

26 *True or false*, with reason:

- (a) The derivative of $\sin^2 x$ is $\cos^2 x$
- (b) The derivative of $\cos(-x)$ is $\sin x$
- (c) A positive function has a negative second derivative.
- (d) If y' is increasing then y'' is positive.

27 Find solutions to $dy/dx = \sin 3x$ and $dy/dx = \cos 3x$.

28 If $y = \sin 5x$ then $y' = 5 \cos 5x$ and $y'' = -25 \sin 5x$. So this function satisfies the differential equation $y'' = \underline{\hspace{2cm}}$.

29 If h is measured in degrees, find $\lim_{h \rightarrow 0} (\sin h)/h$. You could set your calculator in degree mode.

30 Write down a ratio that approaches dy/dx at $x = \pi$. For $y = \sin x$ and $\Delta x = .01$ compute that ratio.

31 By the square rule, the derivative of $(u(x))^2$ is $2u du/dx$. Take the derivative of each term in $\sin^2 x + \cos^2 x = 1$.

32 Give an example of oscillation that does not come from physics. Is it simple harmonic motion (one frequency only)?

33 Explain the second derivative in your own words.

2.5 The Product and Quotient and Power Rules

What are the derivatives of $x + \sin x$ and $x \sin x$ and $1/\sin x$ and $x/\sin x$ and $\sin^n x$? Those are made up from the familiar pieces x and $\sin x$, but we need new rules. Fortunately they are rules that apply to every function, so they can be established once and for all. If we know the separate derivatives of two functions u and v , then the derivatives of $u + v$ and uv and $1/v$ and u/v and u^n are immediately available.

This is a straightforward section, with those five rules to learn. It is also an important section, containing most of the working tools of differential calculus. But I am afraid that five rules and thirteen examples (which we need—the eyes glaze over with formulas alone) make a long list. At least the easiest rule comes first. *When we add functions, we add their derivatives.*

Sum Rule

The derivative of the sum $u(x) + v(x)$ is $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$. (1)

EXAMPLE 1 The derivative of $x + \sin x$ is $1 + \cos x$. That is tremendously simple, but it is fundamental. The interpretation for distances may be more confusing (and more interesting) than the rule itself:

Suppose a train moves with velocity 1. The distance at time t is t . On the train a professor paces back and forth (in simple harmonic motion). His distance from his seat is $\sin t$. Then the total distance from his starting point is $t + \sin t$, and his velocity (train speed plus walking speed) is $1 + \cos t$.

If you add distances, you add velocities. Actually that example is ridiculous, because the professor's maximum speed equals the train speed ($= 1$). He is running like mad, not pacing. Occasionally he is standing still with respect to the ground.

The sum rule is a special case of a bigger rule called "*linearity*." It applies when we add or subtract functions and multiply them by constants—as in $3x - 4 \sin x$. By linearity the derivative is $3 - 4 \cos x$. The rule works for all functions $u(x)$ and $v(x)$. A *linear combination* is $y(x) = au(x) + bv(x)$, where a and b are any real numbers. Then $\Delta y/\Delta x$ is

$$\frac{au(x + \Delta x) + bv(x + \Delta x) - au(x) - bv(x)}{\Delta x} = a \frac{u(x + \Delta x) - u(x)}{\Delta x} + b \frac{v(x + \Delta x) - v(x)}{\Delta x}.$$

The limit on the left is dy/dx . The limit on the right is $a du/dx + b dv/dx$. We are allowed to take limits separately and add. The result is what we hope for:

Rule of Linearity

The derivative of $au(x) + bv(x)$ is $\frac{d}{dx}(au + bv) = a \frac{du}{dx} + b \frac{dv}{dx}$. (2)

The **product rule** comes next. It can't be so simple—products are not linear. The sum rule is what you would have done anyway, but products give something new. **The derivative of u times v is not du/dx times dv/dx .** Example: The derivative of x^5 is $5x^4$. Don't multiply the derivatives of x^3 and x^2 . ($3x^2$ times $2x$ is not $5x^4$.) *For a product of two functions, the derivative has two terms.*

Product Rule (the key to this section)

The derivative of $u(x)v(x)$ is $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$. (3)

EXAMPLE 2 $u = x^3$ times $v = x^2$ is $uv = x^5$. The product rule leads to $5x^4$:

$$x^3 \frac{dv}{dx} + x^2 \frac{du}{dx} = x^3(2x) + x^2(3x^2) = 2x^4 + 3x^4 = 5x^4.$$

EXAMPLE 3 In the slope of $x \sin x$, I don't write $dx/dx = 1$ but it's there:

$$\frac{d}{dx}(x \sin x) = x \cos x + \sin x.$$

EXAMPLE 4 If $u = \sin x$ and $v = \sin x$ then $uv = \sin^2 x$. We get two equal terms:

$$\sin x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(\sin x) = 2 \sin x \cos x.$$

This confirms the “square rule” $2u du/dx$, when u is the same as v . Similarly the slope of $\cos^2 x$ is $-2 \cos x \sin x$ (minus sign from the slope of the cosine).

Question Those answers for $\sin^2 x$ and $\cos^2 x$ have opposite signs, so the derivative of $\sin^2 x + \cos^2 x$ is zero (sum rule). How do you see that more quickly?

EXAMPLE 5 The derivative of uvw is $uvw' + uv'w + u'vw$ —one derivative at a time. The derivative of xxx is $xx + xx + xx$.

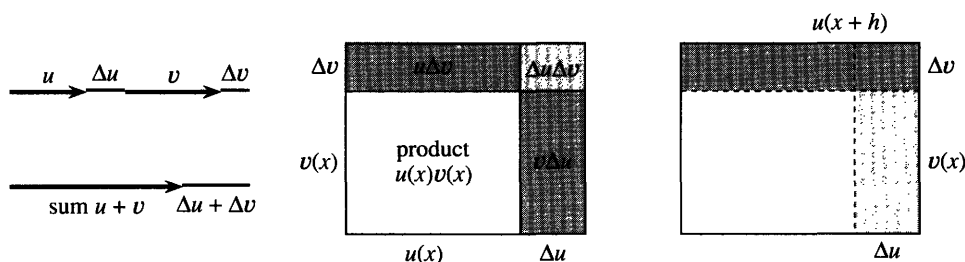


Fig. 2.13 Change in length = $\Delta u + \Delta v$. Change in area = $u\Delta v + v\Delta u + \Delta u\Delta v$.

After those examples we prove the product rule. Figure 2.13 explains it best. The area of the big rectangle is uv . **The important changes in area are the two strips $u \Delta v$ and $v \Delta u$.** The corner area $\Delta u \Delta v$ is much smaller. When we divide by Δx , the strips give $u \Delta v / \Delta x$ and $v \Delta u / \Delta x$. The corner gives $\Delta u \Delta v / \Delta x$, which approaches zero.

Notice how the sum rule is in one dimension and the product rule is in two dimensions. The rule for uvw would be in three dimensions.

The extra area comes from the whole top strip plus the side strip. By algebra,

$$u(x+h)v(x+h) - u(x)v(x) = u(x+h)[v(x+h) - v(x)] + v(x)[u(x+h) - u(x)]. \quad (4)$$

This increase is $u(x+h)\Delta v + v(x)\Delta u$ —top plus side. Now divide by h (or Δx) and let $h \rightarrow 0$. The left side of equation (4) becomes the derivative of $u(x)v(x)$. The right side becomes $u(x)$ times dv/dx —we can multiply the two limits—plus $v(x)$ times du/dx . That proves the product rule—definitely useful.

We could go immediately to the quotient rule for $u(x)/v(x)$. But start with $u = 1$. The derivative of $1/x$ is $-1/x^2$ (known). What is the derivative of $1/v(x)$?

Reciprocal Rule

$$\text{The derivative of } \frac{1}{v(x)} \text{ is } \frac{-dv/dx}{v^2}. \quad (5)$$

The proof starts with $(v)(1/v) = 1$. The derivative of 1 is 0. Apply the product rule:

$$v \frac{d}{dx} \left(\frac{1}{v} \right) + \frac{1}{v} \frac{dv}{dx} = 0 \quad \text{so that} \quad \frac{d}{dx} \left(\frac{1}{v} \right) = \frac{-dv/dx}{v^2}. \quad (6)$$

It is worth checking the units—in the reciprocal rule and others. A test of dimensions is automatic in science and engineering, and a good idea in mathematics. The test ignores constants and plus or minus signs, but it prevents bad errors. If v is in dollars and x is in hours, dv/dx is in *dollars per hour*. Then dimensions agree:

$$\frac{d}{dx} \left(\frac{1}{v} \right) \approx \frac{(1/\text{dollars})}{\text{hour}} \quad \text{and also} \quad \frac{-dv/dx}{v^2} \approx \frac{\text{dollars/hour}}{(\text{dollars})^2}.$$

From this test, the derivative of $1/v$ cannot be $1/(dv/dx)$. A similar test shows that Einstein's formula $e = mc^2$ is dimensionally possible. The theory of relativity might be correct! Both sides have the dimension of $(\text{mass})(\text{distance})^2/(\text{time})^2$, when mass is converted to energy.†

EXAMPLE 6 The derivatives of x^{-1} , x^{-2} , x^{-n} are $-1x^{-2}$, $-2x^{-3}$, $-nx^{-n-1}$.

Those come from the reciprocal rule with $v = x$ and x^2 and any x^n :

$$\frac{d}{dx} (x^{-n}) = \frac{d}{dx} \left(\frac{1}{x^n} \right) = -\frac{nx^{n-1}}{(x^n)^2} = -nx^{-n-1}.$$

The beautiful thing is that this answer $-nx^{-n-1}$ fits into the same pattern as x^n . **Multiply by the exponent and reduce it by one.**

$$\text{For negative and positive exponents the derivative of } x^n \text{ is } nx^{n-1}. \quad (7)$$

†But only Einstein knew that the constant is 1.

