

44 Show that the spin field $\mathbf{S}$ does work around every simple closed curve.

45 For $\mathbf{F} = f(x)\mathbf{j}$ and $R = \text{unit square } 0 \leq x \leq 1, 0 \leq y \leq 1$, integrate both sides of Green's Theorem (1). What formula is required from one-variable calculus?

46 A region R is "simply connected" when every closed curve

inside R can be squeezed to a point without leaving R . Test these regions:

1. xy plane without $(0, 0)$
2. xyz space without $(0, 0, 0)$
3. sphere $x^2 + y^2 + z^2 = 1$
4. a torus (or doughnut)
5. a sweater
6. a human body
7. the region between two spheres
8. xyz space with circle removed.

15.4 Surface Integrals

The double integral in Green's Theorem is over a flat surface R . Now the region moves out of the plane. It becomes a **curved surface** S , part of a sphere or cylinder or cone. When the surface has only one z for each (x, y) , it is the graph of a function $z(x, y)$. In other cases S can twist and close up—a sphere has an upper z and a lower z . In all cases we want to compute area and flux. This is a necessary step (it is our last step) before moving Green's Theorem to three dimensions.

First a quick review. The basic integrals are $\int dx$ and $\iint dx dy$ and $\iiint dx dy dz$. The one that didn't fit was $\int ds$ —the length of a curve. When we go from curves to surfaces, ds becomes dS . **Area is $\iint dS$ and flux is $\iint \mathbf{F} \cdot \mathbf{n} dS$** , with double integrals because the surfaces are two-dimensional. The main difficulty is in dS .

All formulas are summarized in a table at the end of the section.

There are two ways to deal with ds (along curves). The same methods apply to dS (on surfaces). The first is in xyz coordinates; the second uses parameters. Before this subject gets complicated, I will explain those two methods.

Method 1 is for the graph of a function: curve $y(x)$ or surface $z(x, y)$.

A small piece of the curve is almost straight. It goes across by dx and up by dy :

$$\text{length } ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (dy/dx)^2} dx. \quad (1)$$

A small piece of the surface is practically flat. Think of a tiny sloping rectangle. One side goes across by dx and up by $(\partial z/\partial x)dx$. The neighboring side goes along by dy and up by $(\partial z/\partial y)dy$. Computing the area is a linear problem (from Chapter 11), because the flat piece is in a plane.

Two vectors $\mathbf{A}$ and $\mathbf{B}$ form a parallelogram. **The length of their cross product is the area.** In the present case, the vectors are $\mathbf{A} = \mathbf{i} + (\partial z/\partial x)\mathbf{k}$ and $\mathbf{B} = \mathbf{j} + (\partial z/\partial y)\mathbf{k}$. Then $\mathbf{A}dx$ and $\mathbf{B}dy$ are the sides of the small piece, and we compute $\mathbf{A} \times \mathbf{B}$:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \partial z/\partial x \\ 0 & 1 & \partial z/\partial y \end{vmatrix} = -\partial z/\partial x \mathbf{i} - \partial z/\partial y \mathbf{j} + \mathbf{k}. \quad (2)$$

This is exactly the **normal vector** $\mathbf{N}$ to the tangent plane and the surface, from Chapter 13. Please note: The small flat piece is actually a parallelogram (not always

a rectangle). Its area dS is much like ds , but the length of $\mathbf{N} = \mathbf{A} \times \mathbf{B}$ involves two derivatives:

$$\text{area } dS = |\mathbf{A}dx \times \mathbf{B}dy| = |\mathbf{N}|dx dy = \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2} dx dy. \quad (3)$$

EXAMPLE 1 Find the area on the plane $z = x + 2y$ above a base area A .

This is the example to visualize. The area down in the xy plane is A . The area up on the sloping plane is greater than A . A roof has more area than the room underneath it. If the roof goes up at a 45° angle, the ratio is $\sqrt{2}$. Formula (3) yields the correct ratio for any surface—including our plane $z = x + 2y$.

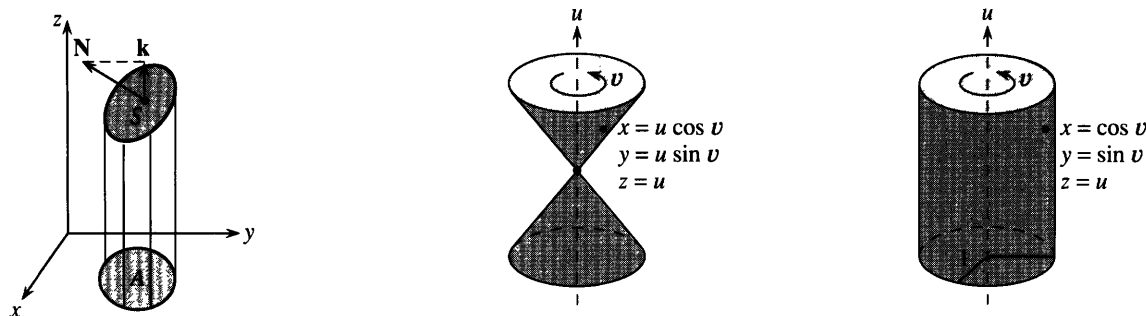


Fig. 15.14 Roof area = base area times $|\mathbf{N}|$. Cone and cylinder with parameters u and v .

The derivatives are $\partial z/\partial x = 1$ and $\partial z/\partial y = 2$. They are constant (planes are easy). The square root in (3) contains $1 + 1^2 + 2^2 = 6$. Therefore $dS = \sqrt{6} dx dy$. An area in the xy plane is multiplied by $\sqrt{6}$ up in the surface (Figure 15.14a). The vectors $\mathbf{A}$ and $\mathbf{B}$ are no longer needed—their work was done when we reached formula (3)—but here they are:

$$\mathbf{A} = \mathbf{i} + (\partial z/\partial x)\mathbf{k} = \mathbf{i} + \mathbf{k} \quad \mathbf{B} = \mathbf{j} + (\partial z/\partial y)\mathbf{k} = \mathbf{j} + 2\mathbf{k} \quad \mathbf{N} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

The length of $\mathbf{N} = \mathbf{A} \times \mathbf{B}$ is $\sqrt{6}$. The angle between $\mathbf{k}$ and $\mathbf{N}$ has $\cos \theta = 1/\sqrt{6}$. *That is the angle between base plane and sloping plane.* Therefore the sloping area is $\sqrt{6}$ times the base area. For curved surfaces the idea is the same, except that the square root in $|\mathbf{N}| = 1/\cos \theta$ changes as we move around the surface.

Method 2 is for curves $x(t)$, $y(t)$ and surfaces $x(u, v)$, $y(u, v)$, $z(u, v)$ with parameters.

A curve has one parameter t . A surface has two parameters u and v (it is two-dimensional). One advantage of parameters is that x , y , z get equal treatment, instead of picking out z as $f(x, y)$. Here are the first two examples:

$$\text{cone } x = u \cos v, y = u \sin v, z = u \quad \text{cylinder } x = \cos v, y = \sin v, z = u. \quad (4)$$

Each choice of u and v gives a point on the surface. By making all choices, we get the complete surface. Notice that a parameter can equal a coordinate, as in $z = u$. Sometimes both parameters are coordinates, as in $x = u$ and $y = v$ and $z = f(u, v)$. That is just $z = f(x, y)$ in disguise—the surface without parameters. In other cases *we find the xyz equation by eliminating u and v :*

$$\text{cone } (u \cos v)^2 + (u \sin v)^2 = u^2 \quad \text{or} \quad x^2 + y^2 = z^2 \quad \text{or} \quad z = \sqrt{x^2 + y^2}$$

$$\text{cylinder } (\cos v)^2 + (\sin v)^2 = 1 \quad \text{or} \quad x^2 + y^2 = 1.$$

The cone is the graph of $f = \sqrt{x^2 + y^2}$. The cylinder is *not* the graph of any function. There is a line of z 's through each point on the circle $x^2 + y^2 = 1$. That is what $z = u$ tells us: Give u all values, and you get the whole line. Give u and v all values, and you get the whole cylinder. Parameters allow a surface to close up and even go through itself—which the graph of $f(x, y)$ can never do.

Actually $z = \sqrt{x^2 + y^2}$ gives only the top half of the cone. (A function produces only one z .) The parametric form gives the bottom half also. Similarly $y = \sqrt{1 - x^2}$ gives only the top of a circle, while $x = \cos t$, $y = \sin t$ goes all the way around.

Now we find dS , using parameters. Small movements give a piece of the surface, practically flat. One side comes from the change du , the neighboring side comes from dv . The two sides are given by small vectors $\mathbf{A}du$ and $\mathbf{B}dv$:

$$\mathbf{A} = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k} \quad \text{and} \quad \mathbf{B} = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k}. \quad (5)$$

To find the area dS of the parallelogram, start with the cross product $\mathbf{N} = \mathbf{A} \times \mathbf{B}$:

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \left(\frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \frac{\partial y}{\partial v} \right) \mathbf{i} + \left(\frac{\partial z}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial x}{\partial u} \frac{\partial z}{\partial v} \right) \mathbf{j} + \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right) \mathbf{k}. \quad (6)$$

Admittedly this looks complicated—actual examples are often fairly simple. The area dS of the small piece of surface is $|\mathbf{N}| du dv$. The length $|\mathbf{N}|$ is a square root:

$$dS = \sqrt{\left(\frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \frac{\partial y}{\partial v} \right)^2 + \left(\frac{\partial z}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial x}{\partial u} \frac{\partial z}{\partial v} \right)^2 + \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right)^2} du dv. \quad (7)$$

EXAMPLE 2 Find $\mathbf{A}$ and $\mathbf{B}$ and $\mathbf{N} = \mathbf{A} \times \mathbf{B}$ and dS for the cone and cylinder.

The cone has $x = u \cos v$, $y = u \sin v$, $z = u$. The u derivatives produce $\mathbf{A} = \partial \mathbf{R} / \partial u = \cos v \mathbf{i} + \sin v \mathbf{j} + \mathbf{k}$. The v derivatives produce the other tangent vector $\mathbf{B} = \partial \mathbf{R} / \partial v = -u \sin v \mathbf{i} + u \cos v \mathbf{j}$. The normal vector is $\mathbf{A} \times \mathbf{B} = -u \cos v \mathbf{i} - u \sin v \mathbf{j} + u \mathbf{k}$. Its length gives dS :

$$dS = |\mathbf{A} \times \mathbf{B}| du dv = \sqrt{(u \cos v)^2 + (u \sin v)^2 + u^2} du dv = \sqrt{2} u du dv.$$

The cylinder is even simpler: $dS = du dv$. In these and many other examples, $\mathbf{A}$ is perpendicular to $\mathbf{B}$. **The small piece is a rectangle.** Its sides have length $|\mathbf{A}| du$ and $|\mathbf{B}| dv$. (The cone has $|\mathbf{A}| = u$ and $|\mathbf{B}| = \sqrt{2}$, the cylinder has $|\mathbf{A}| = |\mathbf{B}| = 1$). The cross product is hardly needed for area, when we can just multiply $|\mathbf{A}| du$ times $|\mathbf{B}| dv$.

Remark on the two methods Method 1 also used parameters, but a very special choice— u is x and v is y . The parametric equations are $x = x$, $y = y$, $z = f(x, y)$. If you go through the long square root in (7), changing u to x and v to y , it simplifies to the square root in (3). (The terms $\partial y / \partial x$ and $\partial x / \partial y$ are zero; $\partial x / \partial x$ and $\partial y / \partial y$ are 1.) Still it pays to remember the shorter formula from Method 1.

Don't forget that after computing dS , you have to integrate it. Many times the good way is with polar coordinates. Surfaces are often symmetric around an axis or a point. Those are the *surfaces of revolution*—which we saw in Chapter 8 and will come back to.

Strictly speaking, the integral starts with ΔS (not dS). A flat piece has area $|\mathbf{A} \times \mathbf{B}| \Delta x \Delta y$ or $|\mathbf{A} \times \mathbf{B}| \Delta u \Delta v$. The area of a curved surface is properly defined as a limit. The key step of calculus, from sums of ΔS to the integral of dS , is safe for

smooth surfaces. In examples, the hard part is computing the double integral and substituting the limits on x , y or u , v .

EXAMPLE 3 Find the surface area of the cone $z = \sqrt{x^2 + y^2}$ up to the height $z = a$.

We use Method 1 (no parameters). The derivatives of z are computed, squared, and added:

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} \quad |\mathbf{N}|^2 = 1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} = 2.$$

Conclusion: $|\mathbf{N}| = \sqrt{2}$ and $dS = \sqrt{2} \, dx \, dy$. The cone is on a 45° slope, so the area $dx \, dy$ in the base is multiplied by $\sqrt{2}$ in the surface above it (Figure 15.15). The square root in dS accounts for the extra area due to slope. A horizontal surface has $dS = \sqrt{1} \, dx \, dy$, as we have known all year.

Now for a key point. **The integration is down in the base plane.** The limits on x and y are given by the “shadow” of the cone. To locate that shadow set $z = \sqrt{x^2 + y^2}$ equal to $z = a$. The plane cuts the cone at the circle $x^2 + y^2 = a^2$. We integrate over the inside of that circle (where the shadow is):

$$\text{surface area of cone} = \iint_{\text{shadow}} \sqrt{2} \, dx \, dy = \sqrt{2} \pi a^2.$$

EXAMPLE 4 Find the same area using $dS = \sqrt{2} \, u \, du \, dv$ from Example 2.

