) and $\mathbf{F}\cdot d\mathbf{R}$ directly.

(c) What is the work from $(2, 1)$ to $(4, 4)$?

24 If $M(x, y, z)\mathbf{i}+N(x, y, z)\mathbf{j}$ is the gradient of $f(x, y, z)$, show that none of these functions can depend on z.

25 Find all gradient fields of the form $M(y)\mathbf{i}+N(x)\mathbf{j}$.

26 Compute the work $W(x, y)=\int M\,dx+N\,dy$ on the straight line path (xt, yt) from $t=0$ to $t=1$. Test to see if $\partial W/\partial x=M$ and $\partial W/\partial y=N$.

(a) $M=y^3$, $N=3xy^2$ (b) $M=x^3$, $N=3yx^2$

(c) $M=x/y$, $N=y/x$ (d) $M=e^{x+y}$, $N=e^{x+y}$

27 Find a field $\mathbf{F}$ whose work around the unit square ($y=0$ then $x=1$ then $y=1$ then $x=0$) equals 4.

28 Find a nonconservative $\mathbf{F}$ whose work around the unit circle $x^2+y^2=1$ is zero.

In 29–34 compute $\int \mathbf{F}\cdot d\mathbf{R}$ along the straight line $\mathbf{R}=t\mathbf{i}+t\mathbf{j}$ and the parabola $\mathbf{R}=t\mathbf{i}+t^2\mathbf{j}$, from (0, 0) to (1, 1). When F is a gradient field, use its potential *f*(*x*, *y*).

29 $\mathbf{F}=\mathbf{i}-2\mathbf{j}$

30 $\mathbf{F}=x^2\mathbf{j}$

31 $\mathbf{F}=2xy^2\mathbf{i}+2yx^2\mathbf{j}$

32 $\mathbf{F}=x^2y\mathbf{i}+xy^2\mathbf{j}$

33 $\mathbf{F}=y\mathbf{i}-x\mathbf{j}$

34 $\mathbf{F}=(x\mathbf{i}+y\mathbf{j})/(x^2+y^2+1)$

35 For which numbers a and b is $\mathbf{F}=axy\mathbf{i}+(x^2+by)\mathbf{j}$ a gradient field?

36 Compute $\int -y\,dx+x\,dy$ from $(1, 0)$ to $(0, 1)$ on the line $x=1-t^2$, $y=t^2$ and the quarter-circle $x=\cos 2t$, $y=\sin 2t$. Example 4 found 1 and $\pi/2$ with different parameters.

Apply the test $N_x = M_y$ to 37–42. Find f when test D is passed.

37 $\mathbf{F} = y^2 e^{-x}\mathbf{i} - 2ye^{-x}\mathbf{j}$

38 $\mathbf{F} = y^2 e^{x}\mathbf{i} - 2ye^{x}\mathbf{j}$

39 $\mathbf{F} = \dfrac{x\mathbf{i} + y\mathbf{j}}{|x\mathbf{i} + y\mathbf{j}|}$

40 $\mathbf{F} = \dfrac{\text{grad } xy}{|\text{grad } xy|}$

41 $\mathbf{F} = \mathbf{R} + \mathbf{S}$

42 $\mathbf{F} = (ax + by)\mathbf{i} + (cx + dy)\mathbf{j}$

43 Around the unit circle find $\oint ds$ and $\oint dx$ and $\oint x ds$.

44 *True or false*, with reason:

(a) When $\mathbf{F} = y\mathbf{i}$ the line integral $\int \mathbf{F} \cdot d\mathbf{R}$ along a curve from P to Q equals the usual area under the curve.

(b) That line integral depends only on P and Q, not on the curve.

(c) That line integral around the unit circle equals π.

15.3 Green's Theorem

This section contains the Fundamental Theorem of Calculus, extended to two dimensions. That sounds important and it is. The formula was discovered 150 years after Newton and Leibniz, by an ordinary mortal named George Green. His theorem connects a ***double integral over a region*** R to a ***line integral along its boundary*** C.

The integral of df/dx equals $f(b) - f(a)$. This connects a one-dimensional integral to a zero-dimensional integral. The boundary only contains two points a and b! The answer $f(b) - f(a)$ is some kind of a "point integral." It is this absolutely crucial idea—to integrate a derivative from information *at the boundary*—that Green's Theorem extends into two dimensions.

There are two important integrals around C. The ***work*** is $\int \mathbf{F} \cdot \mathbf{T}\, ds = \int M\,dx + N\,dy$. The ***flux*** is $\int \mathbf{F} \cdot \mathbf{n}\, ds = \int M\,dy - N\,dx$ (notice the switch). The first is for a force field, the second is for a flow field. The tangent vector $\mathbf{T}$ turns 90° clockwise to become the normal vector $\mathbf{n}$. Green's Theorem handles both, in two dimensions. In three dimensions they split into the Divergence Theorem (15.5) and Stokes' Theorem (15.6).

Green's Theorem applies to "smooth" functions $M(x, y)$ and $N(x, y)$, with continuous first derivatives in a region slightly bigger than R. Then all integrals are well defined. M and N will have a definite and specific meaning in each application—to electricity or magnetism or fluid flow or mechanics. The purpose of a *theorem* is to capture the central ideas once and for all. We do that now, and the applications follow.

15E *Green's Theorem* Suppose the region R is bounded by the simple closed piecewise smooth curve C. Then an integral over R equals a line integral around C:

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy. \qquad (1)$$

A curve is "simple" if it doesn't cross itself (figure 8's are excluded). It is "closed" if its endpoint Q is the same as its starting point P. This is indicated by the closed circle on the integral sign. The curve is "smooth" if its tangent $\mathbf{T}$ changes continuously—the word "piecewise" allows a finite number of corners. Fractals are not allowed, but all reasonable curves are acceptable (later we discuss figure 8's and rings). First comes an understanding of the formula, by testing it on special cases.