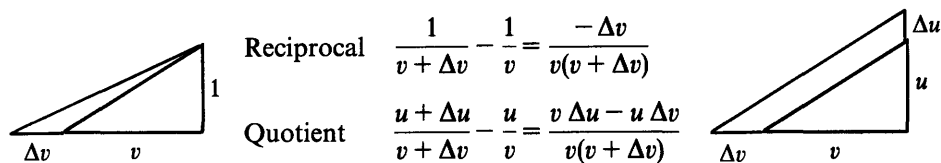


Fig. 2.14 Reciprocal rule from $(-\Delta v)/v^2$. Quotient rule from $(v \Delta u - u \Delta v)/v^2$.

EXAMPLE 7 The derivatives of $\frac{1}{\cos x}$ and $\frac{1}{\sin x}$ are $\frac{+\sin x}{\cos^2 x}$ and $\frac{-\cos x}{\sin^2 x}$.

Those come directly from the reciprocal rule. In trigonometry, $1/\cos x$ is the *secant* of the angle x , and $1/\sin x$ is the *cosecant* of x . Now we have their derivatives:

$$\frac{d}{dx}(\sec x) = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \frac{\sin x}{\cos x} = \sec x \tan x. \quad (8)$$

$$\frac{d}{dx}(\csc x) = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x} \frac{\cos x}{\sin x} = -\csc x \cot x. \quad (9)$$

Those formulas are often seen in calculus. If you have a good memory they are worth storing. Like most mathematicians, I have to check them every time before using them (maybe once a year). It is really the rules that are basic, not the formulas.

The next rule applies to the quotient $u(x)/v(x)$. That is u times $1/v$. Combining the product rule and reciprocal rule gives something new and important:

Quotient Rule

$$\text{The derivative of } \frac{u(x)}{v(x)} \text{ is } \frac{1}{v} \frac{du}{dx} - u \frac{dv/dx}{v^2} = \frac{v \, du/dx - u \, dv/dx}{v^2}.$$

You *must* memorize that last formula. The v^2 is familiar. The rest is new, but not very new. If $v = 1$ the result is du/dx (of course). For $u = 1$ we have the reciprocal rule. Figure 2.14b shows the difference $(u + \Delta u)/(v + \Delta v) - (u/v)$. The denominator $v(v + \Delta v)$ is responsible for v^2 .

EXAMPLE 8 (only practice) If $u/v = x^5/x^3$ (which is x^2) the quotient rule gives $2x$:

$$\frac{d}{dx} \left(\frac{x^5}{x^3} \right) = \frac{x^3(5x^4) - x^5(3x^2)}{x^6} = \frac{5x^7 - 3x^7}{x^6} = 2x.$$

EXAMPLE 9 (important) For $u = \sin x$ and $v = \cos x$, the quotient is $\sin x/\cos x = \tan x$. *The derivative of $\tan x$ is $\sec^2 x$.* Use the quotient rule and $\cos^2 x + \sin^2 x = 1$:

$$\frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x. \quad (11)$$

Again to memorize: $(\tan x)' = \sec^2 x$. At $x = 0$, this slope is 1. The graphs of $\sin x$ and x and $\tan x$ all start with this slope (then they separate). At $x = \pi/2$ the sine curve is flat ($\cos x = 0$) and the tangent curve is vertical ($\sec^2 x = \infty$).

The slope generally blows up faster than the function. We divide by $\cos x$, once for the tangent and twice for its slope. The slope of $1/x$ is $-1/x^2$. The slope is more sensitive than the function, because of the square in the denominator.

EXAMPLE 10

$$\frac{d}{dx} \left(\frac{\sin x}{x} \right) = \frac{x \cos x - \sin x}{x^2}.$$

That one I hesitate to touch at $x = 0$. Formally it becomes $0/0$. In reality it is more like $0^3/0^2$, and the true derivative is zero. Figure 2.10 showed graphically that $(\sin x)/x$ is flat at the center point. The function is *even* (symmetric across the y axis) so its derivative can only be zero.

This section is full of rules, and I hope you will allow one more. It goes beyond x^n to $(u(x))^n$. A power of x changes to a power of $u(x)$ —as in $(\sin x)^6$ or $(\tan x)^7$ or $(x^2 + 1)^8$. The derivative contains nu^{n-1} (copying nx^{n-1}), but **there is an extra factor** du/dx . Watch that factor in $6(\sin x)^5 \cos x$ and $7(\tan x)^6 \sec^2 x$ and $8(x^2 + 1)^7(2x)$:

Power Rule

$$\text{The derivative of } [u(x)]^n \text{ is } n[u(x)]^{n-1} \frac{du}{dx}. \quad (12)$$

For $n = 1$ this reduces to $du/dx = du/dx$. For $n = 2$ we get the square rule $2u du/dx$. Next comes u^3 . The best approach is to use **mathematical induction**, which goes from each n to the next power $n + 1$ by the product rule:

$$\frac{d}{dx}(u^{n+1}) = \frac{d}{dx}(u^n u) = u^n \frac{du}{dx} + u \left(nu^{n-1} \frac{du}{dx} \right) = (n+1)u^n \frac{du}{dx}.$$

That is exactly equation (12) for the power $n + 1$. We get all positive powers this way, going up from $n = 1$ —then the negative powers come from the reciprocal rule.

Figure 2.15 shows the power rule for $n = 1, 2, 3$. The cube makes the point best. The three thin slabs are u by u by Δu . **The change in volume is essentially** $3u^2 \Delta u$. From multiplying out $(u + \Delta u)^3$, the exact change in volume is $3u^2 \Delta u + 3u(\Delta u)^2 + (\Delta u)^3$ —which also accounts for three narrow boxes and a midget cube in the corner. This is the binomial formula in a picture.

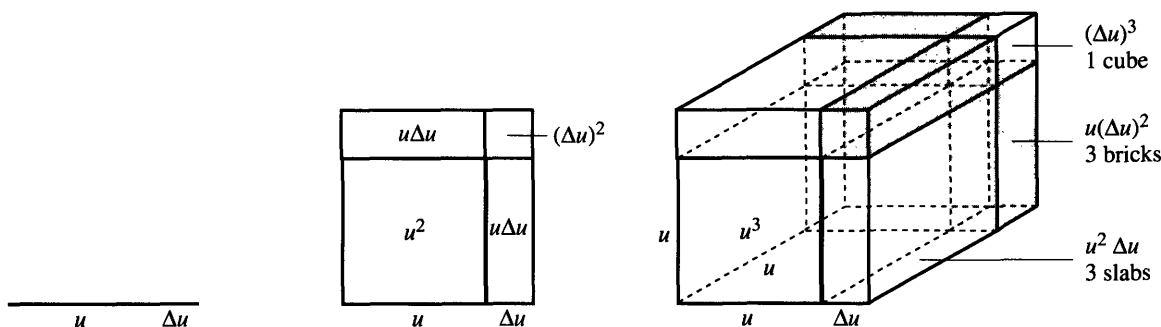


Fig. 2.15 Length change $= \Delta u$; area change $\approx 2u \Delta u$; volume change $\approx 3u^2 \Delta u$.

EXAMPLE 11 $\frac{d}{dx}(\sin x)^n = n(\sin x)^{n-1} \cos x$. The extra factor $\cos x$ is du/dx .

Our last step finally escapes from a very undesirable restriction—that n must be a whole number. We want to allow fractional powers $n = p/q$, and keep the same formula. *The derivative of x^n is still nx^{n-1} .*

To deal with square roots I can write $(\sqrt{x})^2 = x$. Its derivative is $2\sqrt{x}(\sqrt{x})' = 1$. Therefore $(\sqrt{x})'$ is $1/2\sqrt{x}$, which fits the formula when $n = \frac{1}{2}$. Now try $n = p/q$:

Fractional powers Write $u = x^{p/q}$ as $u^q = x^p$. Take derivatives, assuming they exist:

$$qu^{q-1} \frac{du}{dx} = px^{p-1} \quad (\text{power rule on both sides})$$

$$\frac{du}{dx} = \frac{px^{-1}}{qu^{-1}} \quad (\text{cancel } x^p \text{ with } u^q)$$

$$\frac{du}{dx} = nx^{n-1} \quad (\text{replace } p/q \text{ by } n \text{ and } u \text{ by } x^n)$$

EXAMPLE 12 The slope of $x^{1/3}$ is $\frac{1}{3}x^{-2/3}$. The slope is infinite at $x = 0$ and zero at $x = \infty$. But the curve in Figure 2.16 keeps climbing. It doesn't stay below an "asymptote."

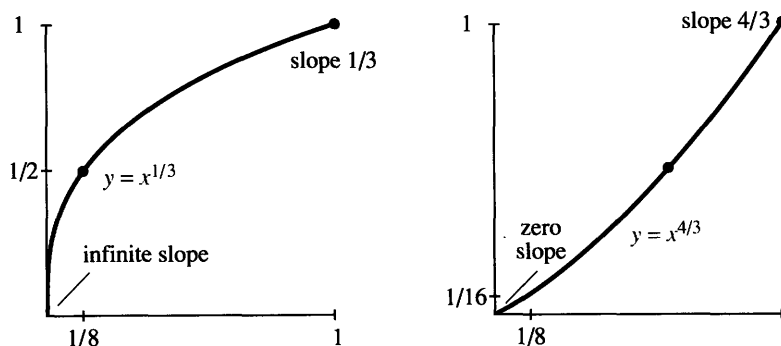


Fig. 2.16 Infinite slope of x^n versus zero slope: the difference between $0 < n < 1$ and $n > 1$.

EXAMPLE 13 The slope of $x^{4/3}$ is $\frac{4}{3}x^{1/3}$. The slope is zero at $x = 0$ and infinite at $x = \infty$. The graph climbs faster than a line and slower than a parabola ($\frac{4}{3}$ is between 1 and 2). Its slope follows the cube root curve (times $\frac{4}{3}$).

WE STOP NOW! I am sorry there were so many rules. A computer can memorize them all, but it doesn't know what they mean and you do. Together with the chain rule that dominates Chapter 4, they achieve virtually all the derivatives ever computed by mankind. We list them in one place for convenience.