With parameters, dS looks different and the shadow in the base looks different. The circle $x^2 + y^2 = a^2$ becomes $u^2 \cos^2 v + u^2 \sin^2 v = a^2$. In other words $u = a$. (The cone has $z = u$, the plane has $z = a$, they meet when $u = a$.) The angle parameter v goes from 0 to 2π . The effect of these parameters is to switch us “automatically” to polar coordinates, where area is $r \, dr \, d\theta$:

$$\text{surface area of cone} = \iint dS = \int_0^{2\pi} \int_0^a \sqrt{2} \, u \, du \, dv = \sqrt{2} \pi a^2.$$

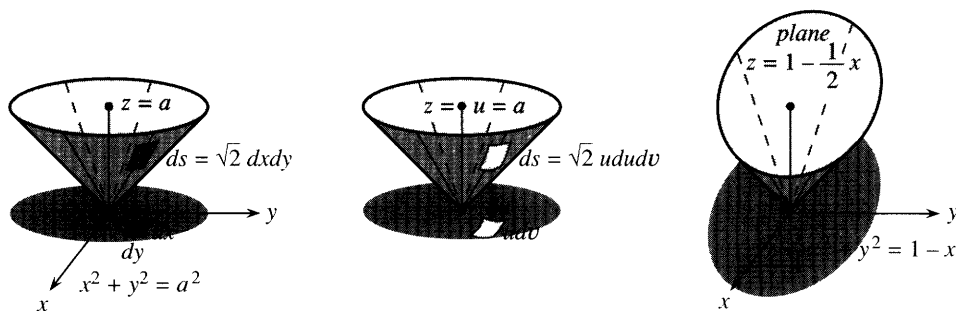


Fig. 15.15 Cone cut by plane leaves shadow in the base. Integrate over the shadow.

EXAMPLE 5 Find the area of the same cone up to the sloping plane $z = 1 - \frac{1}{2}x$.

Solution The cone still has $dS = \sqrt{2} \, dx \, dy$, but the limits of integration are changed. The plane cuts the cone in an ellipse. Its shadow down in the xy plane is another ellipse (Figure 15.15c). **To find the edge of the shadow, set $z = \sqrt{x^2 + y^2}$ equal to $z = 1 - \frac{1}{2}x$.** We square both sides:

$$x^2 + y^2 = 1 - x + \frac{1}{4}x^2 \quad \text{or} \quad \frac{3}{4}(x + \frac{2}{3})^2 + y^2 = \frac{4}{3}.$$

This is the ellipse in the base—where height makes no difference and z is gone. The area of an ellipse is πab , when the equation is in the form $(x/a)^2 + (y/b)^2 = 1$. After multiplying by $3/4$ we find $a = 4/3$ and $b = \sqrt{4/3}$. Then $\iint \sqrt{2} \, dx \, dy = \sqrt{2} \pi ab$ is the surface area of the cone.

The hard part was finding the shadow ellipse (I went quickly). Its area πab came from Example 15.3.2. The new part is $\sqrt{2}$ from the slope.

EXAMPLE 6 Find the surface area of a sphere of radius a (known to be $4\pi a^2$).

This is a good example, because both methods almost work. The equation of the sphere is $x^2 + y^2 + z^2 = a^2$. Method 1 writes $z = \sqrt{a^2 - x^2 - y^2}$. The x and y derivatives are $-x/z$ and $-y/z$:

$$1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{z^2}{z^2} + \frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{a^2}{z^2} = \frac{a^2}{a^2 - x^2 - y^2}.$$

The square root gives $dS = a \, dx \, dy / \sqrt{a^2 - x^2 - y^2}$. Notice that z is gone (as it should be). Now integrate dS over the shadow of the sphere, which is a circle. Instead of $dx \, dy$, switch to polar coordinates and $r \, dr \, d\theta$:

$$\iint_{\text{shadow}} dS = \int_0^{2\pi} \int_0^a \frac{a r \, dr \, d\theta}{\sqrt{a^2 - r^2}} = -2\pi a \sqrt{a^2 - r^2} \Big|_0^a = 2\pi a^2. \quad (8)$$

This calculation is successful but wrong. $2\pi a^2$ is the area of the *half-sphere* above the xy plane. The lower half takes the negative square root of $z^2 = a^2 - x^2 - y^2$. This shows the danger of Method 1, when the surface is not the graph of a function.

EXAMPLE 7 (same sphere by Method 2: *use parameters*) The natural choice is spherical coordinates. Every point has an angle $u = \phi$ down from the North Pole and an angle $v = \theta$ around the equator. The xyz coordinates from Section 14.4 are $x = a \sin \phi \cos \theta$, $y = a \sin \phi \sin \theta$, $z = a \cos \phi$. The radius $\rho = a$ is fixed (not a parameter). Compute the first term in equation (6), noting $\partial z / \partial \theta = 0$:

$$(\partial y / \partial \phi)(\partial z / \partial \theta) - (\partial z / \partial \phi)(\partial y / \partial \theta) = -(-a \sin \phi)(a \sin \phi \cos \theta) = a^2 \sin^2 \phi \cos \theta.$$

The other terms in (6) are $a^2 \sin^2 \phi \sin \theta$ and $a^2 \sin \phi \cos \phi$. Then dS in equation (7) squares these three components and adds. We factor out a^4 and simplify:

$$a^4(\sin^4 \phi \cos^2 \theta + \sin^4 \phi \sin^2 \theta + \sin^2 \phi \cos^2 \phi) = a^4(\sin^4 \phi + \sin^2 \phi \cos^2 \phi) = a^4 \sin^2 \phi.$$

Conclusion: $dS = a^2 \sin \phi \, d\phi \, d\theta$. A spherical person will recognize this immediately. It is the volume element $dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$, except $d\rho$ is missing. The small box has area dS and thickness $d\rho$ and volume dV . Here we only want dS :

$$\text{area of sphere} = \iint dS = \int_0^{2\pi} \int_0^\pi a^2 \sin \phi \, d\phi \, d\theta = 4\pi a^2. \quad (9)$$

Figure 15.16a shows a small surface with sides $a \, d\phi$ and $a \sin \phi \, d\theta$. Their product is dS . Figure 15.16b goes back to Method 1, where equation (8) gave $dS = (a/z) \, dx \, dy$.

I doubt that you will like Figure 15.16c—and you don't need it. With parameters ϕ and θ , the shadow of the sphere is a rectangle. The equator is the line down the middle, where $\phi = \pi/2$. The height is $z = a \cos \phi$. The area $d\phi \, d\theta$ in the base is the shadow of $dS = a^2 \sin \phi \, d\phi \, d\theta$ up in the sphere. Maybe this figure shows what we don't have to know about parameters.

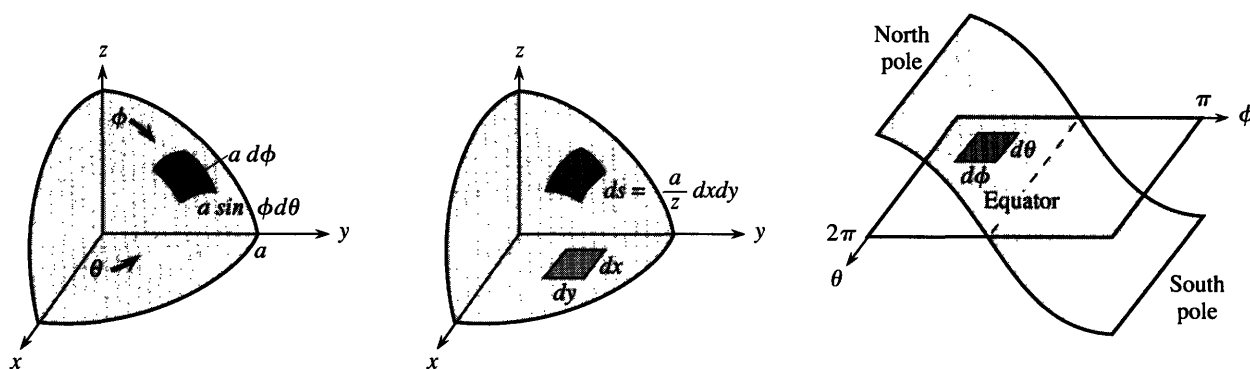


Fig. 15.16 Surface area on a sphere: (a) spherical coordinates (b) xyz coordinates (c) $\phi\theta$ space.

EXAMPLE 8 Rotate $y = x^2$ around the x axis. Find the surface area using parameters.

The first parameter is x (from a to b). The second parameter is the rotation angle θ (from 0 to 2π). The points on the surface in Figure 15.17 are $x = x$, $y = x^2 \cos \theta$, $z = x^2 \sin \theta$. Equation (7) leads after much calculation to $dS = x^2 \sqrt{1 + 4x^2} dx d\theta$.

Main point: dS agrees with Section 8.3, where the area was $\int 2\pi y \sqrt{1 + (dy/dx)^2} dx$. The 2π comes from the θ integral and y is x^2 . Parameters give this formula automatically.

VECTOR FIELDS AND THE INTEGRAL OF $\mathbf{F} \cdot \mathbf{n}$

Formulas for surface area are dominated by square roots. There is a square root in dS , as there was in ds . Areas are like arc lengths, one dimension up. The good point about line integrals $\int \mathbf{F} \cdot \mathbf{n} ds$ is that the square root disappears. It is in the denominator of $\mathbf{n}$, where ds cancels it: $\mathbf{F} \cdot \mathbf{n} ds = M dy - N dx$. The same good thing will now happen for surface integrals $\iint \mathbf{F} \cdot \mathbf{n} dS$.

15I Through the surface $z = f(x, y)$, the vector field $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ has

$$\text{flux} = \iint_{\text{surface}} \mathbf{F} \cdot \mathbf{n} dS = \iint_{\text{shadow}} \left(-M \frac{\partial f}{\partial x} - N \frac{\partial f}{\partial y} + P \right) dx dy. \quad (10)$$

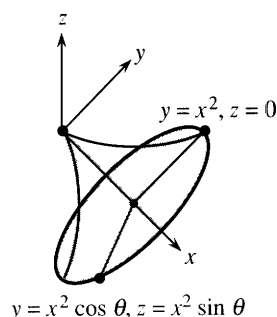
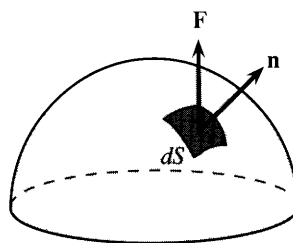
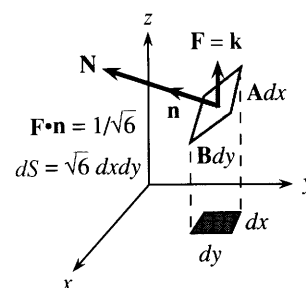
This formula tells what to integrate, given the surface and the vector field (f and $\mathbf{F}$). The xy limits come from the shadow. Formula (10) takes the normal vector from Method 1:

$$\mathbf{N} = -\partial f / \partial x \mathbf{i} - \partial f / \partial y \mathbf{j} + \mathbf{k} \text{ and } |\mathbf{N}| = \sqrt{1 + (\partial f / \partial x)^2 + (\partial f / \partial y)^2}.$$

For the *unit* normal vector $\mathbf{n}$, divide $\mathbf{N}$ by its length: $\mathbf{n} = \mathbf{N} / |\mathbf{N}|$. The square root is in the denominator, and the same square root is in dS . See equation (3):

$$\mathbf{F} \cdot \mathbf{n} dS = \frac{\mathbf{F} \cdot \mathbf{N}}{\sqrt{\quad}} \sqrt{\quad} dx dy = \left(-M \frac{\partial f}{\partial x} - N \frac{\partial f}{\partial y} + P \right) dx dy. \quad (11)$$

That is formula (10), with cancellation of square roots. The expression $\mathbf{F} \cdot \mathbf{n} dS$ is often written as $\mathbf{F} \cdot d\mathbf{S}$, again relying on boldface to make $d\mathbf{S}$ a vector. Then $d\mathbf{S}$ equals $\mathbf{n} dS$, with direction $\mathbf{n}$ and magnitude dS .

Fig. 15.17 Surface of revolution: parameters x, θ .Fig. 15.18 $\mathbf{F} \cdot \mathbf{n} dS$ gives flow through dS .

EXAMPLE 9 Find $\mathbf{n} dS$ for the plane $z = x + 2y$. Then find $\mathbf{F} \cdot \mathbf{n} dS$ for $\mathbf{F} = \mathbf{k}$.

This plane produced $\sqrt{6}$ in Example 1 (for area). For flux the $\sqrt{6}$ disappears:

$$\mathbf{n} dS = \frac{\mathbf{N}}{|\mathbf{N}|} dS = \frac{-\mathbf{i} - 2\mathbf{j} + \mathbf{k}}{\sqrt{6}} \sqrt{6} dx dy = (-\mathbf{i} - 2\mathbf{j} + \mathbf{k}) dx dy.$$

For the flow field $\mathbf{F} = \mathbf{k}$, the dot product $\mathbf{k} \cdot \mathbf{n} dS$ reduces to $1 dx dy$. The slope of the plane makes no difference! **The flow through the base also flows through the plane.** The areas are different, but flux is like rain. Whether it hits a tent or the ground below, it is the same rain (Figure 15.18). In this case $\iint \mathbf{F} \cdot \mathbf{n} dS = \iint dx dy = \text{shadow area in the base.}$

EXAMPLE 10 Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through the cone $z = \sqrt{x^2 + y^2}$.

$$\text{Solution } \mathbf{F} \cdot \mathbf{n} dS = \left[-x \left(\frac{x}{\sqrt{x^2 + y^2}} \right) - y \left(\frac{y}{\sqrt{x^2 + y^2}} \right) + \sqrt{x^2 + y^2} \right] dx dy = 0.$$

The zero comes as a surprise, but it shouldn't. The cone goes straight out from the origin, and so does $\mathbf{F}$. The vector $\mathbf{n}$ that is perpendicular to the cone is also perpendicular to $\mathbf{F}$. There is no flow *through* the cone, because $\mathbf{F} \cdot \mathbf{n} = 0$. The flow travels out along rays.

$\iint \mathbf{F} \cdot \mathbf{n} dS$ FOR A SURFACE WITH PARAMETERS

In Example 10 the cone was $z = f(x, y) = \sqrt{x^2 + y^2}$. We found dS by Method 1. Parameters were not needed (more exactly, they were x and y). For surfaces that fold and twist, the formulas with u and v look complicated but the actual calculations can be simpler. This was certainly the case for $dS = du dv$ on the cylinder.