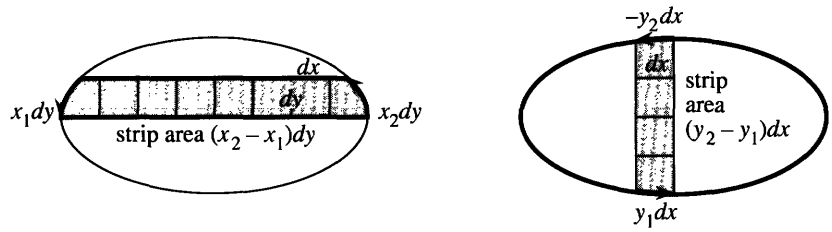

Fig. 15.8 Area of R adds up strips: $\oint x\,dy = \iint dx\,dy$ and $\oint y\,dx = -\iint dy\,dx$.

Special case 1: $M = 0$ and $N = x$. Green's Theorem with $\partial N/\partial x = 1$ becomes

$$\oint_C x\,dy = \iint_R 1\,dx\,dy \quad \text{(which is the area of } R\text{).} \tag{2}$$

The integrals look equal, because the inner integral of dx is x. Then both integrals have $x\,dy$—but we need to go carefully. The area of a layer of R is dy times the difference in x (the length of the strip). The line integral in Figure 15.8 agrees. It has an upward dy times x (at the right) plus a downward $-dy$ times x (at the left). The integrals add up the strips, to give the total area.

Special case 2: $M = y$ and $N = 0$ and $\oint_C y\,dx = \iint_R (-1)\,dx\,dy = -(\text{area of } R)$.

Now Green's formula has a minus sign, because the line integral is *counterclockwise*. The top of each slice has $dx < 0$ (going left) and the bottom has $dx > 0$ (going right). Then $y\,dx$ at the top and bottom combine to give *minus* the area of the slice in Figure 15.8b.

Special case 3: $\oint 1\,dx = 0$. The dx's to the right cancel the dx's to the left (the curve is closed). With $M = 1$ and $N = 0$, Green's Theorem is $0 = 0$.

Most important case: $M\mathbf{i} + N\mathbf{j}$ is a *gradient field*. It has a potential function $f(x, y)$. Green's Theorem is $0 = 0$, because $\partial M/\partial y = \partial N/\partial x$. This is test **D**:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \textit{ is the same as } \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right). \tag{3}$$

The cross derivatives always satisfy $f_{yx} = f_{xy}$. That is why gradient fields pass test **D**.

When the double integral is zero, the line integral is also zero: $\oint_C M\,dx + N\,dy = 0$. The work is zero. ***The field is conservative***! This last step in $\mathbf{A} \Rightarrow \mathbf{B} \Rightarrow \mathbf{C} \Rightarrow \mathbf{D} \Rightarrow \mathbf{A}$ will be complete when Green's Theorem is proved.

Conservative examples are $\oint x\,dx = 0$ and $\oint y\,dy = 0$. Area is not involved.

Remark The special cases $\oint x\,dy$ and $-\oint y\,dx$ led to the area of R. As long as $1 = \partial N/\partial x - \partial M/\partial y$, the double integral becomes $\iint 1\,dx\,dy$. This gives a way to compute area by a line integral.

$$\textit{The area of } R \textit{ is } \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx). \tag{4}$$

EXAMPLE 1 The area of the triangle in Figure 15.9 is 2. Check Green's Theorem.

The last area formula in (4) uses $\frac{1}{2}\mathbf{S}$, half the spin field. $N = \frac{1}{2}x$ and $M = -\frac{1}{2}y$ yield $N_x - M_y = \frac{1}{2} + \frac{1}{2} = 1$. On one side of Green's Theorem is $\iint 1\,dx\,dy = $ area of triangle. On the other side, the line integral has three pieces.

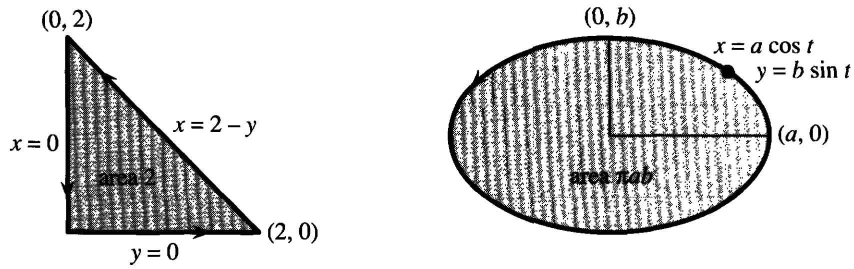

Fig. 15.9 Green's Theorem: Line integral around triangle, area integral for ellipse.

Two pieces are zero: $x\,dy - y\,dx = 0$ on the sides where $x = 0$ and $y = 0$. The sloping side $x = 2 - y$ has $dx = -dy$. The line integral agrees with the area, confirming Green's Theorem:

$$\frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_{y=0}^{2}(2-y)dy + y\,dy = \frac{1}{2}\int_0^2 2dy = 2.$$

EXAMPLE 2 ***The area of an ellipse is πab when the semiaxes have lengths a and b.***

This is a classical example, which all authors like. The points on the ellipse are $x = a\cos t$, $y = b\sin t$, as t goes from 0 to 2π. (The ellipse has $(x/a)^2 + (y/b)^2 = 1$.) By computing the boundary integral, we discover the area inside. Note that the differential $x\,dy - y\,dx$ is just $ab\,dt$:

$$(a\cos t)(b\cos t\,dt) - (b\sin t)(-a\sin t\,dt) = ab(\cos^2 t + \sin^2 t)dt = ab\,dt.$$

The line integral is $\frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab$. This area πab is πr^2, for a circle with $a = b = r$.

