CHAPTER 3

Applications of the Derivative

Chapter 2 concentrated on computing derivatives. This chapter concentrates on *using* them. Our computations produced dy/dx for functions built from x^n and $\sin x$ and $\cos x$. Knowing the slope, and if necessary also the second derivative, we can answer the questions about $y = f(x)$ that this subject was created for:

1. How does y change when x changes?
2. What is the maximum value of y? Or the minimum?
3. How can you tell a maximum from a minimum, using derivatives?

The information in dy/dx is entirely *local*. It tells what is happening close to the point and nowhere else. In Chapter 2, Δx and Δy went to zero. Now we want to get them back. The local information explains the larger picture, ***because*** Δy ***is approximately*** dy/dx ***times*** Δx.

The problem is to connect the finite to the infinitesimal—the average slope to the instantaneous slope. Those slopes are close, and occasionally they are equal. Points of equality are assured by the Mean Value Theorem—which is the local-global connection at the center of differential calculus. But we cannot predict *where* dy/dx equals $\Delta y/\Delta x$. Therefore we now find other ways to recover a function from its derivatives—or to estimate distance from velocity and acceleration.

It may seem surprising that we learn about y from dy/dx. All our work has been going the other way! We struggled with y to squeeze out dy/dx. Now we use dy/dx to study y. That's life. Perhaps it really is life, to understand one generation from later generations.

3.1 Linear Approximation

The book started with a straight line $f = vt$. The distance is linear when the velocity is constant. As soon as v begins to change, $f = vt$ falls apart. Which velocity do we choose, when $v(t)$ is not constant? The solution is to take very short time intervals, in which v is nearly constant:

$$f = vt \quad \textit{is completely false}$$

$$\Delta f = v\Delta t \quad \textit{is nearly true}$$

$$df = vdt \quad \textit{is exactly true.}$$

For a brief moment the function $f(t)$ is linear—and stays near its tangent line.

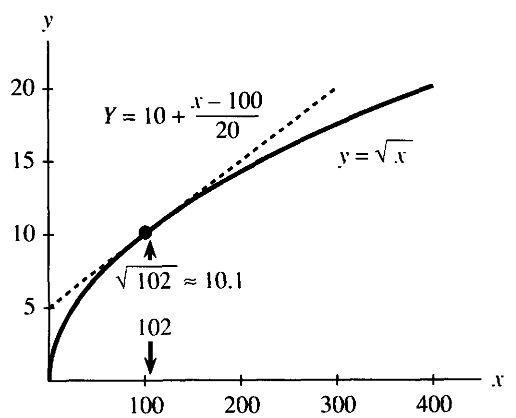
In Section 2.3 we found the tangent line to $y = f(x)$. At $x = a$, the slope of the curve and the slope of the line are $f'(a)$. For points on the line, start at $y = f(a)$. Add the slope times the "increment" $x - a$:

$$Y = f(a) + f'(a)(x - a). \tag{1}$$

We write a capital Y for the line and a small y for the curve. The whole point of tangents is that they are close (*provided we don't move too far from* a):

$$y \approx Y \quad \text{or} \quad f(x) \approx f(a) + f'(a)(x - a). \tag{2}$$

That is the all-purpose ***linear approximation***. Figure 3.1 shows the square root function $y = \sqrt{x}$ and its tangent line at $x = a = 100$. At the point $y = \sqrt{100} = 10$, the slope is $1/2\sqrt{x} = 1/20$. The table beside the figure compares $y(x)$ with $Y(x)$.



x	Y	$y = \sqrt{x}$
100	10	10
→102	10.1	10.0995
110	10.5	10.49
200	15	14.1
400	25	20

Fig. 3.1 $Y(x)$ is the linear approximation to $\sqrt{x}$ near $x = a = 100$.

The accuracy gets worse as x departs from 100. The tangent line leaves the curve. The arrow points to a good approximation at 102, and at 101 it would be even better. In this example Y is larger than y—the straight line is above the curve. The slope of the line stays constant, and the slope of the curve is decreasing. Such a curve will soon be called "concave downward," and its tangent lines are above it.