A small piece of surface has area $dS = |\mathbf{A} \times \mathbf{B}| du dv$. The vectors along the sides are $\mathbf{A} = x_u \mathbf{i} + y_u \mathbf{j} + z_u \mathbf{k}$ and $\mathbf{B} = x_v \mathbf{i} + y_v \mathbf{j} + z_v \mathbf{k}$. They are tangent to the surface. Now we put their cross product $\mathbf{N} = \mathbf{A} \times \mathbf{B}$ to another use, because $\mathbf{F} \cdot \mathbf{n} dS$ involves not only area but *direction*. We need the unit vector $\mathbf{n}$ to see how much flow goes through.

The direction vector is $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$. Equation (7) is $dS = |\mathbf{N}| du dv$, so the square root $|\mathbf{N}|$ cancels in $\mathbf{n} dS$. This leaves a nice formula for the "normal component" of flow:

15J Through a surface with parameters u and v , the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ has

$$\text{flux} = \iint \mathbf{F} \cdot \mathbf{n} dS = \iint \mathbf{F} \cdot \mathbf{N} du dv = \iint \mathbf{F} \cdot (\mathbf{A} \times \mathbf{B}) du dv. \quad (12)$$

EXAMPLE 11 Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through the cylinder $x^2 + y^2 = 1$, $0 \leq z \leq b$.

Solution The surface of the cylinder is $x = \cos u$, $y = \sin u$, $z = v$. The tangent vectors from (5) are $\mathbf{A} = (-\sin u)\mathbf{i} + (\cos u)\mathbf{j}$ and $\mathbf{B} = \mathbf{k}$. The normal vector in Figure 15.19 goes straight out through the cylinder:

$$\mathbf{N} = \mathbf{A} \times \mathbf{B} = \cos u \mathbf{i} + \sin u \mathbf{j} \quad (\text{check } \mathbf{A} \cdot \mathbf{N} = 0 \text{ and } \mathbf{B} \cdot \mathbf{N} = 0).$$

To find $\mathbf{F} \cdot \mathbf{N}$, switch $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ to the parameters u and v . Then $\mathbf{F} \cdot \mathbf{N} = 1$:

$$\mathbf{F} \cdot \mathbf{N} = (\cos u \mathbf{i} + \sin u \mathbf{j} + v \mathbf{k}) \cdot (\cos u \mathbf{i} + \sin u \mathbf{j}) = \cos^2 u + \sin^2 u.$$

For the flux, integrate $\mathbf{F} \cdot \mathbf{N} = 1$ and apply the limits on $u = \theta$ and $v = z$:

$$\text{flux} = \int_0^b \int_0^{2\pi} 1 \, du \, dv = 2\pi b = \text{surface area of the cylinder}.$$

Note that the top and bottom were not included! We can find those fluxes too. The outward direction is $\mathbf{n} = \mathbf{k}$ at the top and $\mathbf{n} = -\mathbf{k}$ down through the bottom. Then $\mathbf{F} \cdot \mathbf{n}$ is $+z = b$ at the top and $-z = 0$ at the bottom. The bottom flux is zero, the top flux is b times the area (or πb). The total flux is $2\pi b + \pi b = 3\pi b$. Hold that answer for the next section.

Apology: I made u the angle and v the height. Then $\mathbf{N}$ goes outward not inward.

EXAMPLE 12 Find the flux of $\mathbf{F} = \mathbf{k}$ out the top half of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution Use spherical coordinates. Example 7 had $u = \phi$ and $v = \theta$. We found

$$\mathbf{N} = \mathbf{A} \times \mathbf{B} = a^2 \sin^2 \phi \cos \theta \mathbf{i} + a^2 \sin^2 \phi \sin \theta \mathbf{j} + a^2 \sin \phi \cos \phi \mathbf{k}.$$

The dot product with $\mathbf{F} = \mathbf{k}$ is $\mathbf{F} \cdot \mathbf{N} = a^2 \sin \phi \cos \phi$. The integral goes from the pole to the equator, $\phi = 0$ to $\phi = \pi/2$, and around from $\theta = 0$ to $\theta = 2\pi$:

$$\text{flux} = \int_0^{2\pi} \int_0^{\pi/2} a^2 \sin \phi \cos \phi \, d\phi \, d\theta = 2\pi a^2 \left[\frac{\sin^2 \phi}{2} \right]_0^{\pi/2} = \pi a^2.$$

The next section will show that the flux remains at πa^2 through *any surface* (!) that is bounded by the equator. A special case is a flat surface—the disk of radius a at the equator. Figure 15.18 shows $\mathbf{n} = \mathbf{k}$ pointing directly up, so $\mathbf{F} \cdot \mathbf{n} = \mathbf{k} \cdot \mathbf{k} = 1$. The flux is $\iint 1 \, dS = \text{area of disk} = \pi a^2$. *All fluid goes past the equator and out through the sphere.*

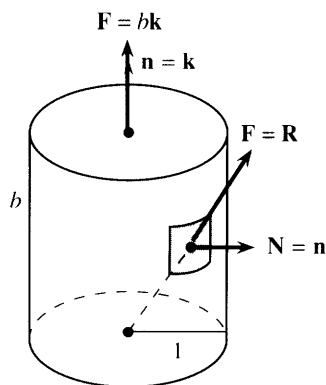


Fig. 15.19 Flow through cylinder.

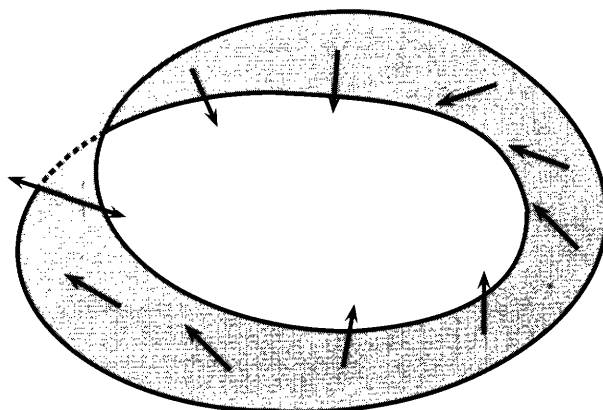


Fig. 15.20 Möbius strip (no way to choose $\mathbf{n}$).

I have to mention one more problem. It might not occur to a reasonable person, but sometimes a surface has only *one side*. The famous example is the **Möbius strip**, for which you take a strip of paper, twist it once, and tape the ends together. Its special property appears when you run a pen along the “inside.” The pen in Figure 15.20 suddenly goes “outside.” After another round trip it goes back “inside.” Those words are in quotation marks, because on a Möbius strip they have no meaning.

Suppose the pen represents the normal vector. On a sphere $\mathbf{n}$ points outward. Alternatively $\mathbf{n}$ could point inward; we are free to choose. But the Möbius strip makes the choice impossible. After moving the pen continuously, it comes back in the opposite direction. **This surface is not orientable.** We cannot integrate $\mathbf{F} \cdot \mathbf{n}$ to compute the flux, because we cannot decide the direction of $\mathbf{n}$.

A surface is *oriented* when we can and do choose $\mathbf{n}$. This uses the final property of cross products, that they have length and direction and also a *right-hand rule*. We can tell $\mathbf{A} \times \mathbf{B}$ from $\mathbf{B} \times \mathbf{A}$. Those give the two orientations of $\mathbf{n}$. For an open surface (like a wastebasket) you can select either one. For a closed surface (like a sphere) it is conventional for $\mathbf{n}$ to be outward. By making that decision once and for all, the sign of the flux is established: *outward flux is positive*.

FORMULAS FOR SURFACE INTEGRALS

Method 1: Parameters x, y	Method 2: Parameters u, v
Coordinates $x, y, z(x, y)$	$x(u, v), y(u, v), z(u, v)$ on surface
$\mathbf{A} = \mathbf{i} + \partial z / \partial x \mathbf{k} \quad \mathbf{N} = \mathbf{A} \times \mathbf{B}$	$\mathbf{A} = \partial x / \partial u \mathbf{i} + \partial y / \partial u \mathbf{j} + \partial z / \partial u \mathbf{k}$
$\mathbf{B} = \mathbf{j} + \partial z / \partial y \mathbf{k} \quad \mathbf{n} = \mathbf{N} / \mathbf{N} $	$\mathbf{B} = \partial x / \partial v \mathbf{i} + \partial y / \partial v \mathbf{j} + \partial z / \partial v \mathbf{k}$
$dS = \mathbf{N} dx dy = \sqrt{1 + z_x^2 + z_y^2} dx dy$	$dS = \mathbf{N} du dv$
$\mathbf{n} dS = \mathbf{N} dx dy = (-\partial z / \partial x \mathbf{i} - \partial z / \partial y \mathbf{j} + \mathbf{k}) dx dy$	$\mathbf{n} dS = \mathbf{N} du dv$

15.4 EXERCISES

Read-through questions

A small piece of the surface $z = f(x, y)$ is nearly a. When we go across by dx , we go up by b. That movement is $\mathbf{A}dx$, where the vector $\mathbf{A}$ is $\mathbf{i} + \underline{\text{c}}$. The other side of the piece is $\mathbf{B}dy$, where $\mathbf{B} = \mathbf{j} + \underline{\text{d}}$. The cross product $\mathbf{A} \times \mathbf{B}$ is $\mathbf{N} = \underline{\text{e}}$. The area of the piece is $dS = |\mathbf{N}| dx dy$. For the surface $z = xy$, the vectors are $\mathbf{A} = \underline{\text{f}}$ and $\mathbf{B} = \underline{\text{g}}$ and $\mathbf{N} = \underline{\text{h}}$. The area integral is $\iint dS = \underline{\text{i}} dx dy$.

With parameters u and v , a typical point on a 45° cone is $x = u \cos v$, $y = \underline{\text{j}}$, $z = \underline{\text{k}}$. A change in u moves that point by $\mathbf{A}du = (\cos v \mathbf{i} + \underline{\text{l}})du$. A change in v moves the point by $\mathbf{B}dv = \underline{\text{m}}$. The normal vector is $\mathbf{N} = \mathbf{A} \times \mathbf{B} = \underline{\text{n}}$. The area is $dS = \underline{\text{o}} du dv$. In this example $\mathbf{A} \cdot \mathbf{B} = \underline{\text{p}}$ so the small piece is a q and $dS = |\mathbf{A}| |\mathbf{B}| du dv$.

For flux we need $\mathbf{n}dS$. The r vector $\mathbf{n}$ is $\mathbf{N} = \mathbf{A} \times \mathbf{B}$ divided by s. For a surface $z = f(x, y)$, the product $\mathbf{n}dS$ is the vector t (to memorize from table). The particular

surface $z = xy$ has $\mathbf{n}dS = \underline{\text{u}} dx dy$. For $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ the flux through $z = xy$ is $\mathbf{F} \cdot \mathbf{n}dS = \underline{\text{v}} dx dy$.

On a 30° cone the points are $x = 2u \cos v$, $y = 2u \sin v$, $z = u$. The tangent vectors are $\mathbf{A} = \underline{\text{w}}$ and $\mathbf{B} = \underline{\text{x}}$. This cone has $\mathbf{n}dS = \mathbf{A} \times \mathbf{B} du dv = \underline{\text{y}}$. For $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the flux element through the cone is $\mathbf{F} \cdot \mathbf{n}dS = \underline{\text{z}}$. The reason for this answer is A. The reason we don't compute flux through a Möbius strip is B.

In 1–14 find $\mathbf{N}$ and $dS = |\mathbf{N}| dx dy$ and the surface area $\iint dS$. Integrate over the xy shadow which ends where the z 's are equal ($x^2 + y^2 = 4$ in Problem 1).

- 1 Paraboloid $z = x^2 + y^2$ below the plane $z = 4$.
- 2 Paraboloid $z = x^2 + y^2$ between $z = 4$ and $z = 8$.
- 3 Plane $z = x - y$ inside the cylinder $x^2 + y^2 = 1$.
- 4 Plane $z = 3x + 4y$ above the square $0 \leq x \leq 1, 0 \leq y \leq 1$.

- 5 Spherical cap $x^2 + y^2 + z^2 = 1$ above $z = 1/\sqrt{2}$.
- 6 Spherical band $x^2 + y^2 + z^2 = 1$ between $z = 0$ and $1/\sqrt{2}$.
- 7 Plane $z = 7y$ above a triangle of area A .
- 8 Cone $z^2 = x^2 + y^2$ between planes $z = a$ and $z = b$.
- 9 The monkey saddle $z = \frac{1}{3}x^3 - xy^2$ inside $x^2 + y^2 = 1$.
- 10 $z = x + y$ above triangle with vertices $(0, 0)$, $(2, 2)$, $(0, 2)$.
- 11 Plane $z = 1 - 2x - 2y$ inside $x \geq 0$, $y \geq 0$, $z \geq 0$.
- 12 Cylinder $x^2 + z^2 = a^2$ inside $x^2 + y^2 = a^2$. Only set up $\iint dS$.
- 13 Right circular cone of radius a and height h . Choose $z = f(x, y)$ or parameters u and v .
- 14 Gutter $z = x^2$ below $z = 9$ and between $y = \pm 2$.

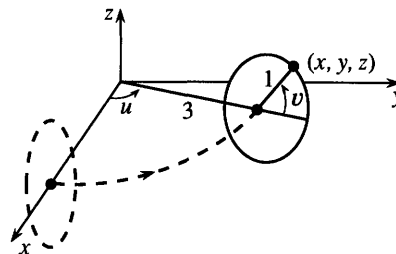
In 15–18 compute the surface integrals $\iint g(x, y, z) dS$.