Proof of Green's Theorem: In our special cases, the two sides of the formula were equal. We now show that they are always equal. The proof uses the Fundamental Theorem to integrate $(\partial N/\partial x)dx$ and $(\partial M/\partial y)dy$. Frankly speaking, this one-dimensional theorem is all we have to work with—since we don't know M and N.

The proof is a step up in mathematics, to work with symbols M and N instead of specific functions. The integral in (6) below has no numbers. The idea is to deal with M and N in two separate parts, which added together give Green's Theorem:

$$\oint_C M\,dx = \iint_R -\frac{\partial M}{\partial y}\,dx\,dy \quad \text{and separately} \quad \oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dx\,dy. \qquad (5)$$

Start with a "very simple" region (Figure 15.10a). Its top is given by $y = f(x)$ and its bottom by $y = g(x)$. In the double integral, integrate $-\partial M/\partial y$ first with respect to y. The inner integral is

$$\int_{g(x)}^{f(x)} -\frac{\partial M}{\partial y}dy = -M(x, y)\Big]_{g(x)}^{f(x)} = -M(x, f(x)) + M(x, g(x)). \qquad (6)$$

The Fundamental Theorem (in the y variable) gives this answer that depends on x. If we knew M and f and g, we could do the outer integral—from $x = a$ to $x = b$. But we have to leave it and go to the other side of Green's Theorem—the line integral:

$$\oint M\,dx = \int_{\text{top}} M(x, y)dx + \int_{\text{bottom}} M(x, y)dx = \int_b^a M(x, f(x))dx + \int_a^b M(x, g(x))dx. \qquad (7)$$

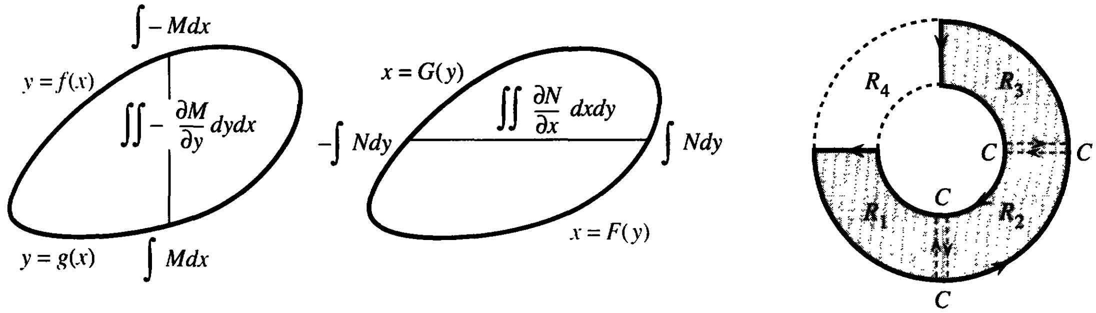

Fig. 15.10 Very simple region (a–b). Simple region (c) is a union of very simple regions.

Compare (7) with (6). The integral of $M(x, g(x))$ is the same for both. The integral of $M(x, f(x))$ has a minus sign from (6). In (7) it has a plus sign but the integral is from b to a. So life is good.

The part for N uses the same idea. Now the x integral comes first, because $(\partial N/\partial x)dx$ is practically asking to be integrated—from $x = G(y)$ at the left to $x = F(y)$ at the right. We reach $N(F(y), y) - N(G(y), y)$. Then the y integral matches $\oint N dy$ and completes (5). Adding the two parts of (5) proves Green's Theorem.

Finally we discuss the shape of R. The broken ring in Figure 15.10 is not "very simple," because horizontal lines go in and out and in and out. Vertical lines do the same. The x and y strips break into pieces. Our reasoning assumed no break between $y = f(x)$ at the top and $y = g(x)$ at the bottom.

There is a nice idea that saves Green's Theorem. Separate the broken ring into three very simple regions R_1, R_2, R_3. The three double integrals equal the three line integrals around the R's. Now *add these separate results*, to produce the double integral over all of R. When we add the line integrals, ***the crosscuts CC are covered twice and they cancel.*** The cut between R_1 and R_2 is covered upward (around R_1) and downward (around R_2). That leaves the integral around the boundary equal to the double integral inside—which is Green's Theorem.

When R is a complete ring, including the piece R_4, the theorem is still true. The integral around the outside is still counterclockwise. But the integral is *clockwise* around the inner circle. ***Keep the region R to your left as you go around C.*** The complete ring is "doubly" connected, not "simply" connected. Green's Theorem allows any finite number of regions R_i and crosscuts CC and holes.

EXAMPLE 3 The area under a curve is $\int_a^b y\,dx$, as we always believed.

In computing area we never noticed the whole boundary. The true area is a line integral $-\oint y\,dx$ around the ***closed curve*** in Figure 15.11a. But $y = 0$ on the x axis. Also $dx = 0$ on the vertical lines (up and down at b and a). Those parts contribute zero to the integral of $y\,dx$. The only nonzero part is back along the curve—which is the area $-\int_b^a y\,dx$ or $\int_a^b y\,dx$ that we know well.

What about signs, when the curve dips below the x axis? That area has been counted as negative since Chapter 1. I saved the proof for Chapter 15. The reason lies in the arrows on C.

The line integral around that part *goes the other way*. The arrows are clockwise, the region is on the *right*, and the area counts as negative. With the correct rules, a figure 8 is allowed after all.

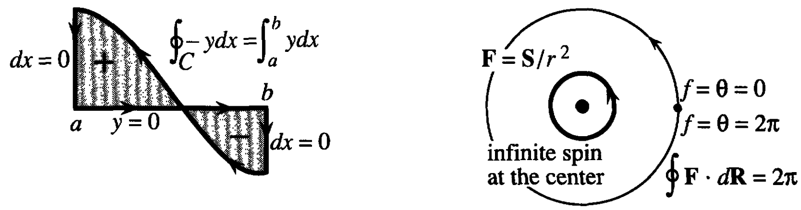

Fig. 15.11 Closed path gives the sign of the area. Nonconservative field because of hole.