Rule of Linearity	$(au + bv)' = au' + bv'$
Product Rule	$(uv)' = uv' + vu'$
Reciprocal Rule	$(1/v)' = -v'/v^2$
Quotient Rule	$(u/v)' = (vu' - uv')/v^2$
Power Rule	$(u^n)' = nu^{n-1}u'$

The power rule applies when n is *negative*, or a *fraction*, or *any real number*. The derivative of x^π is $\pi x^{\pi-1}$, according to Chapter 6. The derivative of $(\sin x)^\pi$ is _____. And the derivatives of all six trigonometric functions are now established:

$$\begin{aligned} (\sin x)' &= \cos x & (\tan x)' &= \sec^2 x & (\sec x)' &= \sec x \tan x \\ (\cos x)' &= -\sin x & (\cot x)' &= -\csc^2 x & (\csc x)' &= -\csc x \cot x \end{aligned}$$

2.5 EXERCISES

Read-through questions

The derivatives of $\sin x$, $\cos x$ and $1/\cos x$ and $\sin x/\cos x$ and $\tan^3 x$ come from the a rule, b rule, c rule, and d rule. The product of $\sin x$ times $\cos x$ has $(uv)' = uv' + \underline{e} = \underline{f}$. The derivative of $1/v$ is g, so the slope of $\sec x$ is h. The derivative of u/v is i, so the slope of $\tan x$ is j. The derivative of $\tan^3 x$ is k. The slope of x^n is l and the slope of $(u(x))^n$ is m. With $n = -1$ the derivative of $(\cos x)^{-1}$ is n, which agrees with the rule for $\sec x$.

Even simpler is the rule of o, which applies to $au(x) + bv(x)$. The derivative is p. The slope of $3 \sin x + 4 \cos x$ is q. The derivative of $(3 \sin x + 4 \cos x)^2$ is r. The derivative of s is $4 \sin^3 x \cos x$.

Find the derivatives of the functions in 1–26.

- | | |
|--|--|
| 1 $(x+1)(x-1)$ | 2 $(x^2+1)(x^2-1)$ |
| 3 $\frac{1}{1+x} + \frac{1}{1+\sin x}$ | 4 $\frac{1}{1+x^2} + \frac{1}{1-\sin x}$ |
| 5 $(x-1)(x-2)(x-3)$ | 6 $(x-1)^2(x-2)^2$ |
| 7 $x^2 \cos x + 2x \sin x$ | 8 $x^{1/2}(x + \sin x)$ |
| 9 $\frac{x^3+1}{x+1} + \frac{\cos x}{\sin x}$ | 10 $\frac{x^2+1}{x^2-1} + \frac{\sin x}{\cos x}$ |
| 11 $x^{1/2} \sin^2 x + (\sin x)^{1/2}$ | 12 $x^{3/2} \sin^3 x + (\sin x)^{3/2}$ |
| 13 $x^4 \cos x + x \cos^4 x$ | 14 $\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+2)$ |
| 15 $\frac{1}{2}x^2 \sin x - x \cos x + \sin x$ | 16 $(x-6)^{10} + \sin^{10} x$ |
| 17 $\sec^2 x - \tan^2 x$ | 18 $\csc^2 x - \cot^2 x$ |
| 19 $\frac{4}{(x-5)^{2/3}} + \frac{4}{(5-x)^{2/3}}$ | 20 $\frac{\sin x - \cos x}{\sin x + \cos x}$ |
| 21 $(\sin x \cos x)^3 + \sin 2x$ | 22 $x \cos x \csc x$ |
| 23 $u(x)v(x)w(x)z(x)$ | 24 $[u(x)]^2[v(x)]^2$ |
| 25 $\frac{1}{\tan x} - \frac{1}{\cot x}$ | 26 $x \sin x + \cos x$ |

27 A growing box has length t , width $1/(1+t)$, and height $\cos t$.

- (a) What is the rate of change of the volume?
 (b) What is the rate of change of the surface area?

28 With two applications of the product rule show that the derivative of uvw is $uvw' + uv'w + u'vw$. When a box with sides u, v, w grows by $\Delta u, \Delta v, \Delta w$, three slabs are added with volume $uv \Delta w$ and _____ and _____.

29 Find the velocity if the distance is $f(t) =$

$$5t^2 \text{ for } t \leq 10, \quad 500 + 100\sqrt{t-10} \text{ for } t \geq 10.$$

30 A cylinder has radius $r = \frac{t^{3/2}}{1+t^{3/2}}$ and height $h = \frac{t}{1+t}$.

- (a) What is the rate of change of its volume?
 (b) What is the rate of change of its surface area (including top and base)?

31 The height of a model rocket is $f(t) = t^3/(1+t)$.

- (a) What is the velocity $v(t)$?
 (b) What is the acceleration dv/dt ?

32 Apply the product rule to $u(x)u^2(x)$ to find the power rule for $u^3(x)$.

33 Find the *second* derivative of the product $u(x)v(x)$. Find the *third* derivative. Test your formulas on $u = v = x$.

34 Find functions $y(x)$ whose derivatives are

- (a) x^3 (b) $1/x^3$ (c) $(1-x)^{3/2}$ (d) $\cos^2 x \sin x$.

35 Find the distances $f(t)$, starting from $f(0) = 0$, to match these velocities:

- (a) $v(t) = \cos t \sin t$ (b) $v(t) = \tan t \sec^2 t$
 (c) $v(t) = \sqrt{1+t}$

36 Apply the quotient rule to $(u(x))^3/(u(x))^2$ and $-v'/v^2$. The latter gives the second derivative of _____.

37 Draw a figure like 2.13 to explain the *square rule*.

38 Give an example where $u(x)/v(x)$ is increasing but $du/dx = dv/dx = 1$.

39 *True or false*, with a good reason:

- (a) The derivative of x^{2n} is $2nx^{2n-1}$.
 (b) By linearity the derivative of $a(x)u(x) + b(x)v(x)$ is $a(x)du/dx + b(x)dv/dx$.
 (c) The derivative of $|x|^3$ is $3|x|^2$.
 (d) $\tan^2 x$ and $\sec^2 x$ have the same derivative.
 (e) $(uv)' = u'v'$ is true when $u(x) = 1$.

40 The cost of u shares of stock at v dollars per share is uv dollars. Check dimensions of $d(uv)/dt$ and $u dv/dt$ and $v du/dt$.

41 If $u(x)/v(x)$ is a ratio of polynomials of degree n , what are the degrees for its derivative?

42 For $y = 5x + 3$, is $(dy/dx)^2$ the same as $d^2 y/dx^2$?

43 If you change from $f(t) = t \cos t$ to its tangent line at $t = \pi/2$, find the two-part function df/dt .

44 Explain in your own words why the derivative of $u(x)v(x)$ has two terms.

45 A plane starts its descent from height $y = h$ at $x = -L$ to land at $(0, 0)$. Choose a, b, c, d so its landing path $y = ax^3 + bx^2 + cx + d$ is *smooth*. With $dx/dt = V = \text{constant}$, find dy/dt and $d^2 y/dt^2$ at $x = 0$ and $x = -L$. (To keep $d^2 y/dt^2$ small, a coast-to-coast plane starts down $L > 100$ miles from the airport.)

2.6 Limits

You have seen enough limits to be ready for a definition. It is true that we have survived this far without one, and we could continue. But this seems a reasonable time to define limits more carefully. The goal is to achieve rigor without rigor mortis.

First you should know that limits of $\Delta y/\Delta x$ are by no means the only limits in mathematics. Here are five completely different examples. They involve $n \rightarrow \infty$, not $\Delta x \rightarrow 0$:

1. $a_n = (n - 3)/(n + 3)$ (for large n , ignore the 3's and find $a_n \rightarrow 1$)
2. $a_n = \frac{1}{2}a_{n-1} + 4$ (start with any a_1 and always $a_n \rightarrow 8$)
3. a_n = probability of living to year n (unfortunately $a_n \rightarrow 0$)
4. a_n = fraction of zeros among the first n digits of π ($a_n \rightarrow \frac{1}{10}$?)
5. $a_1 = .4, a_2 = .49, a_3 = .493, \dots$ No matter what the remaining decimals are, the a 's converge to a limit. Possibly $a_n \rightarrow .493000 \dots$, but not likely.

The problem is to say what the limit symbol $\rightarrow$ really means.

A good starting point is to ask about convergence to zero. When does a sequence of positive numbers approach zero? What does it mean to write $a_n \rightarrow 0$? The numbers $a_1, a_2, a_3, \dots$, must become "small," but that is too vague. We will propose four definitions of *convergence to zero*, and I hope the right one will be clear.

1. All the numbers a_n are below 10^{-10} . That may be enough for practical purposes, but it certainly doesn't make the a_n approach zero.
2. The sequence is getting closer to zero—each a_{n+1} is smaller than the preceding a_n . This test is met by 1.1, 1.01, 1.001, ... which converges to 1 instead of 0.
3. For any small number you think of, at least one of the a_n 's is smaller. That pushes something toward zero, but not necessarily the whole sequence. The condition would be satisfied by $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \dots$, which does not approach zero.
4. For any small number you think of, the a_n 's eventually go below that number and stay below. This is the correct definition.

I want to repeat that. To test for convergence to zero, start with a small number—say 10^{-10} . The a_n 's must go *below that number*. They may come back up and go below again—the first million terms make absolutely no difference. Neither do the next billion, but eventually all terms must go below 10^{-10} . After waiting longer (possibly a lot longer), all terms drop below 10^{-20} . The tail end of the sequence decides everything.

Question 1 Does the sequence $10^{-3}, 10^{-2}, 10^{-6}, 10^{-5}, 10^{-9}, 10^{-8}, \dots$ approach 0?
Answer Yes. These up and down numbers eventually stay below any ε .