- 15 $g = xy$ over the triangle $x + y + z = 1$, $x, y, z \geq 0$.
- 16 $g = x^2 + y^2$ over the top half of $x^2 + y^2 + z^2 = 1$ (use ϕ, θ).
- 17 $g = xyz$ on $x^2 + y^2 + z^2 = 1$ above $z^2 = x^2 + y^2$ (use ϕ, θ).
- 18 $g = x$ on the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 3$.

In 19–22 calculate A , B , N , and dS .

- 19 $x = u$, $y = v + u$, $z = v + 2u + 1$.
 - 20 $x = uv$, $y = u + v$, $z = u - v$.
 - 21 $x = (3 + \cos u) \cos v$, $y = (3 + \cos u) \sin v$, $z = \sin u$.
 - 22 $x = u \cos v$, $y = u \sin v$, $z = v$ (not $z = u$).
- 23–26 In Problems 1–4 respectively find the flux $\iint \mathbf{F} \cdot \mathbf{n} dS$ for $\mathbf{F} = xi + yj + zk$.

27–28 In Problems 19–20 respectively compute $\iint \mathbf{F} \cdot \mathbf{n} dS$ for $\mathbf{F} = yi - xj$ through the region $u^2 + v^2 \leq 1$.



29 A unit circle is rotated around the z axis to give a torus (see figure). The center of the circle stays a distance 3 from the z axis. Show that Problem 21 gives a typical point (x, y, z) on the torus and find the surface area $\iint dS = \iint |\mathbf{N}| du dv$.

30 The surface $x = r \cos \theta$, $y = r \sin \theta$, $z = a^2 - r^2$ is bounded by the equator ($r = a$). Find $\mathbf{N}$ and the flux $\iint \mathbf{k} \cdot \mathbf{n} dS$, and compare with Example 12.

31 Make a “double Möbius strip” from a strip of paper by twisting it twice and taping the ends. Does a normal vector (use a pen) have the same direction after a round trip?

32 Make a “triple Möbius strip” with three twists. Is it orientable—does the normal vector come back in the same or opposite direction?

33 If a very wavy surface stays close to a smooth surface, are their areas close?

34 Give the equation of a plane with roof area $dS = 3$ times base area $dx dy$.

35 The points $(x, f(x) \cos \theta, f(x) \sin \theta)$ are on the surface of revolution: $y = f(x)$ revolved around the x axis, parameters $u = x$ and $v = \theta$. Find $\mathbf{N}$ and compare $dS = |\mathbf{N}| dx d\theta$ with Example 8 and Section 8.3.

15.5 The Divergence Theorem

This section returns to the fundamental law (*flow out*) – (*flow in*) = (*source*). In two dimensions, the flow was in and out through a closed curve C . The plane region inside was R . In three dimensions, the flow enters and leaves through a closed surface S . The solid region inside is V . Green’s Theorem in its normal form (for the flux of a smooth vector field) now becomes the great three-dimensional balance equation—the *Divergence Theorem*:

15K The flux of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ through the boundary surface S equals the integral of the divergence of $\mathbf{F}$ inside V . *The Divergence Theorem is*

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_V \operatorname{div} \mathbf{F} dV = \iiint_V \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right) dx dy dz. \quad (1)$$

In Green's Theorem the divergence was $\partial M/\partial x + \partial N/\partial y$. The new term $\partial P/\partial z$ accounts for upward flow. Notice that a constant upward component P adds nothing to the divergence (its derivative is zero). It also adds nothing to the flux (flow up through the top equals flow up through the bottom). When the whole field $\mathbf{F}$ is constant, the theorem becomes $0 = 0$.

There are other vector fields with $\operatorname{div} \mathbf{F} = 0$. They are of the greatest importance. The Divergence Theorem for those fields is again $0 = 0$, and there is conservation of fluid. *When $\operatorname{div} \mathbf{F} = 0$, flow in equals flow out.* We begin with examples of these “divergence-free” fields.

EXAMPLE 1 The spin fields $-y\mathbf{i} + x\mathbf{j} + 0\mathbf{k}$ and $0\mathbf{i} - z\mathbf{j} + y\mathbf{k}$ have zero divergence.

The first is an old friend, spinning around the z axis. The second is new, spinning around the x axis. Three-dimensional flow has a great variety of spin fields. The separate terms $\partial M/\partial x$, $\partial N/\partial y$, $\partial P/\partial z$ are all zero, so $\operatorname{div} \mathbf{F} = 0$. The flow goes around in circles, and whatever goes out through S comes back in. (We might have put a circle on $\oint_S$ as we did on $\oint_C$, to emphasize that S is closed.)

EXAMPLE 2 The position field $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ has $\operatorname{div} \mathbf{R} = 1 + 1 + 1 = 3$.

This is radial flow, straight out from the origin. Mass has to be added *at every point* to keep the flow going. On the right side of the divergence theorem is $\iiint 3 dV$. Therefore the flux is three times the volume.

Example 11 in Section 15.4 found the flux of $\mathbf{R}$ through a cylinder. The answer was $3\pi b$. Now we also get $3\pi b$ from the Divergence Theorem, since the volume is πb . This is one of many cases in which the triple integral is easier than the double integral.

EXAMPLE 3 An *electrostatic field* $\mathbf{R}/\rho^3$ or *gravity field* $-\mathbf{R}/\rho^3$ almost has $\operatorname{div} \mathbf{F} = 0$.

The vector $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ has length $\sqrt{x^2 + y^2 + z^2} = \rho$. Then $\mathbf{F}$ has length ρ/ρ^3 (inverse square law). Gravity from a point mass pulls *inward* (minus sign). The electric field from a point charge repels *outward*. The three steps almost show that $\operatorname{div} \mathbf{F} = 0$:

Step 1. $\partial \rho/\partial x = x/\rho$, $\partial \rho/\partial y = y/\rho$, $\partial \rho/\partial z = z/\rho$ —but do not add those three. $\mathbf{F}$ is not ρ or $1/\rho^2$ (these are scalars). The vector field is $\mathbf{R}/\rho^3$. We need $\partial M/\partial x$, $\partial N/\partial y$, $\partial P/\partial z$.

Step 2. $\partial M/\partial x = \partial/\partial x(x/\rho^3)$ is equal to $1/\rho^3 - (3x \partial \rho/\partial x)/\rho^4 = 1/\rho^3 - 3x^2/\rho^5$. For $\partial N/\partial y$ and $\partial P/\partial z$, replace $3x^2$ by $3y^2$ and $3z^2$. Now add those three.

Step 3. $\operatorname{div} \mathbf{F} = 3/\rho^3 - 3(x^2 + y^2 + z^2)/\rho^5 = 3/\rho^3 - 3/\rho^3 = 0$.

The calculation $\operatorname{div} \mathbf{F} = 0$ leaves a puzzle. One side of the Divergence Theorem seems to give $\iiint 0 dV = 0$. Then the other side should be $\iint \mathbf{F} \cdot \mathbf{n} dS = 0$. But the flux is *not* zero when all flow is outward:

The unit normal vector to the sphere $\rho = \text{constant}$ is $\mathbf{n} = \mathbf{R}/\rho$.

The outward flow $\mathbf{F} \cdot \mathbf{n} = (\mathbf{R}/\rho^3) \cdot (\mathbf{R}/\rho) = \rho^2/\rho^4$ is always positive.

Then $\iint \mathbf{F} \cdot \mathbf{n} dS = \iint dS/\rho^2 = 4\pi\rho^2/\rho^2 = 4\pi$. *We have reached $4\pi = 0$.*

This paradox in three dimensions is the same as for $\mathbf{R}/r^2$ in two dimensions. Section 15.3 reached $2\pi = 0$, and the explanation was a point source at the origin. Same explanation here: M, N, P are infinite when $\rho = 0$. The divergence is a “delta function” times 4π , from the point source. The Divergence Theorem does not apply (unless we allow delta functions). That single point makes all the difference.

Every surface enclosing the origin has flux $= 4\pi$. Our calculation was for a sphere. The surface integral is much harder when S is twisted (Figure 15.21a). But the Divergence Theorem takes care of everything, because $\text{div } \mathbf{F} = 0$ in the volume V between these surfaces. Therefore $\iint \mathbf{F} \cdot \mathbf{n} dS = 0$ for the two surfaces together. The flux $\iint \mathbf{F} \cdot \mathbf{n} dS = -4\pi$ into the sphere must be balanced by $\iint \mathbf{F} \cdot \mathbf{n} dS = 4\pi$ out of the twisted surface.

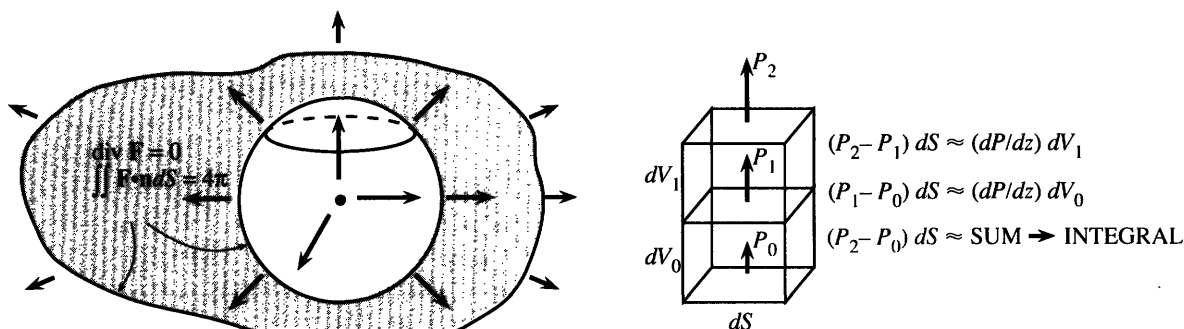


Fig. 15.21 Point source: flux 4π through all enclosing surfaces. Net flux upward $= \iiint (\partial P / \partial z) dV$.

Instead of a paradox $4\pi = 0$, this example leads to Gauss's Law. A mass M at the origin produces a gravity field $\mathbf{F} = -GM\mathbf{R}/\rho^3$. A charge q at the origin produces an electric field $\mathbf{E} = (q/4\pi\epsilon_0)\mathbf{R}/\rho^3$. The physical constants are G and ϵ_0 , the mathematical constant is the relation between divergence and flux. Equation (1) yields equation (2), in which the mass densities $M(x, y, z)$ and charge densities $q(x, y, z)$ need not be concentrated at the origin:

15L Gauss's law in differential form: $\text{div } \mathbf{F} = -4\pi GM$ and $\text{div } \mathbf{E} = q/\epsilon_0$.
Gauss's law in integral form: Flux is proportional to total mass or charge:

$$\iint \mathbf{F} \cdot \mathbf{n} dS = - \iiint 4\pi GM dV \text{ and } \iint \mathbf{E} \cdot \mathbf{n} dS = \iiint q dV / \epsilon_0. \quad (2)$$

THE REASONING BEHIND THE DIVERGENCE THEOREM

The general principle is clear: Flow out minus flow in equals source. Our goal is to see why *the divergence of $\mathbf{F}$ measures the source*. In a small box around each point, we show that $\text{div } \mathbf{F} dV$ balances $\mathbf{F} \cdot \mathbf{n} dS$ through the six sides.

So consider a small box. Its center is at (x, y, z) . Its edges have length $\Delta x, \Delta y, \Delta z$. Out of the top and bottom, the normal vectors are $\mathbf{k}$ and $-\mathbf{k}$. The dot product with $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is $+P$ or $-P$. The area ΔS is $\Delta x \Delta y$. So the two fluxes are close to $P(x, y, z + \frac{1}{2}\Delta z)\Delta x \Delta y$ and $-P(x, y, z - \frac{1}{2}\Delta z)\Delta x \Delta y$. When the top is combined with the bottom, the difference of those P 's is ΔP :

$$\text{net flux upward} \approx \Delta P \Delta x \Delta y = (\Delta P / \Delta z) \Delta x \Delta y \Delta z \approx (\partial P / \partial z) \Delta V. \quad (3)$$

Similarly, the combined flux on two side faces is approximately $(\partial N/\partial y)\Delta V$. On the front and back it is $(\partial M/\partial x)\Delta V$. Adding the six faces, we reach the key point:

$$\text{flux out of the box} \approx (\partial M/\partial x + \partial N/\partial y + \partial P/\partial z)\Delta V. \quad (4)$$

This is $(\operatorname{div} \mathbf{F})\Delta V$. For a constant field both sides are zero—the flow goes straight through. For $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, a little more goes out than comes in. The divergence is 3, so $3\Delta V$ is created inside the box. By the balance equation the flux is also $3\Delta V$.

The approximation symbol $\approx$ means that the leading term is correct (probably not the next term). The ratio $\Delta P/\Delta z$ is not exactly $\partial P/\partial z$. The difference is of order Δz , so the error in (3) is of higher order $\Delta V\Delta z$. Added over many boxes (about $1/\Delta V$ boxes), this error disappears as $\Delta z \rightarrow 0$.

The sum of $(\operatorname{div} \mathbf{F})\Delta V$ over all the boxes approaches $\iiint (\operatorname{div} \mathbf{F})dV$. On the other side of the equation is a sum of fluxes. There is $\mathbf{F} \cdot \mathbf{n}\Delta S$ out of the top of one box, plus $\mathbf{F} \cdot \mathbf{n}\Delta S$ out of the bottom of the box above. The first has $\mathbf{n} = \mathbf{k}$ and the second has $\mathbf{n} = -\mathbf{k}$. *They cancel each other—the flow goes from box to box.* This happens every time two boxes meet. The only fluxes that survive (because nothing cancels them) are at the outer surface S . The final step, as $\Delta x, \Delta y, \Delta z \rightarrow 0$, is that those outside terms approach $\iint \mathbf{F} \cdot \mathbf{n}dS$. Then the local divergence theorem (4) becomes the global Divergence Theorem (1).