CONSERVATIVE FIELDS

We never leave gradients alone! They give conservative fields—the work around a closed path is $f(P)-f(P)=0$. But a potential function $f(x, y)$ is only available when test **D** is passed: ***If*** $\partial f/\partial x = M$ ***and*** $\partial f/\partial y = N$ ***then*** $\partial M/\partial y = \partial N/\partial x$. The reason is that $f_{xy} = f_{yx}$.

Some applications prefer the language of "differentials." Instead of looking for $f(x, y)$, we look for df:

DEFINITION The expression $M(x, y)dx + N(x, y)dy$ is a ***differential form.*** When it agrees with the differential $df = (\partial f/\partial x)\,dx + (\partial f/\partial y)\,dy$ of some function, the form is called ***exact***. *The test for an exact differential is* **D**: $\partial N/\partial x = \partial M/\partial y$.

Nothing is new but the language. Is $y\,dx$ an exact differential? *No*, because $M_y = 1$ and $N_x = 0$. Is $y\,dx + x\,dy$ an exact differential? *Yes*, it is the differential of $f = xy$. That is the product rule! Now comes an important example, to show why R should be ***simply connected*** (a region with no holes).

EXAMPLE 4 The spin field $\mathbf{S}/r^2 = (-y\mathbf{i} + x\mathbf{j})/(x^2+y^2)$ *almost* passes test **D**.

$$N_x = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} = M_y = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2}. \quad (8)$$

Both numerators are $y^2 - x^2$. Test **D** looks good. To find f, integrate $M = \partial f/\partial x$:

$$f(x, y) = \int -y\,dx/(x^2+y^2) = \tan^{-1}(y/x) + C(y).$$

The extra part $C(y)$ can be zero—the y derivative of $\tan^{-1}(y/x)$ gives N with no help from $C(y)$. ***The potential f is the angle*** θ in the usual x, y, r right triangle.

Test **D** is passed and **F** is grad θ. What am I worried about? It is only this, that ***Green's Theorem on a circle seems to give*** $2\pi = 0$. The double integral is $\iint (N_x - M_y)dx\,dy$. According to (8) this is the integral of zero. But the line integral is 2π:

$$\oint \mathbf{F}\cdot d\mathbf{R} = \oint(-y\,dx + x\,dy)/(x^2+y^2) = 2(\text{area of circle})/a^2 = 2\pi a^2/a^2 = 2\pi. \quad (9)$$

With $x = a\cos t$ and $y = a\sin t$ we would find the same answer. ***The work is*** 2π (not zero!) ***when the path goes around the origin.***

We have a paradox. If Green's Theorem is wrong, calculus is in deep trouble. Some requirement must be violated to reach $2\pi = 0$. Looking at $\mathbf{S}/r^2$, the problem is at the origin. The field is not defined when $r = 0$ (it blows up). The derivatives in (8) are not continuous. Test **D** does not apply at the origin, and was not passed. *We could remove* $(0, 0)$, *but then the region where test* **D** *is passed would have a hole.*

It is amazing how one point can change everything. When the path circles the origin, the line integral is not zero. ***The potential function*** $f = \theta$ ***increases by*** 2π. That agrees with $\int \mathbf{F} \cdot d\mathbf{R} = 2\pi$ from (9). It disagrees with $\iint 0\, dx\, dy$. The 2π is right, the zero is wrong. $N_x - M_y$ must be a *"delta function of strength* 2π."

The double integral is 2π from an infinite spike over the origin—even though $N_x = M_y$ everywhere else. In fluid flow the delta function is a "vortex."

FLOW ACROSS A CURVE: GREEN'S THEOREM TURNED BY 90°

A flow field is easier to visualize than a force field, because something is really there and it moves. Instead of gravity in empty space, water has velocity $M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$. At the boundary C it can flow in or out. The new form of Green's Theorem is a fundamental "balance equation" of applied mathematics:

Flow through C (out minus in) = ***replacement in*** R (source minus sink).

The flow is *steady*. Whatever goes out through C must be replaced in R. When there are no sources or sinks (negative sources), the total flow through C must be zero. This balance law is Green's Theorem in its "normal form" (for $\mathbf{n}$) instead of its "tangential form" (for $\mathbf{T}$):

15F For a steady flow field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$, the flux $\int \mathbf{F} \cdot \mathbf{n}\, ds$ through the boundary C balances the replacement of fluid inside R:

$$\oint_C M\, dy - N\, dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy. \tag{10}$$

Figure 15.12 shows the 90° turn. $\mathbf{T}$ becomes $\mathbf{n}$ and "circulation" along C becomes flux through C. In the original form of Green's Theorem, change N and M to M and $-N$ to obtain the flux form:

$$\oint M\, dx + N\, dy \rightarrow \oint -N\, dx + M\, dy \qquad \iint (N_x - M_y) dx\, dy \rightarrow \iint (M_x + N_y) dx\, dy. \tag{11}$$

Playing with letters has proved a new theorem! The two left sides in (11) are equal, so the right sides are equal—which is Green's Theorem (10) for the flux. The components M and N can be chosen freely and named freely.