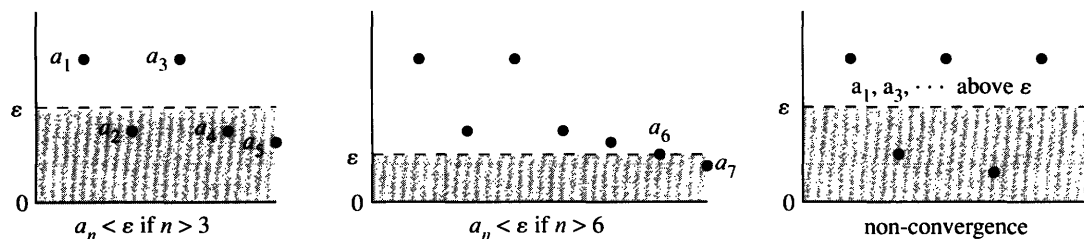


Fig. 2.17 Convergence means: Only a finite number of a 's are outside any strip around L .

Question 2 Does $10^{-4}, 10^{-6}, 10^{-4}, 10^{-8}, 10^{-4}, 10^{-10}, \dots$ approach zero?

Answer No. This sequence goes below 10^{-4} but does not stay below.

There is a recognized symbol for “an arbitrarily small positive number.” By worldwide agreement, it is the Greek letter ε (*epsilon*). Convergence to zero means that *the sequence eventually goes below ε and stays there*. The smaller the ε , the tougher the test and the longer we wait. Think of ε as the tolerance, and keep reducing it.

To emphasize that ε comes from outside, Socrates can choose it. Whatever ε he proposes, the a 's must eventually be smaller. *After some a_N , all the a 's are below the tolerance ε .* Here is the exact statement:

for any ε there is an N such that $a_n < \varepsilon$ if $n > N$.

Once you see that idea, the rest is easy. Figure 2.17 has $N = 3$ and then $N = 6$.

EXAMPLE 1 The sequence $\frac{1}{2}, \frac{4}{8}, \frac{9}{8}, \dots$ starts upward but goes to zero. Notice that $1, 4, 9, \dots, 100, \dots$ are squares, and $2, 4, 8, \dots, 1024, \dots$ are powers of 2. Eventually 2^n grows faster than n^2 , as in $a_{10} = 100/1024$. The ratio goes below any ε .

EXAMPLE 2 $1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \dots$ approaches zero. These a 's do not decrease steadily (the mathematical word for steadily is “monotonically”) but still their limit is zero. The choice $\varepsilon = 1/1000$ produces the right response: *Beyond a_{2001} all terms are below $1/1000$.* So $N = 2001$ for that ε .

The sequence $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots$ is much slower—but it also converges to zero.

Next we allow the numbers a_n to be *negative* as well as positive. They can converge upward toward zero, or they can come in from both sides. The test still requires the a_n to go inside any strip near zero (and stay there). But now the strip starts at $-\varepsilon$.

The distance from zero is the absolute value $|a_n|$. Therefore $a_n \rightarrow 0$ means $|a_n| \rightarrow 0$. The previous test can be applied to $|a_n|$:

for any ε there is an N such that $|a_n| < \varepsilon$ if $n > N$.

EXAMPLE 3 $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots$ converges to zero because $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ converges to zero.

It is a short step to limits other than zero. *The limit is L if the numbers $a_n - L$ converge to zero.* Our final test applies to the absolute value $|a_n - L|$:

for any ε there is an N such that $|a_n - L| < \varepsilon$ if $n > N$.

This is the definition of convergence! Only a finite number of a 's are outside any strip around L (Figure 2.18). We write $a_n \rightarrow L$ or $\lim a_n = L$ or $\lim_{n \rightarrow \infty} a_n = L$.

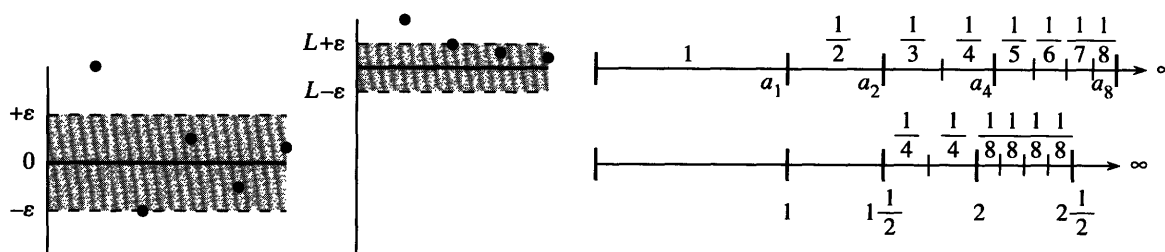


Fig. 2.18 $a_n \rightarrow 0$ in Example 3; $a_n \rightarrow 1$ in Example 4; $a_n \rightarrow \infty$ in Example 5 (but $a_{n+1} - a_n \rightarrow 0$).

EXAMPLE 4 The numbers $\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \dots$ converge to $L = 1$. After subtracting 1 the differences $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots$ converge to zero. Those difference are $|a_n - L|$.

EXAMPLE 5 The sequence $1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \dots$ fails to converge.

The distance between terms is getting smaller. But those numbers $a_1, a_2, a_3, a_4, \dots$ go past any proposed limit L . The second term is $1\frac{1}{2}$. The fourth term adds on $\frac{1}{3} + \frac{1}{4}$, so a_4 goes past 2. The eighth term has four new fractions $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$, totaling more than $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$. Therefore a_8 exceeds $2\frac{1}{2}$. Eight more terms will add more than 8 times $\frac{1}{16}$, so a_{16} is beyond 3. The lines in Figure 2.18c are infinitely long, not stopping at any L .

In the language of Chapter 10, the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \dots$ does not converge. The sum is infinite, because the “partial sums” a_n go beyond every limit L (a_{5000} is past $L = 9$). We will come back to infinite series, but this example makes a subtle point: The steps between the a_n can go to zero while still $a_n \rightarrow \infty$.

Thus the condition $a_{n+1} - a_n \rightarrow 0$ is *not sufficient* for convergence. However this condition is *necessary*. If we do have convergence, then $a_{n+1} - a_n \rightarrow 0$. That is a good exercise in the logic of convergence, emphasizing the difference between “sufficient” and “necessary.” We discuss this logic below, after proving that [statement A] implies [statement B]:

If $[a_n \text{ converges to } L]$ then $[a_{n+1} - a_n \text{ converges to zero}]$. (1)

Proof Because the a_n converge, there is a number N beyond which $|a_n - L| < \varepsilon$ and also $|a_{n+1} - L| < \varepsilon$. Since $a_{n+1} - a_n$ is the sum of $a_{n+1} - L$ and $L - a_n$, its absolute value cannot exceed $\varepsilon + \varepsilon = 2\varepsilon$. Therefore $a_{n+1} - a_n$ approaches zero.

Objection by Socrates: We only got below 2ε and he asked for ε . *Our reply:* If he particularly wants $|a_{n+1} - a_n| < 1/10$, we start with $\varepsilon = 1/20$. Then $2\varepsilon = 1/10$. But this juggling is not necessary. To stay below 2ε is just as convincing as to stay below ε .

THE LOGIC OF “IF” AND “ONLY IF”

The following page is inserted to help with the language of mathematics. In ordinary language we might say “I will come if you call.” Or we might say “I will come only if you call.” That is different! A mathematician might even say “I will come *if and only if* you call.” Our goal is to think through the logic, because it is important and not so familiar.†

Statement A above implies statement B . Statement A is $a_n \rightarrow L$; statement B is $a_{n+1} - a_n \rightarrow 0$. Mathematics has at least five ways of writing down $A \Rightarrow B$, and I though you might like to see them together. It seems excessive to have so many expressions for the same idea, but authors get desperate for a little variety. Here are the five ways that come to mind:

$$A \Rightarrow B$$

A implies B

if A then B

A is a *sufficient* condition for B

B is true *if* A is true

†Logical thinking is much more important than ε and δ .

EXAMPLES *If* [positive numbers are decreasing] **then** [they converge to a limit].

If [sequences a_n and b_n converge] **then** [the sequence $a_n + b_n$ converges].

If [$f(x)$ is the integral of $v(x)$] **then** [$v(x)$ is the derivative of $f(x)$].

Those are all true, but not proved. A is the hypothesis, B is the conclusion.

Now we go in the other direction. (It is called the “converse,” not the inverse.) *We exchange A and B .* Of course stating the converse does not make it true! B might imply A , or it might not. In the first two examples the converse was false—the a_n can converge without decreasing, and $a_n + b_n$ can converge when the separate sequences do not. The converse of the third statement is true—and there are five more ways to state it:

$$A \Leftarrow B$$

A is implied by B

if B then A

A is a *necessary* condition for B

B is true *only if* A is true

Those words “necessary” and “sufficient” are not always easy to master. The same is true of the deceptively short phrase “if and only if.” The two statements $A \Rightarrow B$ and $A \Leftarrow B$ are completely different and *they both require proof*. That means two separate proofs. But they can be stated together for convenience (when both are true):

$$A \Leftrightarrow B$$

A implies B and B implies A

A is *equivalent* to B

A is a *necessary and sufficient* condition for B

A is true *if and only if* B is true

EXAMPLES $[a_n \rightarrow L] \Leftrightarrow [2a_n \rightarrow 2L] \Leftrightarrow [a_n + 1 \rightarrow L + 1] \Leftrightarrow [a_n - L \rightarrow 0]$.

RULES FOR LIMITS

Calculus needs a *definition of limits*, to define dy/dx . That derivative contains two limits: $\Delta x \rightarrow 0$ and $\Delta y/\Delta x \rightarrow dy/dx$. Calculus also needs *rules for limits*, to prove the sum rule and product rule for derivatives. We started on the definition, and now we start on the rules.

Given two convergent sequences, $a_n \rightarrow L$ and $b_n \rightarrow M$, other sequences also converge:

Addition: $a_n + b_n \rightarrow L + M$ Subtraction: $a_n - b_n \rightarrow L - M$

Multiplication: $a_n b_n \rightarrow LM$ Division: $a_n/b_n \rightarrow L/M$ (provided $M \neq 0$)

We check the multiplication rule, which uses a convenient identity:

$$a_n b_n - LM = (a_n - L)(b_n - M) + M(a_n - L) + L(b_n - M). \quad (2)$$

Suppose $|a_n - L| < \varepsilon$ beyond some point N , and $|b_n - M| < \varepsilon$ beyond some other point N' . Then beyond the larger of N and N' , the right side of (2) is small. It is less than $\varepsilon \sim \varepsilon + M\varepsilon + L\varepsilon$. This proves that (2) gives $a_n b_n \rightarrow LM$.