Remark on the proof That “final step” is not easy, because the box surfaces don’t line up with the outer surface S . A formal proof of the Divergence Theorem would imitate the proof of Green’s Theorem. On a very simple region $\iiint (\partial P/\partial z)dx dy dz$ equals $\iint P dx dy$ over the top minus $\iint P dx dy$ over the bottom. After checking the orientation this is $\iint P\mathbf{k} \cdot \mathbf{n}dS$. Similarly the volume integrals of $\partial M/\partial x$ and $\partial N/\partial y$ are the surface integrals $\iint M\mathbf{i} \cdot \mathbf{n}dS$ and $\iint N\mathbf{j} \cdot \mathbf{n}dS$. Adding the three integrals gives the Divergence Theorem. Since Green’s Theorem was already proved in this way, the reasoning behind (4) is more helpful than repeating a detailed proof.

The discoverer of the Divergence Theorem was probably Gauss. His notebooks only contain the outline of a proof—but after all, this is Gauss. Green and Ostrogradsky both published proofs in 1828, one in England and the other in St. Petersburg (now Leningrad). As the theorem was studied, the requirements came to light (smoothness of $\mathbf{F}$ and S , avoidance of one-sided Möbius strips).

New applications are discovered all the time—*when a scientist writes down a balance equation in a small box*. The source is known. The equation is $\operatorname{div} \mathbf{F} = \text{source}$. After Example 5 we explain $\mathbf{F}$.

EXAMPLE 4 If the temperature inside the sun is $T = \ln 1/\rho$, find the heat flow $\mathbf{F} = -\operatorname{grad} T$ and the source $\operatorname{div} \mathbf{F}$ and the flux $\iint \mathbf{F} \cdot \mathbf{n}dS$. The sun is a ball of radius ρ_0 .

Solution $\mathbf{F}$ is $-\operatorname{grad} \ln 1/\rho = +\operatorname{grad} \ln \rho$. Derivatives of $\ln \rho$ bring division by ρ :

$$\mathbf{F} = (\partial \rho/\partial x \mathbf{i} + \partial \rho/\partial y \mathbf{j} + \partial \rho/\partial z \mathbf{k})/\rho = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\rho^2.$$

This flow is radially outward, of magnitude $1/\rho$. The normal vector $\mathbf{n}$ is also radially outward, of magnitude 1. The dot product on the sun’s surface is $1/\rho_0$:

$$\iint \mathbf{F} \cdot \mathbf{n}dS = \iint dS/\rho_0 = (\text{surface area})/\rho_0 = 4\pi\rho_0^2/\rho_0 = 4\pi\rho_0. \quad (5)$$

Check that answer by the Divergence Theorem. Example 5 will find $\operatorname{div} \mathbf{F} = 1/\rho^2$. Integrate over the sun. In spherical coordinates we integrate $d\rho$, $\sin \phi d\phi$, and $d\theta$:

$$\iiint_{\text{sun}} \operatorname{div} \mathbf{F} dV = \int_0^{2\pi} \int_0^\pi \int_0^{\rho_0} \rho^2 \sin \phi d\rho d\phi d\theta/\rho^2 = (\rho_0)(2)(2\pi) \text{ as in (5).}$$

This example illustrates *the basic framework of equilibrium*. The pattern appears everywhere in applied mathematics—electromagnetism, heat flow, elasticity, even relativity. There is usually a constant c that depends on the material (the example has $c = 1$). The names change, but we always take *the divergence of the gradient*:

potential $f \rightarrow$ force field $-c \text{ grad } f$	Then $\text{div}(-c \text{ grad } f) =$ electric charge
temperature $T \rightarrow$ flow field $-c \text{ grad } T$	Then $\text{div}(-c \text{ grad } T) =$ heat source
displacement $u \rightarrow$ stress field $+c \text{ grad } u$	Then $\text{div}(-c \text{ grad } u) =$ outside force.

You are studying calculus, not physics or thermodynamics or elasticity. But please notice the main point. The equation to solve is $\text{div}(-c \text{ grad } f) = \text{known source}$. The divergence and gradient are exactly what the applications need. Calculus teaches the right things.

This framework is developed in many books, including my own text *Introduction to Applied Mathematics* (Wellesley-Cambridge Press). It governs equilibrium, in matrix equations and differential equations.

PRODUCT RULE FOR VECTORS: INTEGRATION BY PARTS

May I go back to basic facts about the divergence? First the definition:

$$\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \text{ has } \text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \partial M/\partial x + \partial N/\partial y + \partial P/\partial z.$$

The divergence is a scalar (not a vector). At each point $\text{div } \mathbf{F}$ is a number. In fluid flow, it is the rate at which mass leaves—the “flux per unit volume” or “flux density.”

The symbol ∇ stands for a vector whose components are *operations not numbers*:

$$\nabla = \text{“del”} = \mathbf{i} \partial/\partial x + \mathbf{j} \partial/\partial y + \mathbf{k} \partial/\partial z. \quad (6)$$

This vector is illegal but very useful. First, apply it to an ordinary function $f(x, y, z)$:

$$\nabla f = \text{“del } f” = \mathbf{i} \partial f/\partial x + \mathbf{j} \partial f/\partial y + \mathbf{k} \partial f/\partial z = \text{gradient of } f. \quad (7)$$

Second, take the dot product $\nabla \cdot \mathbf{F}$ with a vector function $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$:

$$\nabla \cdot \mathbf{F} = \text{“del dot } \mathbf{F}” = \partial M/\partial x + \partial N/\partial y + \partial P/\partial z = \text{divergence of } \mathbf{F}. \quad (8)$$

Third, take the cross product $\nabla \times \mathbf{F}$. This produces the vector curl $\mathbf{F}$ (next section):

$$\nabla \times \mathbf{F} = \text{“del cross } \mathbf{F}” = \dots \text{ (to be defined) } \dots = \text{curl of } \mathbf{F}. \quad (9)$$

The gradient and divergence and curl are ∇ and $\nabla \cdot$ and $\nabla \times$. The three great operations of vector calculus use a single notation! You are free to write ∇ or not—to make equations shorter or to help the memory. Notice that Laplace’s equation shrinks to

$$\nabla \cdot \nabla f = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial z} \right) = 0. \quad (10)$$

Equation (10) gives the potential when the source is zero (very common). $\mathbf{F} = \text{grad } f$ combines with $\text{div } \mathbf{F} = 0$ into Laplace’s equation $\text{div grad } f = 0$. This equation is so important that it shrinks further to $\nabla^2 f = 0$ and even to $\Delta f = 0$. Of course $\Delta f = f_{xx} + f_{yy} + f_{zz}$ has nothing to do with $\Delta f = f(x + \Delta x) - f(x)$. Above all, remember that f is a scalar and $\mathbf{F}$ is a vector: *gradient of scalar is vector* and *divergence of vector is scalar*.

Underlying this chapter is the idea of extending calculus to vectors. So far we have emphasized the Fundamental Theorem. The integral of df/dx is now the integral of $\text{div } \mathbf{F}$. Instead of endpoints a and b , we have a curve C or surface S . But it is the *rules* for derivatives and integrals that make calculus work, and we need them now for vectors. Remember the derivative of u times v and the integral (by parts) of $u dv/dx$:

15M Scalar functions $u(x, y, z)$ and vector fields $\mathbf{V}(x, y, z)$ obey the **product rule**:

$$\text{div}(u\mathbf{V}) = u \text{div } \mathbf{V} + \mathbf{V} \cdot (\text{grad } u). \quad (11)$$

The reverse of the product rule is integration by parts (**Gauss's Formula**):

$$\iiint u \text{div } \mathbf{V} \, dx \, dy \, dz = - \iiint \mathbf{V} \cdot (\text{grad } u) \, dx \, dy \, dz + \iint u \mathbf{V} \cdot \mathbf{n} \, dS. \quad (12)$$

For a plane field this is **Green's Formula** (and $u = 1$ gives Green's Theorem):

$$\iint u \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy = - \iint \left(M \frac{\partial u}{\partial x} + N \frac{\partial u}{\partial y} \right) dx \, dy + \int u (M\mathbf{i} + N\mathbf{j}) \cdot \mathbf{n} \, ds. \quad (13)$$

Those look like heavy formulas. They are too much to memorize, unless you use them often. The important point is to connect vector calculus with “scalar calculus,” which is not heavy. Every product rule yields two terms:

$$(uM)_x = u \partial M / \partial x + M \partial u / \partial x \quad (uN)_y = u \partial N / \partial y + N \partial u / \partial y \quad (uP)_z = u \partial P / \partial z + P \partial u / \partial z.$$

Add those ordinary rules and you have the vector rule (11) for the divergence of $u\mathbf{V}$.

Integrating the two parts of $\text{div}(u\mathbf{V})$ gives $\iint u \mathbf{V} \cdot \mathbf{n} \, dS$ by the Divergence Theorem. Then one part moves to the other side, producing the minus signs in (12) and (13). **Integration by parts leaves a boundary term**, in three and two dimensions as it did in one dimension: $\int uv' \, dx = - \int u'v \, dx + [uv]_a^b$.

EXAMPLE 5 Find the divergence of $\mathbf{F} = \mathbf{R}/\rho^2$, starting from $\text{grad } \rho = \mathbf{R}/\rho$.

Solution Take $\mathbf{V} = \mathbf{R}$ and $u = 1/\rho^2$ in the product rule (11). Then $\text{div } \mathbf{F} = (\text{div } \mathbf{R})/\rho^2 + \mathbf{R} \cdot (\text{grad } 1/\rho^2)$. The divergence of $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is 3. For $\text{grad } 1/\rho^2$ apply the chain rule:

$$\mathbf{R} \cdot (\text{grad } 1/\rho^2) = -2\mathbf{R} \cdot (\text{grad } \rho)/\rho^3 = -2\mathbf{R} \cdot \mathbf{R}/\rho^4 = -2/\rho^2.$$

The two parts of $\text{div } \mathbf{F}$ combine into $3/\rho^2 - 2/\rho^2 = 1/\rho^2$ —as claimed in Example 4.

EXAMPLE 6 Find the balance equation for flow with velocity $\mathbf{V}$ and fluid density ρ .

$\mathbf{V}$ is the rate of movement of fluid, while $\rho\mathbf{V}$ is the rate of movement of *mass*. Comparing the ocean to the atmosphere shows the difference. Air has a greater velocity than water, but a much lower density. So normally $\mathbf{F} = \rho\mathbf{V}$ is larger for the ocean. (Don't confuse the density ρ with the radial distance ρ . The Greeks only used 24 letters.)

There is another difference between water and air. Water is virtually incompressible (meaning $\rho = \text{constant}$). Air is certainly compressible (its density varies). The balance equation is a fundamental law—the conservation of mass or the “*continuity equation*” for fluids. This is a mathematical statement about a physical flow without sources or sinks:

$$\text{Continuity Equation: } \text{div}(\rho\mathbf{V}) + \partial\rho/\partial t = 0. \quad (14)$$

Explanation: The mass in a region is $\iiint \rho \, dV$. Its rate of decrease is $-\iiint \partial\rho/\partial t \, dV$. The decrease comes from flow out through the surface (normal vector $\mathbf{n}$). The dot product $\mathbf{F} \cdot \mathbf{n} = \rho \mathbf{V} \cdot \mathbf{n}$ is the rate of mass flow through the surface. So the integral $\iint \mathbf{F} \cdot \mathbf{n} \, dS$ is the total rate that mass goes out. By the Divergence Theorem this is $\iiint \operatorname{div} \mathbf{F} \, dV$.

To balance $-\iiint \partial\rho/\partial t \, dV$ in every region, $\operatorname{div} \mathbf{F}$ must equal $-\partial\rho/\partial t$ at every point. The figure shows this continuity equation (14) for flow in the x direction.

$$\rho \mathbf{V} \, dS \, dt \rightarrow \boxed{\text{mass } \rho dS \, dx} \rightarrow \text{extra mass out } d(\rho \mathbf{V}) \, dS \, dt = \text{mass loss } -d\rho \, dS \, dx$$

Fig. 15.22 Conservation of mass during time dt : $d(\rho \mathbf{V})/dx + d\rho/dt = 0$.

15.5 EXERCISES

Read-through questions

In words, the basic balance law is a. The flux of $\mathbf{F}$ through a closed surface S is the double integral b. The divergence of $M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is c, and it measures d. The total source is the triple integral e. That equals the flux by the f Theorem.

For $\mathbf{F} = 5z\mathbf{k}$ the divergence is g. If V is a cube of side a then the triple integral equals h. The top surface where $z = a$ has $\mathbf{n} = \mathbf{i}$ and $\mathbf{F} \cdot \mathbf{n} = \mathbf{j}$. The bottom and sides have $\mathbf{F} \cdot \mathbf{n} = \mathbf{k}$. The integral $\iint \mathbf{F} \cdot \mathbf{n} \, dS$ equals i.

The field $\mathbf{F} = \mathbf{R}/\rho^3$ has $\operatorname{div} \mathbf{F} = 0$ except m. $\iint \mathbf{F} \cdot \mathbf{n} \, dS$ equals n over any surface around the origin. This illustrates Gauss's Law o. The field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$ has $\operatorname{div} \mathbf{F} = \mathbf{p}$ and $\iint \mathbf{F} \cdot \mathbf{n} \, dS = \mathbf{q}$. For this $\mathbf{F}$, the flux out through a pyramid and in through its base are r.

The symbol ∇ stands for s. In this notation $\operatorname{div} \mathbf{F}$ is t. The gradient of f is u. The divergence of $\operatorname{grad} f$ is v. The equation $\operatorname{div} \operatorname{grad} f = 0$ is w's equation.

The divergence of a product is $\operatorname{div}(u\mathbf{V}) = \mathbf{x}$. Integration by parts is $\iiint u \operatorname{div} \mathbf{V} \, dx \, dy \, dz = \mathbf{y} + \mathbf{z}$. In two dimensions this becomes A. In one dimension it becomes B. For steady fluid flow the continuity equation is $\operatorname{div} \rho \mathbf{V} = \mathbf{C}$.

In 1–10 compute the flux $\iint \mathbf{F} \cdot \mathbf{n} \, dS$ by the Divergence Theorem.