The change takes $M\mathbf{i} + N\mathbf{j}$ into its perpendicular field $-N\mathbf{i} + M\mathbf{j}$. The field is turned at every point (we are not just turning the plane by 90°). The spin field $\mathbf{S} = -y\mathbf{i} + x\mathbf{j}$ changes to the position field $\mathbf{R} = x\mathbf{i} + y\mathbf{j}$. The position field $\mathbf{R}$ changes to $-\mathbf{S}$. Streamlines of one field are equipotentials of the other field. The new form (10) of Green's

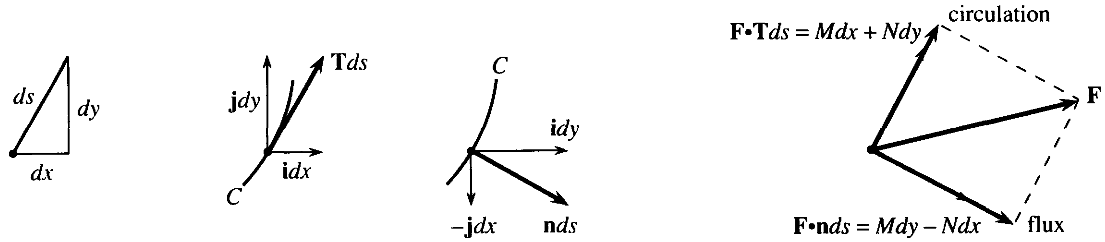

Fig. 15.12 The perpendicular component $\mathbf{F} \cdot \mathbf{n}$ flows through C. Note $\mathbf{n}\, ds = dy\, \mathbf{i} - dx\, \mathbf{j}$.

Theorem is just as important as the old one—in fact I like it better. It is easier to visualize flow across a curve than circulation along it.

The change of letters was just for the proof. From now on $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$.

EXAMPLE 5 Compute both sides of the new form (10) for $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j}$. The region R is a rectangle with sides a and b.

Solution This field has $\partial M/\partial x + \partial N/\partial y = 2 + 3$. The integral over R is $\iint_R 5\,dx\,dy = 5ab$. The line integral has four parts, because R has four sides. Between the left and right sides, $M = 2x$ increases by $2a$. Down the left and up the right, $\int M\,dy = 2ab$ (those sides have length b). Similarly $N = 3y$ changes by $3b$ between the bottom and top. Those sides have length a, so they contribute $3ab$ to a total line integral of $5ab$.

Important: ***The "divergence" of a flow field is*** $\partial M/\partial x + \partial N/\partial y$. The example has divergence $= 5$. To maintain this flow we must replace 5 units continually—not just at the origin but everywhere. (A one-point source is in example 7.) The divergence is the source strength, because it equals the outflow. ***To understand Green's Theorem for any vector field*** $M\mathbf{i} + N\mathbf{j}$, ***look at a tiny rectangle*** (sides dx and dy):

Flow out the right side minus flow in the left side = (change in M) times dy

Flow out the top minus flow in the bottom = (change in N) times dx

Total flow out of rectangle: $dM\,dy + dN\,dx = (\partial M/\partial x + \partial N/\partial y)dx\,dy$.

The divergence times the area $dx\,dy$ ***equals the total flow out.*** Section 15.5 gives more detail with more care in three dimensions. The divergence is $M_x + N_y + P_z$.

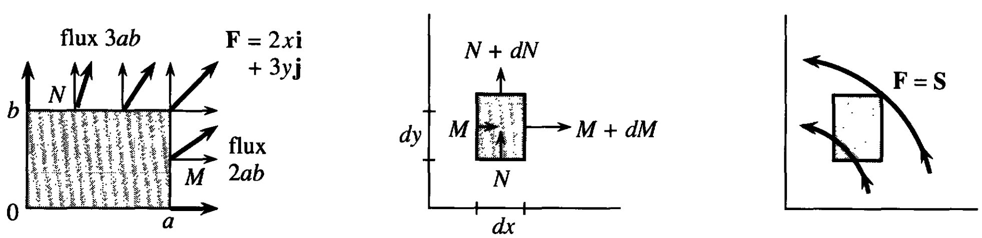

Fig. 15.13 $M_x + N_y = 2 + 3 = 5$ yields flux = 5(area) = $5ab$. The flux is $dM\,dy + dN\,dx = (M_x + N_y)\,dx\,dy$. The spin field has no flux.

EXAMPLE 6 Find the flux through a closed curve C of the spin field $\mathbf{S} = -y\mathbf{i} + x\mathbf{j}$.

Solution The field has $M = -y$ and $N = x$ and $M_x + N_y = 0$. *The double integral is zero.* Therefore the total flow (out minus in) is also zero—through any closed curve. Figure 15.13 shows flow entering and leaving a square. No fluid is added or removed. There is no rain and no evaporation. ***When the divergence*** $M_x + N_y$ ***is zero, there is no source or sink.***

FLOW FIELDS WITHOUT SOURCES

This is really quite important. Remember that conservative fields do no work around C, they have a potential f, and they have "zero curl." Now turn those statements through 90°, to find their twins. ***Source-free fields have no flux through*** C, ***they have stream functions*** g, ***and they have "zero divergence."*** The new statements **E–F–G–H** describe fields without sources.

15G The field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is source-free if it has these properties:

E The total flux $\oint \mathbf{F} \cdot \mathbf{n}\, ds$ through every closed curve is zero.

F Across all curves from P to Q, the flux $\int_P^Q \mathbf{F} \cdot \mathbf{n}\, ds$ is the same.

G There is a ***stream function*** $g(x, y)$, for which $M = \partial g/\partial y$ and $N = -\partial g/\partial x$.

H The components satisfy $\partial M/\partial x + \partial N/\partial y = 0$ (***the divergence is zero***).

A field with one of these properties has them all. **H** is the quick test.

The spin field $-y\mathbf{i} + x\mathbf{j}$ passed this test (Example 6 was source-free). The field $2x\mathbf{i} + 3y\mathbf{j}$ does not pass (Example 5 had $M_x + N_y = 5$). ***Example 7 almost passes.***

EXAMPLE 7 The radial field $\mathbf{R}/r^2 = (x\mathbf{i} + y\mathbf{j})/(x^2 + y^2)$ has a ***point source*** at $(0, 0)$.