An important special case is $ca_n \rightarrow cL$. (The sequence of b 's is $c, c, c, c, \dots$.) Thus a constant can be brought “outside” the limit, to give $\lim ca_n = c \lim a_n$.

THE LIMIT OF $f(x)$ AS $x \rightarrow a$

The final step is to replace sequences by functions. Instead of $a_1, a_2, \dots$ there is a continuum of values $f(x)$. The limit is taken as x approaches a specified point a (instead of $n \rightarrow \infty$). Example: As x approaches $a = 0$, the function $f(x) = 4 - x^2$ approaches $L = 4$. As x approaches $a = 2$, the function $5x$ approaches $L = 10$. Those statements are fairly obvious, but we have to say what they mean. Somehow it must be this:

if x is close to a then $f(x)$ is close to L .

If $x - a$ is small, then $f(x) - L$ should be small. As before, the word *small* does not say everything. We really mean “arbitrarily small,” or “below any ε .” The difference $f(x) - L$ must become *as small as anyone wants*, when x gets near a . In that case $\lim_{x \rightarrow a} f(x) = L$. Or we write $f(x) \rightarrow L$ as $x \rightarrow a$.

The statement is awkward because it involves *two limits*. The limit $x \rightarrow a$ is forcing $f(x) \rightarrow L$. (Previously $n \rightarrow \infty$ forced $a_n \rightarrow L$.) But it is wrong to expect the same ε in both limits. We do not and cannot require that $|x - a| < \varepsilon$ produces $|f(x) - L| < \varepsilon$. *It may be necessary to push x extremely close to a* (closer than ε). We must guarantee that if x is close enough to a , then $|f(x) - L| < \varepsilon$.

We have come to the “*epsilon-delta definition*” of limits. First, Socrates chooses ε . He has to be shown that $f(x)$ is within ε of L , for every x near a . Then somebody else (maybe Plato) replies with a number δ . That gives the meaning of “near a .” Plato’s goal is to get $f(x)$ within ε of L , by keeping x within δ of a :

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon. \quad (3)$$

The input tolerance is δ (delta), the output tolerance is ε . When Plato can find a δ for every ε , Socrates concedes that the limit is L .

EXAMPLE Prove that $\lim_{x \rightarrow 2} 5x = 10$. In this case $a = 2$ and $L = 10$.

Socrates asks for $|5x - 10| < \varepsilon$. Plato responds by requiring $|x - 2| < \delta$. What δ should he choose? In this case $|5x - 10|$ is exactly 5 times $|x - 2|$. So Plato picks δ below $\varepsilon/5$ (a smaller δ is always OK). Whenever $|x - 2| < \varepsilon/5$, multiplication by 5 shows that $|5x - 10| < \varepsilon$.

Remark 1 In Figure 2.19, Socrates chooses the height of the box. It extends above and below L , by the small number ε . Second, Plato chooses the width. He must make the box narrow enough for the graph to go *out the sides*. Then $|f(x) - L| < \varepsilon$.

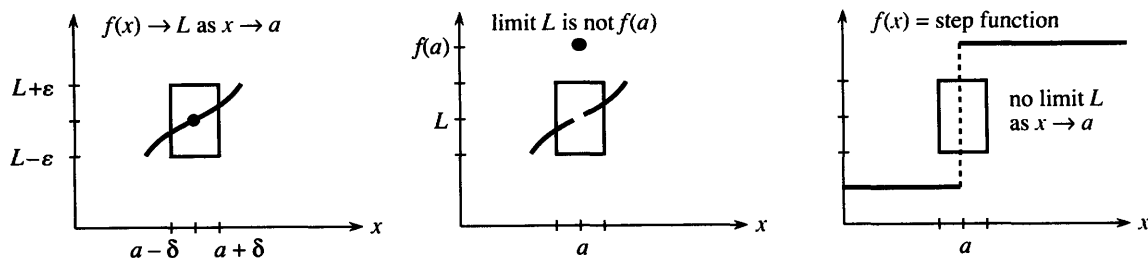


Fig. 2.19 S chooses height 2ε , then P chooses width 2δ . Graph must go out the sides.

When $f(x)$ has a jump, the box can't hold it. A step function has no limit as x approaches the jump, because the graph goes through the top or bottom—no matter how thin the box.

Remark 2 The second figure has $f(x) \rightarrow L$, because in taking limits *we ignore the final point* $x = a$. The value $f(a)$ can be anything, with no effect on L . The first figure has more: $f(a)$ equals L . Then a special name applies— f is *continuous*. The left figure shows a continuous function, the other figures do not.

We soon come back to continuous functions.

Remark 3 In the example with $f = 5x$ and $\delta = \varepsilon/5$, the number 5 was the *slope*. That choice barely kept the graph in the box—it goes out the corners. A little narrower, say $\delta = \varepsilon/10$, and the graph goes safely out the sides. *A reasonable choice is to divide ε by $2|f'(a)|$.* (We double the slope for safety.) I want to say why this δ works—even if the ε - δ test is seldom used in practice.

The ratio of $f(x) - L$ to $x - a$ is distance up over distance across. This is $\Delta f / \Delta x$, close to the slope $f'(a)$. When the distance across is δ , the distance up or down is near $\delta|f'(a)|$. That equals $\varepsilon/2$ for our “reasonable choice” of δ —so we are safely below ε . This choice solves most exercises. But Example 7 shows that a limit might exist even when the slope is infinite.

EXAMPLE 7 $\lim_{x \rightarrow 1^+} \sqrt{x-1} = 0$ (*a one-sided limit*).

Notice the plus sign in the symbol $x \rightarrow 1^+$. The number x approaches $a = 1$ *only from above*. An ordinary limit $x \rightarrow 1$ requires us to accept x on both sides of 1 (the exact value $x = 1$ is not considered). Since negative numbers are not allowed by the square root, we have a *one-sided limit*. It is $L = 0$.

Suppose ε is $1/10$. Then the response could be $\delta = 1/100$. A number below $1/100$ has a square root below $1/10$. In this case the box must be made extremely narrow, δ much smaller than ε , because the square root starts with infinite slope.

Those examples show the point of the ε - δ definition. (Given ε , look for δ . This came from Cauchy in France, not Socrates in Greece.) We also see its bad feature: The test is not convenient. Mathematicians do not go around proposing ε 's and replying with δ 's. We may live a strange life, but not that strange.

It is easier to establish once and for all that $5x$ approaches its obvious limit $5a$. The same is true for other familiar functions: $x^n \rightarrow a^n$ and $\sin x \rightarrow \sin a$ and $(1-x)^{-1} \rightarrow (1-a)^{-1}$ —except at $a = 1$. *The correct limit L comes by substituting $x = a$ into the function.* This is exactly the property of a “*continuous function*.” Before the section on continuous functions, we prove the Squeeze Theorem using ε and δ .

2H Squeeze Theorem Suppose $f(x) \leq g(x) \leq h(x)$ for x near a . If $f(x) \rightarrow L$ and $h(x) \rightarrow L$ as $x \rightarrow a$, then the limit of $g(x)$ is also L .

PROOF $g(x)$ is squeezed between $f(x)$ and $h(x)$. After subtracting L , $g(x) - L$ is between $f(x) - L$ and $h(x) - L$. Therefore

$$|g(x) - L| < \varepsilon \quad \text{if} \quad |f(x) - L| < \varepsilon \quad \text{and} \quad |h(x) - L| < \varepsilon.$$

For any ε , the last two inequalities hold in some region $0 < |x - a| < \delta$. So the first one also holds. This proves that $g(x) \rightarrow L$. Values at $x = a$ are not involved—until we get to continuous functions.

2.6 EXERCISES

Read-through questions

The limit of $a_n = (\sin n)/n$ is a. The limit of $a_n = n^4/2^n$ is b. The limit of $a_n = (-1)^n$ is c. The meaning of $a_n \rightarrow 0$ is: Only d of the numbers $|a_n|$ can be e. The meaning of $a_n \rightarrow L$ is: For every f there is an g such that h if $n > \text{---}$ i. The sequence $1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, \dots$ is not j because eventually those sums go past k.

The limit of $f(x) = \sin x$ as $x \rightarrow a$ is l. The limit of $f(x) = x/|x|$ as $x \rightarrow -2$ is m, but the limit as $x \rightarrow 0$ does not n. This function only has o-sided limits. The meaning of $\lim_{x \rightarrow a} f(x) = L$ is: For every ε there is a δ such that $|f(x) - L| < \varepsilon$ whenever p.

Two rules for limits, when $a_n \rightarrow L$ and $b_n \rightarrow M$, are $a_n + b_n \rightarrow \text{---}$ q and $a_n b_n \rightarrow \text{---}$ r. The corresponding rules for functions, when $f(x) \rightarrow L$ and $g(x) \rightarrow M$ as $x \rightarrow a$, are s and t. In all limits, $|a_n - L|$ or $|f(x) - L|$ must eventually go below and u any positive v.

$A \Rightarrow B$ means that A is a w condition for B . Then B is true x A is true. $A \Leftrightarrow B$ means that A is a y condition for B . Then B is true z A is true.