- 1 $\mathbf{F} = x\mathbf{i} + x\mathbf{j} + x\mathbf{k}$, S : unit sphere $x^2 + y^2 + z^2 = 1$.
- 2 $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$, V : unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.
- 3 $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, S : unit sphere
- 4 $\mathbf{F} = x^2\mathbf{i} + 8y^2\mathbf{j} + z^2\mathbf{k}$, V : unit cube.
- 5 $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j}$, S : sides $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.
- 6 $\mathbf{F} = \mathbf{u}_r = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\rho$, S : sphere $\rho = a$.
- 7 $\mathbf{F} = \rho(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, S : sphere $\rho = a$

8 $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$, S : sphere $\rho = a$.

9 $\mathbf{F} = z^2\mathbf{k}$, V : upper half of ball $\rho \leq a$.

10 $\mathbf{F} = \operatorname{grad}(xe^y \sin z)$, S : sphere $\rho = a$.

11 Find $\iiint \operatorname{div}(x^2\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}) \, dV$ in the cube $0 \leq x, y, z \leq a$. Also compute $\mathbf{n}$ and $\iint \mathbf{F} \cdot \mathbf{n} \, dS$ for all six faces and add.

12 When a is small in problem 11, the answer is close to ca^3 . Find the number c . At what point does $\operatorname{div} \mathbf{F} = c$?

13 (a) Integrate the divergence of $\mathbf{F} = \rho\mathbf{i}$ in the ball $\rho \leq a$.
(b) Compute $\iint \mathbf{F} \cdot \mathbf{n} \, dS$ over the spherical surface $\rho = a$.

14 Integrate $\iint \mathbf{R} \cdot \mathbf{n} \, dS$ over the faces of the box $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 3$ and check by the Divergence Theorem.

15 Evaluate $\iint \mathbf{F} \cdot \mathbf{n} \, dS$ when $\mathbf{F} = x\mathbf{i} + z^2\mathbf{j} + y^2\mathbf{k}$ and:

- (a) S is the cone $z^2 = x^2 + y^2$ bounded above by the plane $z = 1$.
- (b) S is the pyramid with corners $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.

16 Compute all integrals in the Divergence Theorem when $\mathbf{F} = x(\mathbf{i} + \mathbf{j} - \mathbf{k})$ and V is the unit cube $0 \leq x, y, z \leq 1$.

17 Following Example 5, compute the divergence of $(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\rho^7$.

18 $(\operatorname{grad} f) \cdot \mathbf{n}$ is the _____ derivative of f in the direction _____. It is also written $\partial f / \partial n$. If $f_{xx} + f_{yy} + f_{zz} = 0$ in V , derive $\iint \partial f / \partial n \, dS = 0$ from the Divergence Theorem.

19 Describe the closed surface S and outward normal $\mathbf{n}$:

- (a) V = hollow ball $1 \leq x^2 + y^2 + z^2 \leq 9$.
- (b) V = solid cylinder $x^2 + y^2 \leq 1$, $|z| \leq 7$.
- (c) V = pyramid $x \geq 0$, $y \geq 0$, $z \geq 0$, $x + 2y + 3z \leq 1$.
- (d) V = solid cone $x^2 + y^2 \leq z^2 \leq 1$.

20 Give an example where $\iint \mathbf{F} \cdot \mathbf{n} \, dS$ is easier than $\iiint \operatorname{div} \mathbf{F} \, dV$.

- 21 Suppose $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$, R is a region in the xy plane, and (x, y, z) is in V if (x, y) is in R and $|z| \leq 1$.
- Describe V and reduce $\iiint \operatorname{div} \mathbf{F} \, dV$ to a double integral.
 - Reduce $\iint \mathbf{F} \cdot \mathbf{n} \, dS$ to a line integral (check top, bottom, side).
 - Whose theorem says that the double integral equals the line integral?
- 22 Is it possible to have $\mathbf{F} \cdot \mathbf{n} = 0$ at all points of S and also $\operatorname{div} \mathbf{F} = 0$ at all points in V ? $\mathbf{F} = \mathbf{0}$ is not allowed.
- 23 Inside a solid ball (radius a , density 1, mass $M = 4\pi a^3/3$) the gravity field is $\mathbf{F} = -GM\mathbf{R}/a^3$.
- Check $\operatorname{div} \mathbf{F} = -4\pi G$ in Gauss's Law.
 - The force at the surface is the same as if the whole mass M were _____.
 - Find a gradient field with $\operatorname{div} \mathbf{F} = 6$ in the ball $\rho \leq a$ and $\operatorname{div} \mathbf{F} = 0$ outside.
- 24 The outward field $\mathbf{F} = \mathbf{R}/\rho^3$ has magnitude $|\mathbf{F}| = 1/\rho^2$. Through an area A on a sphere of radius ρ , the flux is _____. A spherical box has faces at ρ_1 and ρ_2 with $A = \rho_1^2 \sin \phi \, d\phi \, d\theta$ and $A = \rho_2^2 \sin \phi \, d\phi \, d\theta$. Deduce that the flux out of the box is zero, which confirms $\operatorname{div} \mathbf{F} = 0$.
- 25 In Gauss's Law, what charge distribution $q(x, y, z)$ gives the unit field $\mathbf{E} = \mathbf{u}_r$? What is the flux through the unit sphere?
- 26 If a fluid with velocity $\mathbf{V}$ is incompressible (constant density ρ), then its continuity equation reduces to _____. If it is irrotational then $\mathbf{F} = \operatorname{grad} f$. If it is both then f satisfies _____ equation.
- 27 *True or false*, with a good reason.
- If $\iint \mathbf{F} \cdot \mathbf{n} \, dS = 0$ for every closed surface, $\mathbf{F}$ is constant.
 - If $\mathbf{F} = \operatorname{grad} f$ then $\operatorname{div} \mathbf{F} = 0$.
 - If $|\mathbf{F}| \leq 1$ at all points then $\iiint \operatorname{div} \mathbf{F} \, dV \leq \text{area of the surface } S$.
 - If $|\mathbf{F}| \leq 1$ at all points then $|\operatorname{div} \mathbf{F}| \leq 1$ at all points.
- 28 Write down statements **E–F–G–H** for source-free fields $\mathbf{F}(x, y, z)$ in three dimensions. In statement **F**, paths sharing the same endpoint become surfaces sharing the same boundary curve. In **G**, the stream function becomes a *vector field* such that $\mathbf{F} = \operatorname{curl} \mathbf{g}$.
- 29 Describe two different surfaces bounded by the circle $x^2 + y^2 = 1, z = 0$. The field $\mathbf{F}$ automatically has the same flux through both if _____.
- 30 The boundary of a bounded region R has no boundary. Draw a plane region and explain what that means. What does it mean for a solid ball?

15.6 Stokes' Theorem and the Curl of $\mathbf{F}$

For the Divergence Theorem, the surface was closed. S was the boundary of V . Now the surface is not closed and S has its own boundary—a curve called C . We are back near the original setting for Green's Theorem (region bounded by curve, double integral equal to work integral). But Stokes' Theorem, also called Stokes's Theorem, is in three-dimensional space. There is a *curved surface* S bounded by a *space curve* C . This is our first integral around a space curve.

The move to three dimensions brings a change in the vector field. The plane field $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ becomes a space field $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$. The work $Mdx + Ndy$ now includes Pdz . The critical quantity in the double integral (it was $\partial N/\partial x - \partial M/\partial y$) must change too. We called this scalar quantity “curl $\mathbf{F}$,” but in reality it is only the third component of a vector. Stokes' Theorem needs all three components of that vector—which is curl $\mathbf{F}$.

DEFINITION The curl of a vector field $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is the vector field

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}. \quad (1)$$

The symbol $\nabla \times \mathbf{F}$ stands for a determinant that yields those six derivatives:

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix}. \quad (2)$$

The three products $\mathbf{i} \partial/\partial y P$ and $\mathbf{j} \partial/\partial z M$ and $\mathbf{k} \partial/\partial x N$ have plus signs. The three products like $\mathbf{k} \partial/\partial y M$, down to the left, have minus signs. There is a cyclic symmetry. This determinant helps the memory, even if it looks and is illegal. A determinant is not supposed to have a row of vectors, a row of operators, and a row of functions.

EXAMPLE 1 The plane field $M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ has $P=0$ and $\partial M/\partial z=0$ and $\partial N/\partial z=0$. Only two terms survive: $\text{curl } \mathbf{F} = (\partial N/\partial x - \partial M/\partial y)\mathbf{k}$. Back to Green.

EXAMPLE 2 The cross product $\mathbf{a} \times \mathbf{R}$ is a *spin field* $\mathbf{S}$. Its axis is the fixed vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. The flow in Figure 15.23 turns around $\mathbf{a}$, and its components are

$$\mathbf{S} = \mathbf{a} \times \mathbf{R} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2z - a_3y)\mathbf{i} + (a_3x - a_1z)\mathbf{j} + (a_1y - a_2x)\mathbf{k}. \quad (3)$$

Our favorite spin field $-y\mathbf{i} + x\mathbf{j}$ has $(a_1, a_2, a_3) = (0, 0, 1)$ and its axis is $\mathbf{a} = \mathbf{k}$.

The divergence of a spin field is $M_x + N_y + P_z = 0 + 0 + 0$. Note how the divergence uses M_x while the curl uses N_x and P_x . **The curl of $\mathbf{S}$ is the vector $2\mathbf{a}$:**

$$\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} = 2a_1\mathbf{i} + 2a_2\mathbf{j} + 2a_3\mathbf{k} = 2\mathbf{a}.$$

This example begins to reveal the meaning of the curl. It measures the spin! The direction of $\text{curl } \mathbf{F}$ is the *axis of rotation*—in this case along $\mathbf{a}$. The magnitude of $\text{curl } \mathbf{F}$ is *twice the speed of rotation*. In this case $|\text{curl } \mathbf{F}| = 2|\mathbf{a}|$ and the angular velocity is $|\mathbf{a}|$.

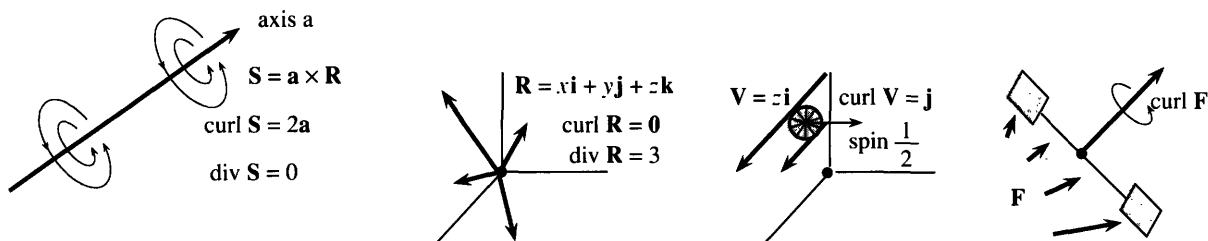


Fig. 15.23 Spin field $\mathbf{S} = \mathbf{a} \times \mathbf{R}$, position field $\mathbf{R}$, velocity field (shear field) $\mathbf{V} = z\mathbf{i}$, any field $\mathbf{F}$.

EXAMPLE 3 (!) Every gradient field $\mathbf{F} = \partial f/\partial x \mathbf{i} + \partial f/\partial y \mathbf{j} + \partial f/\partial z \mathbf{k}$ has $\text{curl } \mathbf{F} = \mathbf{0}$:

$$\text{curl } \mathbf{F} = \left(\frac{\partial}{\partial y} \frac{\partial f}{\partial z} - \frac{\partial}{\partial z} \frac{\partial f}{\partial y}\right)\mathbf{i} + \left(\frac{\partial}{\partial z} \frac{\partial f}{\partial x} - \frac{\partial}{\partial x} \frac{\partial f}{\partial z}\right)\mathbf{j} + \left(\frac{\partial}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial}{\partial y} \frac{\partial f}{\partial x}\right)\mathbf{k} = \mathbf{0}. \quad (4)$$

Always f_{yz} equals f_{zy} . They cancel. Also $f_{xz} = f_{zx}$ and $f_{yx} = f_{xy}$. So $\text{curl grad } f = \mathbf{0}$.

EXAMPLE 4 (twin of Example 3) The divergence of curl $\mathbf{F}$ is also automatically zero:

$$\operatorname{div} \operatorname{curl} \mathbf{F} = \frac{\partial}{\partial x} \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = 0. \quad (5)$$

Again the mixed derivatives give $P_{xy} = P_{yx}$ and $N_{xz} = N_{zx}$ and $M_{zy} = M_{yz}$. The terms cancel in pairs. In “curl grad” and “div curl”, everything is arranged to give zero.

15N The curl of the gradient of every $f(x, y, z)$ is $\operatorname{curl} \operatorname{grad} f = \nabla \times \nabla f = \mathbf{0}$.
The divergence of the curl of every $\mathbf{F}(x, y, z)$ is $\operatorname{div} \operatorname{curl} \mathbf{F} = \nabla \cdot \nabla \times \mathbf{F} = 0$.

The spin field $\mathbf{S}$ has no divergence. The position field $\mathbf{R}$ has no curl. $\mathbf{R}$ is the gradient of $f = \frac{1}{2}(x^2 + y^2 + z^2)$. $\mathbf{S}$ is the curl of a suitable $\mathbf{F}$. Then $\operatorname{div} \mathbf{S} = \operatorname{div} \operatorname{curl} \mathbf{F}$ and $\operatorname{curl} \mathbf{R} = \operatorname{curl} \operatorname{grad} f$ are automatically zero.

You correctly believe that $\operatorname{curl} \mathbf{F}$ measures the “spin” of the field. You may expect that $\operatorname{curl} (\mathbf{F} + \mathbf{G})$ is $\operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G}$. Also correct. Finally you may think that a field of parallel vectors has no spin. That is wrong. Example 5 has parallel vectors, but their different lengths produce spin.