The new test **H** is *divergence* $= \partial M/\partial x + \partial N/\partial y = 0$. Those two derivatives are

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) = \frac{x^2 + y^2 - x(2x)}{(x^2 + y^2)^2} \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right) = \frac{x^2 + y^2 - y(2y)}{(x^2 + y^2)^2}. \qquad (12)$$

They add to zero. There seems to be no source (if the calculation is correct). The flow through a circle $x^2 + y^2 = a^2$ should be zero. But it's not:

$$\oint M\, dy - N\, dx = \oint (x\, dy - y\, dx)/(x^2 + y^2) = 2(\text{area of circle})/a^2 = 2\pi. \qquad (13)$$

A source is hidden somewhere. Looking at $\mathbf{R}/r^2$, the problem is at $(0, 0)$. The field is not defined when $r = 0$ (it blows up). The derivatives in (12) are not continuous. Test **H** does not apply, and was not passed. The divergence $M_x + N_y$ must be a "delta function" of strength 2π. There is a ***point source*** sending flow out through all circles.

I hope you see the analogy with Example 4. The field $\mathbf{S}/r^2$ is curl-free except at $r = 0$. The field $\mathbf{R}/r^2$ is divergence-free except at $r = 0$. The mathematics is parallel and the fields are perpendicular. A potential f and a stream function g require a region without holes.

THE BEST FIELDS: CONSERVATIVE AND SOURCE-FREE

***What if* F *is conservative and also source-free*?** Those are outstandingly important fields. The curl is zero and the divergence is zero. Because the field is conservative, it comes from a potential. Because it is source-free, there is a stream function:

$$M = \frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} \quad \text{and} \quad N = \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}. \qquad (14)$$

Those are the ***Cauchy-Riemann equations***, named after a great mathematician of his time and one of the greatest of all time. I can't end without an example.

EXAMPLE 8 Show that $y\mathbf{i} + x\mathbf{j}$ is both conservative and source-free. Find f and g.

Solution With $M = y$ and $N = x$, check first that $\partial M/\partial y = 1 = \partial N/\partial x$. There must be a potential function. It is $f = xy$, which achieves $\partial f/\partial x = y$ and $\partial f/\partial y = x$. Note that $f_{xx} + f_{yy} = 0$.

Check next that $\partial M/\partial x + \partial N/\partial y = 0 + 0$. There must be a stream function. It is $g = \frac{1}{2}(y^2 - x^2)$, which achieves $\partial g/\partial y = y$ and $\partial g/\partial x = -x$. Note that $g_{xx} + g_{yy} = 0$.

The curves $f=$ constant are the equipotentials. The curves $g=$ constant are the streamlines (Figure 15.4). These are the twin properties—a conservative field with a potential and a source-free field with a stream function. They come together into the fundamental partial differential equation of equilibrium—*Laplace's equation* $f_{xx}+f_{yy}=0$.

15H There is a potential and stream function when $M_y=N_x$ and $M_x=-N_y$. They satisfy ***Laplace's equation***:

$$f_{xx}+f_{yy}=M_x+N_y=0 \quad \text{and} \quad g_{xx}+g_{yy}=-N_x+M_y=0. \qquad (15)$$

If we have f without g, as in $f=x^2+y^2$ and $M=2x$ and $N=2y$, we don't have Laplace's equation: $f_{xx}+f_{yy}=4$. This is a gradient field that needs a source. If we have g without f, as in $g=x^2+y^2$ and $M=2y$ and $N=-2x$, we don't have Laplace's equation. The field is source-free but it has spin. The first field is $2\mathbf{R}$ and the second field is $2\mathbf{S}$.

With no source and no spin, we are with Laplace at the center of mathematics and science.

Green's Theorem: Tangential form $\oint \mathbf{F}\cdot\mathbf{T}\,ds$ and normal form $\oint \mathbf{F}\cdot\mathbf{n}\,ds$

$$\oint_C M\,dx+N\,dy=\iint_R (N_x-M_y)dx\,dy \qquad \oint_C M\,dy-N\,dx=\iint_R (M_x+N_y)dx\,dy$$

work   curl   flux   divergence

Conservative: work = zero, $N_x=M_y$, gradient of a potential: $M=f_x$ and $N=f_y$

Source-free: flux = zero, $M_x=-N_y$, has a stream function: $M=g_y$ and $N=-g_x$

Conservative + source-free: Cauchy-Riemann + Laplace equations for f and g.

15.3 EXERCISES

Read-through questions

The work integral $\oint M\,dx+N\,dy$ equals the double integral __a__ by __b__'s Theorem. For $\mathbf{F}=3\mathbf{i}+4\mathbf{j}$ the work is __c__. For $\mathbf{F}=$ __d__ and __e__, the work equals the area of R. When $M=\partial f/\partial x$ and $N=\partial f/\partial y$, the double integral is zero because __f__. The line integral is zero because __g__. An example is $\mathbf{F}=$ __h__. The direction on C is __i__ around the outside and __j__ around the boundary of a hole. If R is broken into very simple pieces with crosscuts between them, the integrals of __k__ cancel along the crosscuts.

Test **D** for gradient fields is __l__. A field that passes this test has $\oint \mathbf{F}\cdot d\mathbf{R}=$ __m__. There is a solution to $f_x=$ __n__ and $f_y=$ __o__. Then $df=M\,dx+N\,dy$ is an __p__ differential. The spin field $\mathbf{S}/r^2$ passes test **D** except at __q__. Its potential $f=$ __r__ increases by __s__ going around the origin. The integral $\iint(N_x-M_y)dx\,dy$ is not zero but __t__.