1 What is a_4 and what is the limit L ? After which N is $|a_n - L| < \frac{1}{10}$? (Calculator allowed)

- (a) $-1, +\frac{1}{2}, -\frac{1}{3}, \dots$ (b) $\frac{1}{2}, \frac{1}{2} + \frac{1}{4}, \frac{1}{2} + \frac{1}{4} + \frac{1}{6}, \dots$
 (c) $\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \dots$ $a_n = n/2^n$ (d) $1.1, 1.11, 1.111, \dots$
 (e) $a_n = \sqrt[n]{n}$ (f) $a_n = \sqrt{n^2 + n} - n$
 (g) $1 + 1, (1 + \frac{1}{2})^2, (1 + \frac{1}{3})^3, \dots$

2 Show by example that these statements are false:

- (a) If $a_n \rightarrow L$ and $b_n \rightarrow L$ then $a_n/b_n \rightarrow 1$
 (b) $a_n \rightarrow L$ if and only if $a_n^2 \rightarrow L^2$
 (c) If $a_n < 0$ and $a_n \rightarrow L$ then $L < 0$
 (d) If infinitely many a_n 's are inside every strip around zero then $a_n \rightarrow 0$.

3 Which of these statements are equivalent to $B \Rightarrow A$?

- (a) If A is true so is B
 (b) A is true if and only if B is true
 (c) B is a sufficient condition for A
 (d) A is a necessary condition for B .

4 Decide whether $A \Rightarrow B$ or $B \Rightarrow A$ or neither or both:

- (a) $A = [a_n \rightarrow 1]$ $B = [-a_n \rightarrow -1]$
 (b) $A = [a_n \rightarrow 0]$ $B = [a_n - a_{n-1} \rightarrow 0]$
 (c) $A = [a_n \leq n]$ $B = [a_n = n]$
 (d) $A = [a_n \rightarrow 0]$ $B = [\sin a_n \rightarrow 0]$
 (e) $A = [a_n \rightarrow 0]$ $B = [1/a_n \text{ fails to converge}]$
 (f) $A = [a_n < n]$ $B = [a_n/n \text{ converges}]$

*5 If the sequence $a_1, a_2, a_3, \dots$ approaches zero, prove that we can put those numbers in any order and the new sequence still approaches zero.

*6 Suppose $f(x) \rightarrow L$ and $g(x) \rightarrow M$ as $x \rightarrow a$. Prove from the definitions that $f(x) + g(x) \rightarrow L + M$ as $x \rightarrow a$.

Find the limits 7–24 if they exist. An ε - δ test is not required.

- 7 $\lim_{t \rightarrow 2} \frac{t+3}{t^2-2}$ 8 $\lim_{t \rightarrow 2} \frac{t^2+3}{t-2}$
 9 $\lim_{x \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ (careful) 10 $\lim_{h \rightarrow 0} \frac{f(1+h)-f(1)}{h}$
 11 $\lim_{h \rightarrow 0} \frac{\sin^2 h \cos^2 h}{h^2}$ 12 $\lim_{x \rightarrow 0} \frac{2x \tan x}{\sin x}$
 13 $\lim_{x \rightarrow 0^+} \frac{|x|}{x}$ (one-sided) 14 $\lim_{x \rightarrow 0^-} \frac{|x|}{x}$ (one-sided)
 15 $\lim_{x \rightarrow 1} \frac{\sin x}{x}$ 16 $\lim_{c \rightarrow a} \frac{f(c)-f(a)}{c-a}$
 17 $\lim_{x \rightarrow 5} \frac{x^2+25}{x-5}$ 18 $\lim_{x \rightarrow 5} \frac{x^2-25}{x-5}$
 19 $\lim_{x \rightarrow 0} \frac{\sqrt{1+x}-1}{x}$ (test $x = .01$) 20 $\lim_{x \rightarrow 2} \frac{\sqrt{4-x}}{\sqrt{6+x}}$
 21 $\lim_{x \rightarrow a} [f(x)-f(a)](?)$ 22 $\lim_{x \rightarrow \pi/2} (\sec x - \tan x)$
 23 $\lim_{x \rightarrow 0} \frac{\sin x}{\sin x/2}$ 24 $\lim_{x \rightarrow 1} \frac{\sin(x-1)}{x^2-1}$

25 Choose δ so that $|f(x)| < \frac{1}{100}$ if $0 < x < \delta$.

$$f(x) = 10x \quad f(x) = \sqrt{x} \quad f(x) = \sin 2x \quad f(x) = x \sin x$$

26 Which does the definition of a limit require?

- (1) $|f(x) - L| < \varepsilon \Rightarrow 0 < |x - a| < \delta$
 (2) $|f(x) - L| < \varepsilon \Leftrightarrow 0 < |x - a| < \delta$
 (3) $|f(x) - L| < \varepsilon \Leftrightarrow 0 < |x - a| < \delta$

27 The definition of " $f(x) \rightarrow L$ as $x \rightarrow \infty$ " is this: For any ε there is an X such that $< \varepsilon$ if $x > X$. Give an example in which $f(x) \rightarrow 4$ as $x \rightarrow \infty$.

28 Give a correct definition of " $f(x) \rightarrow 0$ as $x \rightarrow -\infty$."

29 The limit of $f(x) = (\sin x)/x$ as $x \rightarrow \infty$ is . For $\varepsilon = .01$ find a point X beyond which $|f(x)| < \varepsilon$.

30 The limit of $f(x) = 2x/(1+x)$ as $x \rightarrow \infty$ is $L = 2$. For $\varepsilon = .01$ find a point X beyond which $|f(x) - 2| < \varepsilon$.

31 The limit of $f(x) = \sin x$ as $x \rightarrow \infty$ does not exist. Explain why not.

32 (Calculator) Estimate the limit of $\left(1 + \frac{1}{x}\right)^x$ as $x \rightarrow \infty$.

33 For the polynomial $f(x) = 2x - 5x^2 + 7x^3$ find

(a) $\lim_{x \rightarrow 1} f(x)$ (b) $\lim_{x \rightarrow \infty} f(x)$

(c) $\lim_{x \rightarrow \infty} \frac{f(x)}{x^3}$ (d) $\lim_{x \rightarrow -\infty} \frac{f(x)}{x^3}$

34 For $f(x) = 6x^3 + 1000x$ find

(a) $\lim_{x \rightarrow \infty} \frac{f(x)}{x}$ (b) $\lim_{x \rightarrow \infty} \frac{f(x)}{x^2}$

(c) $\lim_{x \rightarrow \infty} \frac{f(x)}{x^4}$ (d) $\lim_{x \rightarrow \infty} \frac{f(x)}{x^3 + 1}$

Important rule As $x \rightarrow \infty$ the ratio of polynomials $f(x)/g(x)$ has the same limit as the ratio of their *leading terms*. $f(x) = x^3 - x + 2$ has leading term x^3 and $g(x) = 5x^6 + x + 1$ has leading term $5x^6$. Therefore $f(x)/g(x)$ behaves like $x^3/5x^6 \rightarrow 0$, $g(x)/f(x)$ behaves like $5x^6/x^3 \rightarrow \infty$, $(f(x))^2/g(x)$ behaves like $x^6/5x^6 \rightarrow 1/5$.

35 Find the limit as $x \rightarrow \infty$ if it exists:

$$\frac{3x^2 + 2x + 1}{3 + 2x + x^2} \quad \frac{x^4}{x^3 + x^2} \quad \frac{x^2 + 1000}{x^3 - 1000} \quad x \sin \frac{1}{x}.$$

36 If a particular δ achieves $|f(x) - L| < \varepsilon$, why is it OK to choose a smaller δ ?

37 The sum of $1 + r + r^2 + \cdots + r^{n-1}$ is $a_n = (1 - r^n)/(1 - r)$. What is the limit of a_n as $n \rightarrow \infty$? For which r does the limit exist?

38 If $a_n \rightarrow L$ prove that there is a number N with this property: If $n > N$ and $m > N$ then $|a_n - a_m| < 2\varepsilon$. This is Cauchy's test for convergence.

39 No matter what decimals come later, $a_1 = .4$, $a_2 = .49$, $a_3 = .493$, ... approaches a limit L . How do we know (when we can't know L)? *Cauchy's test* is passed: the a 's get closer to each other.

(a) From a_4 onwards we have $|a_n - a_m| < \underline{\hspace{2cm}}$.

(b) After which a_N is $|a_m - a_n| < 10^{-7}$?

40 Choose decimals in Problem 39 so the limit is $L = .494$. Choose decimals so that your professor can't find L .

41 If every decimal in $.abcde\cdots$ is picked at random from 0, 1, ..., 9, what is the "average" limit L ?

42 If every decimal is 0 or 1 (at random), what is the average limit L ?

43 Suppose $a_n = \frac{1}{2}a_{n-1} + 4$ and start from $a_1 = 10$. Find a_2 and a_3 and a connection between $a_n - 8$ and $a_{n-1} - 8$. Deduce that $a_n \rightarrow 8$.

44 "For every δ there is an ε such that $|f(x)| < \varepsilon$ if $|x| < \delta$." That test is twisted around. Find ε when $f(x) = \cos x$, which does *not* converge to zero.

45 Prove the Squeeze Theorem for sequences, using ε : If $a_n \rightarrow L$ and $c_n \rightarrow L$ and $a_n \leq b_n \leq c_n$ for $n > N$, then $b_n \rightarrow L$.

46 Explain in 110 words the difference between "we will get there if you hurry" and "we will get there only if you hurry" and "we will get there if and only if you hurry."

2.7 Continuous Functions

This will be a brief section. It was originally included with limits, but the combination was too long. We are still concerned with the limit of $f(x)$ as $x \rightarrow a$, but a new number is involved. That number is $f(a)$, the *value of f at $x = a$* . For a "limit," x approached a but never reached it—so $f(a)$ was ignored. For a "continuous function," this final number $f(a)$ must be right.

May I summarize the usual (good) situation as x approaches a ?

1. The number $f(a)$ exists (f is defined at a)
2. The limit of $f(x)$ exists (it was called L)
3. The limit L equals $f(a)$ ($f(a)$ is the right value)

In such a case, $f(x)$ is *continuous* at $x = a$. These requirements are often written in a single line: $f(x) \rightarrow f(a)$ as $x \rightarrow a$. By way of contrast, start with four functions that are *not* continuous at $x = 0$.