EXAMPLE 5 The field $\mathbf{V} = z\mathbf{i}$ in the x direction has $\operatorname{curl} \mathbf{V} = \mathbf{j}$ in the y direction.

If you put a wheel in the xz plane, *this field will turn it*. The velocity $z\mathbf{i}$ at the top of the wheel is greater than $z\mathbf{i}$ at the bottom (Figure 15.23c). So the top goes faster and the wheel rotates. The axis of rotation is $\operatorname{curl} \mathbf{V} = \mathbf{j}$. The turning speed is $\frac{1}{2}$, because this curl has magnitude 1.

Another velocity field $\mathbf{v} = -x\mathbf{k}$ produces the same spin: $\operatorname{curl} \mathbf{v} = \mathbf{j}$. The flow is in the z direction, it varies in the x direction, and the spin is in the y direction. Also interesting is $\mathbf{V} + \mathbf{v}$. The two “shear fields” add to a perfect spin field $\mathbf{S} = z\mathbf{i} - x\mathbf{k}$, whose curl is $2\mathbf{j}$.

THE MEANING OF CURL F

Example 5 put a paddlewheel into the flow. This is possible for any vector field $\mathbf{F}$, and it gives insight into $\operatorname{curl} \mathbf{F}$. The turning of the wheel (if it turns) depends on its location (x, y, z) . The turning also depends on the *orientation* of the wheel. We could put it into a spin field, and if the wheel axis $\mathbf{n}$ is perpendicular to the spin axis $\mathbf{a}$, the wheel won’t turn! The general rule for turning speed is this: *the angular velocity of the wheel is $\frac{1}{2}(\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}$* . This is the “*directional spin*,” just as $(\operatorname{grad} f) \cdot \mathbf{u}$ was the “*directional derivative*”—and $\mathbf{n}$ is a unit vector like $\mathbf{u}$.

There is no spin anywhere in a gradient field. It is *irrotational*: $\operatorname{curl} \operatorname{grad} f = \mathbf{0}$.

The pure spin field $\mathbf{a} \times \mathbf{R}$ has $\operatorname{curl} \mathbf{F} = 2\mathbf{a}$. The angular velocity is $\mathbf{a} \cdot \mathbf{n}$ (note that $\frac{1}{2}$ cancels 2). This turning is everywhere, *not just at the origin*. If you put a penny on a compact disk, it turns once when the disk rotates once. That spin is “*around itself*,” and it is the same whether the penny is at the center or not.

The turning speed is greatest when the wheel axis $\mathbf{n}$ lines up with the spin axis $\mathbf{a}$. Then $\mathbf{a} \cdot \mathbf{n}$ is the full length $|\mathbf{a}|$. The gradient gives the direction of fastest growth, and the curl gives the direction of fastest turning:

maximum growth rate of f is $|\operatorname{grad} f|$ in the direction of $\operatorname{grad} f$

maximum rotation rate of $\mathbf{F}$ is $\frac{1}{2}|\operatorname{curl} \mathbf{F}|$ in the direction of $\operatorname{curl} \mathbf{F}$.

STOKES' THEOREM

Finally we come to the big theorem. It will be like Green's Theorem—a line integral equals a surface integral. The line integral is still the work $\oint \mathbf{F} \cdot d\mathbf{R}$ around a curve. The surface integral in Green's Theorem is $\iint (N_x - M_y) dx dy$. The surface is flat (in the xy plane). Its normal direction is $\mathbf{k}$, and we now recognize $N_x - M_y$ as the $\mathbf{k}$ component of the curl. Green's Theorem uses only this component because the normal direction is always $\mathbf{k}$. For Stokes' Theorem on a curved surface, we need all three components of curl $\mathbf{F}$.

Figure 15.24 shows a hat-shaped surface S and its boundary C (a closed curve). Walking in the positive direction around C , with your head pointing in the direction of $\mathbf{n}$, the surface is *on your left*. You may be standing straight up ($\mathbf{n} = \mathbf{k}$ in Green's Theorem). You may even be upside down ($\mathbf{n} = -\mathbf{k}$ is allowed). In that case you must go the other way around C , to keep the two sides of equation (6) equal. The surface is still on the left. A Möbius strip is not allowed, because its normal direction cannot be established. The unit vector $\mathbf{n}$ determines the “counterclockwise direction” along C .

$$\text{150 (Stokes' Theorem)} \quad \oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS. \quad (6)$$

The right side adds up small spins in the surface. The left side is the total circulation (or work) around C . That is not easy to visualize—this may be the hardest theorem in the book—but notice one simple conclusion. *If* $\text{curl } \mathbf{F} = \mathbf{0}$ *then* $\oint \mathbf{F} \cdot d\mathbf{R} = 0$. *This applies above all to gradient fields*—as we know.

A gradient field has no curl, by (4). A gradient field does no work, by (6). In three dimensions as in two dimensions, *gradient fields are conservative fields*. They will be the focus of this section, after we outline a proof (or two proofs) of Stokes' Theorem.

The first proof shows *why* the theorem is true. The second proof shows that it really is true (and how to compute). You may prefer the first.

First proof Figure 15.24 has a triangle ABC attached to a triangle ACD . Later there can be more triangles. S will be *piecewise flat*, close to a curved surface. Two triangles are enough to make the point. In the plane of each triangle (they have different $\mathbf{n}$'s) Green's Theorem is known:

$$\oint_{AB+BC+CA} \mathbf{F} \cdot d\mathbf{R} = \iint_{ABC} \text{curl } \mathbf{F} \cdot \mathbf{n} dS \quad \quad \oint_{AC+CD+DA} \mathbf{F} \cdot d\mathbf{R} = \iint_{ACD} \text{curl } \mathbf{F} \cdot \mathbf{n} dS.$$

Now add. The right sides give $\iint \text{curl } \mathbf{F} \cdot \mathbf{n} dS$ over the two triangles. On the left, *the integral over CA cancels the integral over AC* . The “crosscut” disappears. That leaves $AB + BC + CD + DA$. This line integral goes around the outer boundary C —which is the left side of Stokes' Theorem.

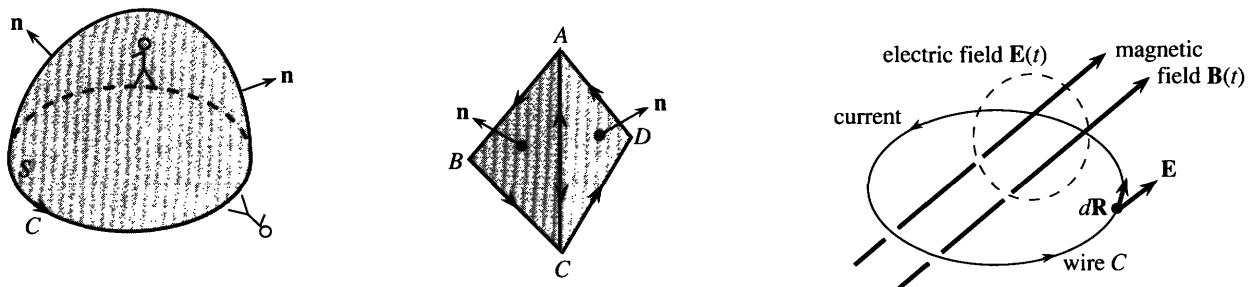


Fig. 15.24 Surfaces S and boundary curves C . Change in $\mathbf{B} \rightarrow \text{curl } \mathbf{E} \rightarrow$ current in C .

Second proof Now the surface can be curved. A new proof may seem excessive, but it brings formulas you could compute with. From $z = f(x, y)$ we have

$$dz = \partial f / \partial x \, dx + \partial f / \partial y \, dy \quad \text{and} \quad \mathbf{n} dS = (-\partial f / \partial x \, \mathbf{i} - \partial f / \partial y \, \mathbf{j} + \mathbf{k}) \, dx \, dy.$$

For $\mathbf{n} dS$, see equation 15.4.11. With this dz , the line integral in Stokes' Theorem is

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \oint_{\text{shadow of } C} M \, dx + N \, dy + P(\partial f / \partial x \, dx + \partial f / \partial y \, dy). \quad (7)$$

The dot product of $\text{curl } \mathbf{F}$ and $\mathbf{n} dS$ gives the surface integral $\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dS$:

$$\iint_{\text{shadow of } S} [(P_y - N_z)(-\partial f / \partial x) + (M_z - P_x)(-\partial f / \partial y) + (N_x - M_y)] \, dx \, dy. \quad (8)$$

To prove (7) = (8), change M in Green's Theorem to $M + P\partial f / \partial x$. Also change N to $N + P\partial f / \partial y$. Then (7) = (8) is Green's Theorem down on the shadow (Problem 47). This proves Stokes' Theorem up on S . Notice how Green's Theorem (flat surface) was the key to both proofs of Stokes' Theorem (curved surface).

EXAMPLE 6 Stokes' Theorem in electricity and magnetism yields Faraday's Law.

Stokes' Theorem is not heavily used for calculations—equation (8) shows why. But the spin or curl or *vorticity* of a flow is absolutely basic in fluid mechanics. The other important application, coming now, is to electric fields. Faraday's Law is to Gauss's Law as Stokes' Theorem is to the Divergence Theorem.

Suppose the curve C is an actual wire. We can produce current along C by varying the magnetic field $\mathbf{B}(t)$. The flux $\varphi = \iint_S \mathbf{B} \cdot \mathbf{n} dS$, passing within C and changing in time, creates an electric field $\mathbf{E}$ that does work:

$$\text{Faraday's Law (integral form): work} = \oint_C \mathbf{E} \cdot d\mathbf{R} = -d\varphi/dt.$$

That is physics. It may be true, it may be an approximation. Now comes mathematics (surely true), which turns this integral form into a differential equation. Information at points is more convenient than information around curves. Stokes converts the line integral of $\mathbf{E}$ into the surface integral of $\text{curl } \mathbf{E}$:

$$\oint_C \mathbf{E} \cdot d\mathbf{R} = \iint_S \text{curl } \mathbf{E} \cdot \mathbf{n} dS \quad \text{and also} \quad -\partial\varphi/\partial t = \iint_S -(\partial\mathbf{B}/\partial t) \cdot \mathbf{n} dS.$$

These are equal for any curve C , however small. So the right sides are equal for any surface S . We squeeze to a point. The right hand sides give one of Maxwell's equations:

$$\text{Faraday's Law (differential form): } \text{curl } \mathbf{E} = -\partial\mathbf{B}/\partial t.$$

CONSERVATIVE FIELDS AND POTENTIAL FUNCTIONS

The chapter ends with our constant and important question: Which fields do no work around closed curves? Remember test **D** for plane curves and plane vector fields:

$$\text{if } \partial M / \partial y = \partial N / \partial x \text{ then } \mathbf{F} \text{ is conservative and } \mathbf{F} = \text{grad } f \text{ and } \oint \mathbf{F} \cdot d\mathbf{R} = 0.$$

Now allow a three-dimensional field like $\mathbf{F} = 2xy \, \mathbf{i} + (x^2 + z) \, \mathbf{j} + y \, \mathbf{k}$. Does it do work around a space curve? Or is it a gradient field? That will require $\partial f / \partial x = 2xy$ and $\partial f / \partial y = x^2 + z$ and $\partial f / \partial z = y$. *We have three equations for one function* $f(x, y, z)$. Normally they can't be solved. When test **D** is passed (now it is the three-dimensional test: $\text{curl } \mathbf{F} = \mathbf{0}$) they *can* be solved. This example passes test **D**, and f is $x^2y + yz$.

15P $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a conservative field if it has these properties:
A. The work $\oint \mathbf{F} \cdot d\mathbf{R}$ around every closed path in space is zero.
B. The work $\int_C \mathbf{F} \cdot d\mathbf{R}$ depends only on P and Q , not on the path in space.
C. $\mathbf{F}$ is a **gradient field**: $M = \partial f / \partial x$ and $N = \partial f / \partial y$ and $P = \partial f / \partial z$.
D. The components satisfy $M_y = N_x$, $M_z = P_x$, and $N_z = P_y$ (**curl $\mathbf{F}$ is zero**).
 A field with one of these properties has them all. **D** is the quick test.

A detailed proof of $\mathbf{A} \Rightarrow \mathbf{B} \Rightarrow \mathbf{C} \Rightarrow \mathbf{D} \Rightarrow \mathbf{A}$ is not needed. Only notice how $\mathbf{C} \Rightarrow \mathbf{D}$: curl grad $\mathbf{F}$ is always zero. The newest part is $\mathbf{D} \Rightarrow \mathbf{A}$. If curl $\mathbf{F} = \mathbf{0}$ then $\oint \mathbf{F} \cdot d\mathbf{R} = 0$. But that is not news. It is Stokes' Theorem.

The interesting problem is to solve the three equations for f , when test **D** is passed. The example above had

$$\partial f / \partial x = 2xy \Rightarrow f = \int 2xy \, dx = x^2 y \text{ plus any function } C(y, z)$$

$$\partial f / \partial y = x^2 + z = x^2 + \partial C / \partial y \Rightarrow C = yz \text{ plus any function } c(z)$$

$$\partial f / \partial z = y = y + dc/dz \Rightarrow c(z) \text{ can be zero.}$$

The first step leaves an arbitrary $C(y, z)$ to fix the second step. The second step leaves an arbitrary $c(z)$ to fix the third step (not needed here). Assembling the three steps, $f = x^2 y + C = x^2 y + yz + c = x^2 y + yz$. Please recognize that the "fix-up" is only possible when curl $\mathbf{F} = \mathbf{0}$. Test **D** must be passed.

EXAMPLE 7 Is $\mathbf{F} = (z - y)\mathbf{i} + (x - z)\mathbf{j} + (y - x)\mathbf{k}$ the gradient of any f ?

Test **D** says *no*. This $\mathbf{F}$ is a spin field $\mathbf{a} \times \mathbf{R}$. Its curl is $2\mathbf{a} = (2, 2, 2)$, which is not zero. A search for f is bound to fail, but we can try. To match $\partial f / \partial x = z - y$, we must have $f = zx - yx + C(y, z)$. The y derivative is $-x + \partial C / \partial y$. That never matches $N = x - z$, so f can't exist.