The flow form of Green's Theorem is __u__ = __v__. The normal vector in $\mathbf{F}\cdot\mathbf{n}\,ds$ points __w__ and $|\mathbf{n}|=$ __x__ and $\mathbf{n}\,ds$ equals $dy\,\mathbf{i}-dx\,\mathbf{j}$. The divergence of $M\mathbf{i}+N\mathbf{j}$ is __y__. For $\mathbf{F}=x\mathbf{i}$ the double integral is __z__. There (is)(is not) a source. For $\mathbf{F}=y\mathbf{i}$ the divergence is __A__. The divergence of $\mathbf{R}/r^2$ is zero except at __B__. This field has a __C__ source.

A field with no source has properties $\mathbf{E}=$ __D__, $\mathbf{F}=$ __E__, $\mathbf{G}=$ __F__, $\mathbf{H}=$ zero divergence. The stream function g satisfies the equations __G__. Then $\partial M/\partial x+\partial N/\partial y=0$ because $\partial^2 g/\partial x\partial y=$ __H__. The example $\mathbf{F}=y\mathbf{i}$ has $g=$ __I__. There (is)(is not) a potential function. The example $\mathbf{F}=x\mathbf{i}-y\mathbf{j}$ has $g=$ __J__ and also $f=$ __K__. This f satisfies Laplace's equation __L__, because the field $\mathbf{F}$ is both __M__ and __N__. The functions f and g are connected by the __O__ equations $\partial f/\partial x=\partial g/\partial y$ and __P__.

Compute the line integrals 1–6 and (separately) the double integrals in Green's Theorem (1). The circle has $x=a\cos t$, $y=a\sin t$. The triangle has sides $x=0$, $y=0$, $x+y=1$.

1 $\oint x\,dy$ along the circle

2 $\oint x^2y\,dy$ along the circle

3 $\oint x\,dx$ along the triangle

4 $\oint y\,dx$ along the triangle

5 $\oint x^2y\,dx$ along the circle

6 $\oint x^2y\,dx$ along the triangle

7 Compute both sides of Green's Theorem in the form (10):

(a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, R = upper half of the disk $x^2 + y^2 \leqslant 1$.

(b) $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$, C = square with sides $y = 0, x = 1, y = 1, x = 0$.

8 Show that $\oint_C (x^2y + 2x)dy + xy^2dx$ depends only on the area of R. Does it equal the area?

9 Find the area inside the hypocycloid $x = \cos^3 t$, $y = \sin^3 t$ from $\frac{1}{2}\oint x\,dy - y\,dx$.

10 For constants b and c, how is $\oint by\,dx + cx\,dy$ related to the area inside C? If $b = 7$, which c makes the integral zero?

11 For $\mathbf{F} = \text{grad}\sqrt{x^2 + y^2}$, show in three ways that $\oint \mathbf{F}\cdot d\mathbf{R} = 0$ around $x = \cos t$, $y = \sin t$.

(a) $\mathbf{F}$ is a gradient field so ________.

(b) Compute $\mathbf{F}$ and directly integrate $\mathbf{F}\cdot d\mathbf{R}$.

(c) Compute the double integral in Green's Theorem.

12 Devise a way to find the one-dimensional theorem $\int_a^b (df/dx)dx = f(b) - f(a)$ as a special case of Green's Theorem when R is a square.

13 (a) Choose $x(t)$ and $y(t)$ so that the path goes from $(1, 0)$ to $(1, 0)$ after circling the origin *twice*.

(b) Compute $\oint y\,dx$ and compare with the area inside your path.

(c) Compute $\oint (y\,dx - x\,dy)/(x^2 + y^2)$ and compare with 2π in Example 7.

14 In Example 4 of the previous section, the work $\int \mathbf{S}\cdot d\mathbf{R}$ between $(1, 0)$ and $(0, 1)$ was 1 for the straight path and $\pi/2$ for the quarter-circle path. Show that the work is always twice the area between the path and the axes.

Compute both sides of $\oint \mathbf{F}\cdot\mathbf{n}\,ds = \iint (M_x + N_y)\,dx\,dy$ in 15–20.

15 $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$ in the unit circle

16 $\mathbf{F} = xy\mathbf{i}$ in the unit square $0 \leqslant x, y \leqslant 1$

17 $\mathbf{F} = \mathbf{R}/r$ in the unit circle

18 $\mathbf{F} = \mathbf{S}/r$ in the unit square

19 $\mathbf{F} = x^2y\mathbf{j}$ in the unit triangle (sides $x = 0$, $y = 0$, $x + y = 1$)

20 $\mathbf{F} = \text{grad}\, r$ in the top half of the unit circle.

21 Suppose $\text{div}\,\mathbf{F} = 0$ except at the origin. Then the flux $\oint \mathbf{F}\cdot\mathbf{n}\,ds$ is the same through any two circles around the origin, because ________. (What is $\iint (M_x + N_y)dx\,dy$ between the circles?)

22 Example 7 has $\text{div}\,\mathbf{F} = 0$ except at the origin. The flux through every circle $x^2 + y^2 = a^2$ is 2π. The flux through a square around the origin is also 2π because ________. (Compare Problem 21.)

23 Evaluate $\oint a(x, y)dx + b(x, y)dy$ by both forms of Green's Theorem. The choice $M = a$, $N = b$ in the work form gives the double integral ________. The choice $M = b$, $N = -a$ in the flux form gives the double integral ________. There was only one Green.

24 Evaluate $\oint \cos^3 y\,dy - \sin^3 x\,dx$ by Green's Theorem.

25 The field $\mathbf{R}/r^2$ in Example 7 has zero divergence except at $r = 0$. Solve $\partial g/\partial y = x/(x^2 + y^2)$ to find an attempted stream function g. Does g have trouble at the origin?

26 Show that $\mathbf{S}/r^2$ has zero divergence (except at $r = 0$). Find a stream function by solving $\partial g/\partial y = y/(x^2 + y^2)$. Does g have trouble at the origin?