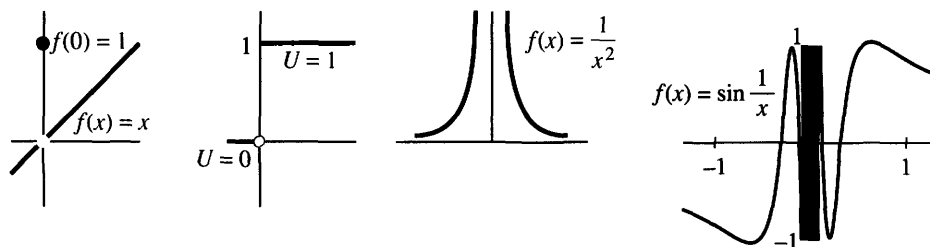


Fig. 2.20 Four types of discontinuity (others are possible) at $x = 0$.

In Figure 2.20, the first function would be continuous if it had $f(0) = 0$. But it has $f(0) = 1$. After changing $f(0)$ to the right value, the problem is gone. The discontinuity is *removable*. Examples 2, 3, 4 are more important and more serious. There is no “correct” value for $f(0)$:

2. $f(x) = \text{step function}$ (jump from 0 to 1 at $x = 0$)
3. $f(x) = 1/x^2$ (infinite limit as $x \rightarrow 0$)
4. $f(x) = \sin(1/x)$ (infinite oscillation as $x \rightarrow 0$).

The graphs show how the limit fails to exist. The step function has a **jump discontinuity**. It has **one-sided limits**, from the left and right. It does not have an ordinary (two-sided) limit. The limit from the left ($x \rightarrow 0^-$) is 0. The limit from the right ($x \rightarrow 0^+$) is 1. Another step function is $x/|x|$, which jumps from -1 to 1 .

In the graph of $1/x^2$, the only reasonable limit is $L = +\infty$. I cannot go on record as saying that this limit exists. Officially, it doesn’t—but we often write it anyway: $1/x^2 \rightarrow \infty$ as $x \rightarrow 0$. This means that $1/x^2$ goes (and stays) above every L as $x \rightarrow 0$.

In the same unofficial way we write one-sided limits for $f(x) = 1/x$:

$$\text{From the left, } \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty. \quad \text{From the right, } \lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty. \quad (1)$$

Remark $1/x$ has a “pole” at $x = 0$. So has $1/x^2$ (a double pole). The function $1/(x^2 - x)$ has poles at $x = 0$ and $x = 1$. In each case the denominator goes to zero and the function goes to $+\infty$ or $-\infty$. Similarly $1/\sin x$ has a pole at every multiple of π (where $\sin x$ is zero). Except for $1/x^2$ these poles are “simple”—the functions are completely smooth at $x = 0$ when we multiply them by x :

$$(x)\left(\frac{1}{x}\right) = 1 \quad \text{and} \quad (x)\left(\frac{1}{x^2 - x}\right) = \frac{1}{x - 1} \quad \text{and} \quad (x)\left(\frac{1}{\sin x}\right) \quad \text{are continuous at } x = 0.$$

$1/x^2$ has a double pole, since it needs multiplication by x^2 (not just x). A ratio of polynomials $P(x)/Q(x)$ has poles where $Q = 0$, provided any common factors like $(x + 1)/(x + 1)$ are removed first.

Jumps and poles are the most basic discontinuities, but others can occur. The fourth graph shows that $\sin(1/x)$ has no limit as $x \rightarrow 0$. This function does not blow up; the sine never exceeds 1. At $x = \frac{1}{3}$ and $\frac{1}{4}$ and $\frac{1}{1000}$ it equals $\sin 3$ and $\sin 4$ and $\sin 1000$. Those numbers are positive and negative and (?). As x gets small and $1/x$ gets large, the sine oscillates faster and faster. Its graph won’t stay in a small box of height ε , no matter how narrow the box.

CONTINUOUS FUNCTIONS

DEFINITION f is “**continuous at** $x = a$ ” if $f(a)$ is defined and $f(x) \rightarrow f(a)$ as $x \rightarrow a$. If f is continuous at every point where it is defined, it is a **continuous function**.

Objection The definition makes $f(x) = 1/x$ a continuous function! It is not defined at $x = 0$, so its continuity can't fail. The logic requires us to accept this, but we don't have to like it. Certainly there is no $f(0)$ that would make $1/x$ continuous at $x = 0$.

It is amazing but true that the definition of "continuous function" is still debated (*Mathematics Teacher*, May 1989). You see the reason—we speak about a discontinuity of $1/x$, and at the same time call it a continuous function. The definition misses the difference between $1/x$ and $(\sin x)/x$. The function $f(x) = (\sin x)/x$ can be made continuous at all x . Just set $f(0) = 1$.

We call a function "**continuable**" if its definition can be extended to all x in a way that makes it continuous. Thus $(\sin x)/x$ and $\sqrt{x}$ are continuable. The functions $1/x$ and $\tan x$ are not continuable. This suggestion may not end the debate, but I hope it is helpful.

EXAMPLE 1 $\sin x$ and $\cos x$ and all polynomials $P(x)$ are continuous functions.

EXAMPLE 2 The absolute value $|x|$ is continuous. Its slope jumps (not continuable).

EXAMPLE 3 Any rational function $P(x)/Q(x)$ is continuous except where $Q = 0$.

EXAMPLE 4 The function that jumps between 1 at fractions and 0 at non-fractions is **discontinuous everywhere**. There is a fraction between every pair of non-fractions and vice versa. (Somehow there are many more non-fractions.)

EXAMPLE 5 The function 0^{x^2} is zero for every x , except that 0^0 is not defined. So define it as zero and this function is continuous. But see the next paragraph where 0^0 has to be 1.

We could fill the book with proofs of continuity, but usually the situation is clear. "A function is continuous if you can draw its graph without lifting up your pen." At a jump, or an infinite limit, or an infinite oscillation, there is no way across the discontinuity except to start again on the other side. The function x^n is continuous for $n > 0$. It is not continuable for $n < 0$. The function x^0 equals 1 for every x , except that 0^0 is not defined. This time continuity requires $0^0 = 1$.

The interesting examples are the close ones—we have seen two of them:

EXAMPLE 6 $\frac{\sin x}{x}$ and $\frac{1 - \cos x}{x}$ are both continuable at $x = 0$.

Those were crucial for the slope of $\sin x$. The first approaches 1 and the second approaches 0. Strictly speaking we must give these functions the correct values (1 and 0) at the limiting point $x = 0$ —which of course we do.

It is important to know what happens when the denominators change to x^2 .

EXAMPLE 7 $\frac{\sin x}{x^2}$ blows up but $\frac{1 - \cos x}{x^2}$ has the limit $\frac{1}{2}$ at $x = 0$.

Since $(\sin x)/x$ approaches 1, dividing by another x gives a function like $1/x$. There is a simple pole. It is an example of $0/0$, in which the zero from x^2 is reached more quickly than the zero from $\sin x$. The "**race to zero**" produces almost all interesting problems about limits.

For $1 - \cos x$ and x^2 the race is almost even. Their ratio is 1 to 2:

$$\frac{1 - \cos x}{x^2} = \frac{1 - \cos^2 x}{x^2(1 + \cos x)} = \frac{\sin^2 x}{x^2} \cdot \frac{1}{1 + \cos x} \rightarrow \frac{1}{1 + 1} \text{ as } x \rightarrow 0.$$

This answer $\frac{1}{2}$ will be found again (more easily) by “l’Hôpital’s rule.” Here I emphasize not the answer but the problem. A central question of differential calculus is *to know how fast the limit is approached. The speed of approach is exactly the information in the derivative.*

These three examples are all continuous at $x = 0$. The race is controlled by the slope—because $f(x) - f(0)$ is nearly $f'(0)$ times x :

derivative of $\sin x$ is 1 $\leftrightarrow$ $\sin x$ decreases like x

derivative of $\sin^2 x$ is 0 $\leftrightarrow$ $\sin^2 x$ decreases faster than x

derivative of $x^{1/3}$ is ∞ $\leftrightarrow$ $x^{1/3}$ decreases more slowly than x .

DIFFERENTIABLE FUNCTIONS

The absolute value $|x|$ is continuous at $x = 0$ but has no derivative. The same is true for $x^{1/3}$. *Asking for a derivative is more than asking for continuity.* The reason is fundamental, and carries us back to the key definitions:

Continuous at x : $f(x + \Delta x) - f(x) \rightarrow 0$ as $\Delta x \rightarrow 0$

Derivative at x : $\frac{f(x + \Delta x) - f(x)}{\Delta x} \rightarrow f'(x)$ as $\Delta x \rightarrow 0$.

In the first case, Δf goes to zero (maybe slowly). In the second case, Δf goes to zero *as fast as* Δx (because $\Delta f/\Delta x$ has a limit). That requirement is stronger:

21 At a point where $f(x)$ has a derivative, the function must be continuous. But $f(x)$ can be continuous with no derivative.

Proof The limit of $\Delta f = (\Delta x)(\Delta f/\Delta x)$ is $(0)(df/dx) = 0$. So $f(x + \Delta x) - f(x) \rightarrow 0$.

The continuous function $x^{1/3}$ has no derivative at $x = 0$, because $\frac{1}{3}x^{-2/3}$ blows up. The absolute value $|x|$ has no derivative because its slope jumps. The remarkable function $\frac{1}{2} \cos 3x + \frac{1}{4} \cos 9x + \dots$ is continuous at *all points* and has a derivative at *no points*. You can draw its graph without lifting your pen (but not easily—it turns at every point). To most people, it belongs with space-filling curves and unmeasurable areas—in a box of curiosities. Fractals used to go into the same box! They are beautiful shapes, with boundaries that have no tangents. The theory of fractals is very alive, for good mathematical reasons, and we touch on it in Section 3.7.

I hope you have a clear idea of these basic definitions of calculus:

1 Limit ($n \rightarrow \infty$ or $x \rightarrow a$) **2 Continuity** (at $x = a$) **3 Derivative** (at $x = a$).

Those go back to ε and δ , but it is seldom necessary to follow them so far. In the same way that economics describes many transactions, or history describes many events, a function comes from many values $f(x)$. A few points may be special, like market crashes or wars or discontinuities. At other points df/dx is the best guide to the function.