EXAMPLE 8 What choice of P makes $\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + P\mathbf{k}$ conservative? Find f .

Solution We need curl $\mathbf{F} = \mathbf{0}$, by test **D**. First check $\partial M / \partial y = z^2 = \partial N / \partial x$. Also

$$\partial P / \partial x = \partial M / \partial z = 2yz \quad \text{and} \quad \partial P / \partial y = \partial N / \partial z = 2xz.$$

$P = 2xyz$ passes all tests. To find f we can solve the three equations, or notice that $f = xyz^2$ is successful. Its gradient is $\mathbf{F}$.

A third method defines $f(x, y, z)$ as *the work to reach* (x, y, z) *from* $(0, 0, 0)$. The path doesn't matter. For practice we integrate $\mathbf{F} \cdot d\mathbf{R} = M \, dx + N \, dy + P \, dz$ along the straight line (xt, yt, zt) :

$$f(x, y, z) = \int_0^1 (yt)(zt)^2(x \, dt) + (xt)(zt)^2(y \, dt) + 2(xt)(yt)(zt)(z \, dt) = xyz^2.$$

EXAMPLE 9 Why is div curl grad f automatically zero (in two ways)?

Solution First, curl grad f is zero (always). Second, div curl $\mathbf{F}$ is zero (always). Those are the key identities of vector calculus. We end with a review.

$$\begin{aligned} \text{Green's Theorem:} \quad \oint \mathbf{F} \cdot d\mathbf{R} &= \iint (\partial N / \partial x - \partial M / \partial y) \, dx \, dy \\ \oint \mathbf{F} \cdot \mathbf{n} \, ds &= \iint (\partial M / \partial x + \partial N / \partial y) \, dx \, dy \end{aligned}$$

$$\text{Divergence Theorem: } \iint \mathbf{F} \cdot \mathbf{n} dS = \iiint (\partial M / \partial x + \partial N / \partial y + \partial P / \partial z) dx dy dz$$

$$\text{Stokes' Theorem: } \oint \mathbf{F} \cdot d\mathbf{R} = \iint \text{curl } \mathbf{F} \cdot \mathbf{n} dS.$$

The first form of Green's Theorem leads to Stokes' Theorem. The second form becomes the Divergence Theorem. You may ask, *why not go to three dimensions in the first place?* The last two theorems contain the first two (take $P = 0$ and a flat surface). We could have reduced this chapter to two theorems, not four. I admit that, but a fundamental principle is involved: "It is easier to generalize than to specialize."

For the same reason df/dx came before partial derivatives and the gradient.

15.6 EXERCISES

Read-through questions

The curl of $M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is the vector a. It equals the 3 by 3 determinant b. The curl of $x^2\mathbf{i} + z^2\mathbf{k}$ is c. For $\mathbf{S} = y\mathbf{i} - (x+z)\mathbf{j} + y\mathbf{k}$ the curl is d. This $\mathbf{S}$ is a e field $\mathbf{a} \times \mathbf{R} = \frac{1}{2}(\text{curl } \mathbf{F}) \times \mathbf{R}$, with axis vector $\mathbf{a} = \mathbf{f}$. For any gradient field $f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$ the curl is g. That is the important identity $\text{curl grad } f = \mathbf{h}$. It is based on $f_{xy} = f_{yx}$ and i and j. The twin identity is k.

The curl measures the l of a vector field. A paddlewheel in the field with its axis along $\mathbf{n}$ has turning speed m. The spin is greatest when $\mathbf{n}$ is in the direction of n. Then the angular velocity is o.

Stokes' Theorem is p = q. The curve C is the r of the s S . This is t Theorem extended to u dimensions. Both sides are zero when $\mathbf{F}$ is a gradient field because v.

The four properties of a conservative field are $\mathbf{A} = \mathbf{w}$, $\mathbf{B} = \mathbf{x}$, $\mathbf{C} = \mathbf{y}$, $\mathbf{D} = \mathbf{z}$. The field $y^2z\mathbf{i} + 2xy^2z\mathbf{k}$ (passes)(fails) test $\mathbf{D}$. This field is the gradient of $f = \mathbf{A}$. The work $\int \mathbf{F} \cdot d\mathbf{R}$ from $(0, 0, 0)$ to $(1, 1, 1)$ is B (on which path?). For every field $\mathbf{F}$, $\iint \text{curl } \mathbf{F} \cdot \mathbf{n} dS$ is the same out through a pyramid and up through its base because C.

In Problems 1–6 find curl F.

- 1 $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$
- 2 $\mathbf{F} = \text{grad}(xe^y \sin z)$
- 3 $\mathbf{F} = (x + y + z)(\mathbf{i} + \mathbf{j} + \mathbf{k})$
- 4 $\mathbf{F} = (x + y)\mathbf{i} - (x + y)\mathbf{k}$
- 5 $\mathbf{F} = \rho^n(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$
- 6 $\mathbf{F} = (\mathbf{i} + \mathbf{j}) \times \mathbf{R}$
- 7 Find a potential f for the field in Problem 3.
- 8 Find a potential f for the field in Problem 5.
- 9 When do the fields $x^m\mathbf{i}$ and $x^n\mathbf{j}$ have zero curl?
- 10 When does $(a_1x + a_2y + a_3z)\mathbf{k}$ have zero curl?

In 11–14, compute $\text{curl } \mathbf{F}$ and find $\oint_C \mathbf{F} \cdot d\mathbf{R}$ by Stokes' Theorem.

11 $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{k}$, C = circle $x^2 + z^2 = 1$, $y = 0$.

12 $\mathbf{F} = \mathbf{i} \times \mathbf{R}$, C = circle $x^2 + z^2 = 1$, $y = 0$.

13 $\mathbf{F} = (\mathbf{i} + \mathbf{j}) \times \mathbf{R}$, C = circle $y^2 + z^2 = 1$, $x = 0$.

14 $\mathbf{F} = (y\mathbf{i} - x\mathbf{j}) \times (\mathbf{x}\mathbf{i} + \mathbf{y}\mathbf{j})$, C = circle $x^2 + y^2 = 1$, $z = 0$.

15 (important) Suppose two surfaces S and T have the same boundary C , and the direction around C is the same.

(a) Prove $\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dS = \iint_T \text{curl } \mathbf{F} \cdot \mathbf{n} dS$.

(b) Second proof: The difference between those integrals is $\iiint \text{div}(\text{curl } \mathbf{F}) dV$. By what Theorem? What region is V ? Why is this integral zero?

16 In 15, suppose S is the top half of the earth ($\mathbf{n}$ goes out) and T is the bottom half ($\mathbf{n}$ comes in). What are C and V ? Show by example that $\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_T \mathbf{F} \cdot \mathbf{n} dS$ is *not* generally true.

17 Explain why $\iint \text{curl } \mathbf{F} \cdot \mathbf{n} dS = 0$ over the closed boundary of any solid V .

18 Suppose $\text{curl } \mathbf{F} = \mathbf{0}$ and $\text{div } \mathbf{F} = 0$. (a) Why is $\mathbf{F}$ the gradient of a potential? (b) Why does the potential satisfy Laplace's equation $f_{xx} + f_{yy} + f_{zz} = 0$?

In 19–22, find a potential f if it exists.

19 $\mathbf{F} = z\mathbf{i} + \mathbf{j} + x\mathbf{k}$.

20 $\mathbf{F} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$

21 $\mathbf{F} = e^{x-z}\mathbf{i} - e^{x-z}\mathbf{k}$

22 $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + (xy + z^2)\mathbf{k}$

23 Find a field with $\text{curl } \mathbf{F} = (1, 0, 0)$.

24 Find all fields with $\text{curl } \mathbf{F} = (1, 0, 0)$.

25 $\mathbf{S} = \mathbf{a} \times \mathbf{R}$ is a spin field. Compute $\mathbf{F} = \mathbf{b} \times \mathbf{S}$ (constant vector $\mathbf{b}$) and find its curl.

26 How fast is a paddlewheel turned by the field $\mathbf{F} = y\mathbf{i} - x\mathbf{k}$ (a) if its axis direction is $\mathbf{n} = \mathbf{j}$? (b) if its axis is lined up with $\text{curl } \mathbf{F}$? (c) if its axis is perpendicular to $\text{curl } \mathbf{F}$?

27 How is $\text{curl } \mathbf{F}$ related to the angular velocity ω in the spin field $\mathbf{F} = \omega(-y\mathbf{i} + x\mathbf{j})$? How fast does a wheel spin, if it is in the plane $x + y + z = 1$?

28 Find a vector field $\mathbf{F}$ whose curl is $\mathbf{S} = y\mathbf{i} - x\mathbf{j}$.

29 Find a vector field $\mathbf{F}$ whose curl is $\mathbf{S} = \mathbf{a} \times \mathbf{R}$.

30 True or false: when two vector fields have the same curl at all points: (a) their difference is a constant field (b) their difference is a gradient field (c) they have the same divergence.

In 31–34, compute $\iint \text{curl } \mathbf{F} \cdot \mathbf{n} dS$ over the top half of the sphere $x^2 + y^2 + z^2 = 1$ and (separately) $\oint \mathbf{F} \cdot d\mathbf{R}$ around the equator.

31 $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$

32 $\mathbf{F} = \mathbf{R}/\rho^2$

33 $\mathbf{F} = \mathbf{a} \times \mathbf{R}$

34 $\mathbf{F} = (\mathbf{a} \times \mathbf{R}) \times \mathbf{R}$

35 The circle C in the plane $x + y + z = 6$ has radius r and center at $(1, 2, 3)$. The field $\mathbf{F}$ is $3z\mathbf{j} + 2y\mathbf{k}$. Compute $\oint \mathbf{F} \cdot d\mathbf{R}$ around C .

36 S is the top half of the unit sphere and $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + xyz\mathbf{k}$. Find $\iint \text{curl } \mathbf{F} \cdot \mathbf{n} dS$.

37 Find $g(x, y)$ so that $\text{curl } g\mathbf{k} = y\mathbf{i} + x^2\mathbf{j}$. What is the name for g in Section 15.3? It exists because $y\mathbf{i} + x^2\mathbf{j}$ has zero _____.

38 Construct $\mathbf{F}$ so that $\text{curl } \mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} - 5z\mathbf{k}$ (which has zero divergence).

39 Split the field $\mathbf{F} = xy\mathbf{i}$ into $\mathbf{V} + \mathbf{W}$ with $\text{curl } \mathbf{V} = 0$ and $\text{div } \mathbf{W} = 0$.

40 Ampère's law for a steady magnetic field $\mathbf{B}$ is $\text{curl } \mathbf{B} = \mu\mathbf{J}$ ($\mathbf{J}$ = current density, μ = constant). Find the work done by $\mathbf{B}$ around a space curve C from the current passing through it.

Maxwell allows varying currents which brings in the electric field.

41 For $\mathbf{F} = (x^2 + y^2)\mathbf{i}$, compute $\text{curl } (\text{curl } \mathbf{F})$ and $\text{grad } (\text{div } \mathbf{F})$ and $\mathbf{F}_{xx} + \mathbf{F}_{yy} + \mathbf{F}_{zz}$.

42 For $\mathbf{F} = v(x, y, z)\mathbf{i}$, prove these useful identities:

(a) $\text{curl}(\text{curl } \mathbf{F}) = \text{grad } (\text{div } \mathbf{F}) - (\mathbf{F}_{xx} + \mathbf{F}_{yy} + \mathbf{F}_{zz})$.

(b) $\text{curl}(f\mathbf{F}) = f \text{curl } \mathbf{F} + (\text{grad } f) \times \mathbf{F}$.

43 If $\mathbf{B} = \mathbf{a} \cos t$ (constant direction $\mathbf{a}$), find $\text{curl } \mathbf{E}$ from Faraday's Law. Then find the alternating spin field $\mathbf{E}$.

44 With $\mathbf{G}(x, y, z) = m\mathbf{i} + n\mathbf{j} + p\mathbf{k}$, write out $\mathbf{F} \times \mathbf{G}$ and take its divergence. Match the answer with $\mathbf{G} \cdot \text{curl } \mathbf{F} - \mathbf{F} \cdot \text{curl } \mathbf{G}$.

45 Write down Green's Theorem in the xz plane from Stokes' Theorem.

46 True or false: $\nabla \times \mathbf{F}$ is perpendicular to $\mathbf{F}$.

47 (a) The second proof of Stokes' Theorem took $M^* = M(x, y, f(x, y)) + P(x, y, f(x, y))\partial f/\partial x$ as the M in Green's Theorem. Compute $\partial M^*/\partial y$ from the chain rule and product rule (there are five terms).

(b) Similarly $N^* = N(x, y, f) + P(x, y, f)\partial f/\partial y$ has the x derivative $N_x + N_z f_x + P_x f_y + P_z f_x f_y + P f_{yx}$. Check that $N_x^* - M_y^*$ matches the right side of equation (8), as needed in the proof.

48 "The shadow of the boundary is the boundary of the shadow." This fact was used in the second proof of Stokes' Theorem, going to Green's Theorem on the shadow. Give two examples of S and C and their shadows.

49 Which integrals are equal when C = boundary of S or S = boundary of V ?

$$\oint \mathbf{F} \cdot d\mathbf{R} \quad \oint (\text{curl } \mathbf{F}) \cdot d\mathbf{R} \quad \oint (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS \quad \iint \mathbf{F} \cdot \mathbf{n} dS \\ \iint \text{div } \mathbf{F} dS \quad \iint (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS \quad \iint (\text{grad div } \mathbf{F}) \cdot \mathbf{n} dS \quad \iiint \text{div } \mathbf{F} dV$$

50 Draw the field $\mathbf{V} = -x\mathbf{k}$ spinning a wheel in the xz plane. What wheels would *not* spin?