27 Which differentials are exact: $y\,dx - x\,dy$, $x^2dx + y^2dy$, $y^2dx + x^2dy$?

28 If $M_x + N_y = 0$ then the equations $\partial g/\partial y =$ ________ and $\partial g/\partial x =$ ________ yield a stream function. If also $N_x = M_y$, show that g satisfies Laplace's equation.

Compute the divergence of each field in 29–36 and solve $g_y = M$ and $g_x = -N$ for a stream function (if possible).

29 $2xy\mathbf{i} - y^2\mathbf{j}$

30 $3xy^2\mathbf{i} - y^3\mathbf{j}$

31 $x^2\mathbf{i} + y^2\mathbf{j}$

32 $y^2\mathbf{i} + x^2\mathbf{j}$

33 $e^x \cos y\,\mathbf{i} - e^x \sin y\,\mathbf{j}$

34 $e^{x+y}(\mathbf{i} - \mathbf{j})$

35 $2y\mathbf{i}/x + y^2\mathbf{j}/x^2$

36 $xy\mathbf{i} - xy\mathbf{j}$

37 Compute $N_x - M_y$ for each field in 29–36 and find a potential function f when possible.

38 The potential $f(Q)$ is the work $\int_P^Q \mathbf{F}\cdot\mathbf{T}\,ds$ to reach Q from a fixed point P (Section 15.2). In the same way, the stream function $g(Q)$ can be constructed from the integral ________. Then $g(Q) - g(P)$ represents ***the flux across the path from P to Q***. Why do all paths give the same answer?

39 The real part of $(x + iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3$ is $f = x^3 - 3xy^2$. Its gradient field is $\mathbf{F} = \text{grad}\, f =$ ________. The divergence of $\mathbf{F}$ is ________. Therefore f satisfies Laplace's equation $f_{xx} + f_{yy} = 0$ (check that it does).

40 Since $\text{div}\,\mathbf{F} = 0$ in Problem 39, we can solve $\partial g/\partial y =$ ________ and $\partial g/\partial x =$ ________. The stream function is $g =$ ________. It is the imaginary part of the same $(x + iy)^3$. Check that f and g satisfy the Cauchy–Riemann equations.

41 The real part f and imaginary part g of $(x + iy)^n$ satisfy the Laplace and Cauchy-Riemann equations for $n = 1, 2, \ldots$. (They give all the polynomial solutions.) Compute f and g for $n = 4$.

42 When is $M\,dy - N\,dx$ an exact differential dg?

43 The potential $f = e^x \cos y$ satisfies Laplace's equation. There must be a g. Find the field $\mathbf{F} = \text{grad}\, f$ and the stream function $g(x, y)$.

44 Show that the spin field **S** does work around every simple closed curve.

45 For $\mathbf{F}=f(x)\mathbf{j}$ and $R=$ unit square $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$, integrate both sides of Green's Theorem (1). What formula is required from one-variable calculus?

46 A region R is "*simply connected*" when every closed curve inside R can be squeezed to a point without leaving R. Test these regions:

1. xy plane without $(0, 0)$
2. xyz space without $(0, 0, 0)$
3. sphere $x^2+y^2+z^2=1$
4. a torus (or doughnut)
5. a sweater
6. a human body
7. the region between two spheres
8. xyz space with circle removed.

15.4 Surface Integrals

The double integral in Green's Theorem is over a flat surface R. Now the region moves out of the plane. It becomes a ***curved surface*** S, part of a sphere or cylinder or cone. When the surface has only one z for each (x, y), it is the graph of a function $z(x, y)$. In other cases S can twist and close up—a sphere has an upper z and a lower z. In all cases we want to compute area and flux. This is a necessary step (it is our last step) before moving Green's Theorem to three dimensions.

First a quick review. The basic integrals are $\int dx$ and $\iint dx\,dy$ and $\iiint dx\,dy\,dz$. The one that didn't fit was $\int ds$—the length of a curve. When we go from curves to surfaces, ds becomes dS. ***Area is*** $\iint dS$ ***and flux is*** $\iint \mathbf{F}\cdot\mathbf{n}\,dS$, with double integrals because the surfaces are two-dimensional. The main difficulty is in dS.

All formulas are summarized in a table at the end of the section.

There are two ways to deal with ds (along curves). The same methods apply to dS (on surfaces). The first is in xyz coordinates; the second uses parameters. Before this subject gets complicated, I will explain those two methods.

***Method* 1 *is for the graph of a function*: *curve* $y(x)$ *or surface* $z(x, y)$.**

A small piece of the curve is almost straight. It goes across by dx and up by dy:

$$\text{length } ds=\sqrt{(dx)^2+(dy)^2}=\sqrt{1+(dy/dx)^2}\,dx. \tag{1}$$

A small piece of the surface is practically flat. Think of a tiny sloping rectangle. One side goes across by dx and up by $(\partial z/\partial x)dx$. The neighboring side goes along by dy and up by $(\partial z/\partial y)dy$. Computing the area is a linear problem (from Chapter 11), because the flat piece is in a plane.

Two vectors **A** and **B** form a parallelogram. ***The length of their cross product is the area.*** In the present case, the vectors are $\mathbf{A}=\mathbf{i}+(\partial z/\partial x)\mathbf{k}$ and $\mathbf{B}=\mathbf{j}+(\partial z/\partial y)\mathbf{k}$. Then $\mathbf{A}dx$ and $\mathbf{B}dy$ are the sides of the small piece, and we compute $\mathbf{A}\times\mathbf{B}$:

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & \partial z/\partial x\\ 0 & 1 & \partial z/\partial y\end{vmatrix}=-\partial z/\partial x\,\mathbf{i}-\partial z/\partial y\,\mathbf{j}+\mathbf{k}. \tag{2}$$

This is exactly the ***normal vector*** **N** to the tangent plane and the surface, from Chapter 13. Please note: The small flat piece is actually a parallelogram (not always