2.7 Continuous Functions

This chapter ends with two essential facts about **a continuous function on a closed interval**. The interval is $a \leq x \leq b$, written simply as $[a, b]$.† At the endpoints a and b we require $f(x)$ to approach $f(a)$ and $f(b)$.

Extreme Value Property A continuous function on the finite interval $[a, b]$ has a maximum value M and a minimum value m . There are points $x_{\max}$ and $x_{\min}$ in $[a, b]$ where it reaches those values:

$$f(x_{\max}) = M \geq f(x) \geq f(x_{\min}) = m \text{ for all } x \text{ in } [a, b].$$

Intermediate Value Property If the number F is between $f(a)$ and $f(b)$, there is a point c between a and b where $f(c) = F$. Thus if F is between the minimum m and the maximum M , there is a point c between $x_{\min}$ and $x_{\max}$ where $f(c) = F$.

Examples show why we require closed intervals and continuous functions. For $0 < x \leq 1$ the function $f(x) = x$ never reaches its minimum (zero). If we close the interval by defining $f(0) = 3$ (discontinuous) the minimum is still not reached. Because of the jump, the intermediate value $F = 2$ is also not reached. The idea of continuity was inescapable, after Cauchy defined the idea of a limit.

2.7 EXERCISES

Read-through questions

Continuity requires the a of $f(x)$ to exist as $x \rightarrow a$ and to agree with b. The reason that $x/|x|$ is not continuous at $x = 0$ is c. This function does have d limits. The reason that $1/\cos x$ is discontinuous at e is f. The reason that $\cos(1/x)$ is discontinuous at $x = 0$ is g. The function $f(x) = \frac{1}{x-3}$ has a simple pole at $x = 3$, where f^2 has a i pole.

The power x^n is continuous at all x provided n is j. It has no derivative at $x = 0$ when n is k. $f(x) = \sin(-x)/x$ approaches l as $x \rightarrow 0$, so this is a m function provided we define $f(0) = \frac{0}{0}$. A "continuous function" must be continuous at all o. A "continuable function" can be extended to every point x so that p.

If f has a derivative at $x = a$ then f is necessarily q at $x = a$. The derivative controls the speed at which $f(x)$ approaches r. On a closed interval $[a, b]$, a continuous f has the s value property and the t value property. It reaches its u M and its v m , and it takes on every value w.

In Problems 1–20, find the numbers c that make $f(x)$ into (A) a continuous function and (B) a differentiable function. In one case $f(x) \rightarrow f(a)$ at every point, in the other case $\Delta f/\Delta x$ has a limit at every point.

$$1 \quad f(x) = \begin{cases} \sin x & x < 1 \\ c & x \geq 1 \end{cases}$$

$$2 \quad f(x) = \begin{cases} \cos^3 x & x \neq \pi \\ c & x = \pi \end{cases}$$

$$3 \quad f(x) = \begin{cases} cx & x < 0 \\ 2cx & x \geq 0 \end{cases}$$

$$4 \quad f(x) = \begin{cases} cx & x < 1 \\ 2cx & x \geq 1 \end{cases}$$

$$5 \quad f(x) = \begin{cases} c + x & x < 0 \\ c^2 + x^2 & x \geq 0 \end{cases}$$

$$6 \quad f(x) = \begin{cases} x^3 & x \neq c \\ -8 & x = c \end{cases}$$

$$7 \quad f(x) = \begin{cases} 2x & x < c \\ x + 1 & x \geq c \end{cases}$$

$$8 \quad f(x) = \begin{cases} x^c & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$9 \quad f(x) = \begin{cases} (\sin x)/x^2 & x \neq 0 \\ c & x = 0 \end{cases}$$

$$10 \quad f(x) = \begin{cases} x + c & x \leq c \\ 1 & x > c \end{cases}$$

$$11 \quad f(x) = \begin{cases} c & x \neq 4 \\ 1/x^3 & x = 4 \end{cases}$$

$$12 \quad f(x) = \begin{cases} c & x \leq 0 \\ \sec x & x \geq 0 \end{cases}$$

$$13 \quad f(x) = \begin{cases} \frac{x^2 + c}{x - 1} & x \neq 1 \\ 2 & x = 1 \end{cases}$$

$$14 \quad f(x) = \begin{cases} \frac{x^2 - 1}{x - c} & x \neq c \\ 2c & x = c \end{cases}$$

$$15 \quad f(x) = \begin{cases} (\tan x)/x & x \neq 0 \\ c & x = 0 \end{cases}$$

$$16 \quad f(x) = \begin{cases} x^2 & x \leq c \\ 2x & x > c \end{cases}$$

$$17 \quad f(x) = \begin{cases} (c + \cos x)/x & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$18 \quad f(x) = |x + c|$$

†The interval $[a, b]$ is **closed** (endpoints included). The interval (a, b) is **open** (a and b left out). The infinite interval $[0, \infty)$ contains all $x \geq 0$.

$$19 \quad f(x) = \begin{cases} (\sin x - x)/x^c & x \neq 0 \\ 0 & x = 0 \end{cases} \quad 20 \quad f(x) = |x^2 + c^2|$$

Construct your own $f(x)$ with these discontinuities at $x = 1$.

21 Removable discontinuity

22 Infinite oscillation

23 Limit for $x \rightarrow 1^+$, no limit for $x \rightarrow 1^-$

24 A double pole

$$25 \quad \lim_{x \rightarrow 1^-} f(x) = 4 + \lim_{x \rightarrow 1^+} f(x)$$

$$26 \quad \lim_{x \rightarrow 1} f(x) = \infty \text{ but } \lim_{x \rightarrow 1} (x-1)f(x) = 0$$

$$27 \quad \lim_{x \rightarrow 1} (x-1)f(x) = 5$$

28 The statement “ $3x \rightarrow 7$ as $x \rightarrow 1$ ” is false. Choose an ε for which no δ can be found. The statement “ $3x \rightarrow 3$ as $x \rightarrow 1$ ” is true. For $\varepsilon = \frac{1}{2}$ choose a suitable δ .

29 How many derivatives f' , f'' , ... are continuable functions?

$$(a) f = x^{3/2} \quad (b) f = x^{3/2} \sin x \quad (c) f = (\sin x)^{5/2}$$

30 Find one-sided limits at points where there is no two-sided limit. Give a 3-part formula for function (c).

$$(a) \frac{|x|}{7x} \quad (b) \sin |x| \quad (c) \frac{d}{dx} |x^2 - 1|$$

31 Let $f(1) = 1$ and $f(-1) = 1$ and $f(x) = (x^2 - x)/(x^2 - 1)$ otherwise. Decide whether f is continuous at

$$(a) x = 1 \quad (b) x = 0 \quad (c) x = -1.$$

*32 Let $f(x) = x^2 \sin 1/x$ for $x \neq 0$ and $f(0) = 0$. If the limits exist, find

$$(a) \lim_{x \rightarrow 0} f(x) \quad (b) df/dx \text{ at } x = 0 \quad (c) \lim_{x \rightarrow 0} f'(x).$$

33 If $f(0) = 0$ and $f'(0) = 3$, rank these functions from smallest to largest as x decreases to zero:

$$f(x), \quad x, \quad xf(x), \quad f(x) + 2x, \quad 2(f(x) - x), \quad (f(x))^2.$$

34 Create a discontinuous function $f(x)$ for which $f^2(x)$ is continuous.

35 True or false, with an example to illustrate:

(a) If $f(x)$ is continuous at all x , it has a maximum value M .

(b) If $f(x) \leq 7$ for all x , then f reaches its maximum.

(c) If $f(1) = 1$ and $f(2) = -2$, then somewhere $f(x) = 0$.

(d) If $f(1) = 1$ and $f(2) = -2$ and f is continuous on $[1, 2]$, then somewhere on that interval $f(x) = 0$.

36 The functions $\cos x$ and $2x$ are continuous. Show from the _____ property that $\cos x = 2x$ at some point between 0 and 1.

37 Show by example that these statements are false:

(a) If a function reaches its maximum and minimum then the function is continuous.

(b) If $f(x)$ reaches its maximum and minimum and all values between $f(0)$ and $f(1)$, it is continuous at $x = 0$.

(c) (mostly for instructors) If $f(x)$ has the intermediate value property between all points a and b , it must be continuous.

38 Explain with words and a graph why $f(x) = x \sin(1/x)$ is continuous but has no derivative at $x = 0$. Set $f(0) = 0$.

39 Which of these functions are *continuable*, and why?

$$f_1(x) = \begin{cases} \sin x & x < 0 \\ \cos x & x > 1 \end{cases} \quad f_2(x) = \begin{cases} \sin 1/x & x < 0 \\ \cos 1/x & x > 1 \end{cases}$$

$$f_3(x) = \frac{x}{\sin x} \text{ when } \sin x \neq 0 \quad f_4(x) = x^0 + 0^{x^2}$$

40 Explain the difference between a continuous function and a continuable function. Are continuous functions always continuable?

*41 $f(x)$ is any continuous function with $f(0) = f(1)$.

(a) Draw a typical $f(x)$. Mark where $f(x) = f(x + \frac{1}{2})$.

(b) Explain why $g(x) = f(x + \frac{1}{2}) - f(x)$ has $g(\frac{1}{2}) = -g(0)$.

(c) Deduce from (b) that (a) is always possible: There *must* be a point where $g(x) = 0$ and $f(x) = f(x + \frac{1}{2})$.

42 Create an $f(x)$ that is continuous only at $x = 0$.

43 If $f(x)$ is continuous and $0 \leq f(x) \leq 1$ for all x , then there is a point where $f(x^*) = x^*$. Explain with a graph and prove with the intermediate value theorem.

44 In the ε - δ definition of a limit, change $0 < |x - a| < \delta$ to $|x - a| < \delta$. Why is $f(x)$ now *continuous* at $x = a$?

45 A function has a _____ at $x = 0$ if and only if $(f(x) - f(0))/x$ is _____ at $x = 0$.