Look again at $x = 102$, where the approximation is good. In Chapter 2, when we were approaching dy/dx, we started with $\Delta y/\Delta x$:

$$\text{slope} \approx \frac{\sqrt{102} - \sqrt{100}}{102 - 100}. \tag{3}$$

Now that is turned around! The slope is $1/20$. ***What we don't know is*** $\sqrt{102}$:

$$\sqrt{102} \approx \sqrt{100} + (\text{slope})(102 - 100). \tag{4}$$

You work with what you have. Earlier we didn't know dy/dx, so we used (3). Now we are experts at dy/dx, and we use (4). After computing $y' = 1/20$ once and for all, the tangent line stays near $\sqrt{x}$ for every number near 100. When that nearby number is $100 + \Delta x$, notice the error as the approximation is squared:

$$\left(\sqrt{100} + \frac{1}{20}\Delta x\right)^2 = 100 + \Delta x + \frac{1}{400}(\Delta x)^2.$$

The desired answer is $100 + \Delta x$, and we are off by the last term involving $(\Delta x)^2$. The whole point of linear approximation is to ignore every term after Δx.

There is nothing magic about $x = 100$, except that it has a nice square root. Other points and other functions allow $y \approx Y$. I would like to express this same idea in different symbols. *Instead of starting from a and going to x, we start from x and go a distance* Δx *to* $x + \Delta x$. The letters are different but the mathematics is identical.

3A At any point x, and for any smooth function $y = f(x)$,

$$\text{slope at } x \;\approx\; \frac{f(x+\Delta x) - f(x)}{\Delta x}. \tag{5}$$

For the approximation to $f(x + \Delta x)$, multiply both sides by Δx and add $f(x)$:

$$f(x + \Delta x) \;\approx\; f(x) + (\text{slope at } x)(\Delta x). \tag{6}$$

EXAMPLE 1 ***An important linear approximation***: $(1 + x)^n \approx 1 + nx$ for x near zero.

EXAMPLE 2 ***A second important approximation***: $1/(1 + x)^n \approx 1 - nx$ for x near zero.

Discussion Those are really the same. By changing n to $-n$ in Example 1, it becomes Example 2. These are linear approximations using the slopes n and $-n$ at $x = 0$:

$$(1 + x)^n \;\approx\; 1 + (\textit{slope at zero}) \text{ times } (x - 0) = 1 + nx.$$

Here is the same thing with $f(x) = x^n$. The basepoint in equation (6) is now 1 or x:

$$(1 + \Delta x)^n \;\approx\; 1 + n\Delta x \qquad (x + \Delta x)^n \;\approx\; x^n + nx^{n-1}\Delta x.$$

Better than that, here are numbers. For $n = 3$ and -1 and 100, take $\Delta x = .01$:

$$(1.01)^3 \approx 1.03 \qquad \frac{1}{1.01} \approx .99 \qquad \left(1 + \frac{1}{100}\right)^{100} \approx 2$$

Actually that last number is no good. The 100th power is too much. Linear approximation gives $1 + 100\Delta x = 2$, but a calculator gives $(1.01)^{100} = 2.7\ldots.$ This is close to e, the all-important number in Chapter 6. The binomial formula shows why the approximation failed:

$$(1 + \Delta x)^{100} = 1 + 100\,\Delta x + \frac{(100)(99)}{(2)(1)}(\Delta x)^2 + \cdots.$$

Linear approximation forgets the $(\Delta x)^2$ term. For $\Delta x = 1/100$ that error is nearly $\frac{1}{2}$. It is too big to overlook. The exact error is $\frac{1}{2}(\Delta x)^2 f''(c)$, where the Mean Value Theorem in Section 3.8 places c between x and $x + \Delta x$. You already see the point:

$y - Y$ ***is of order*** $(\Delta x)^2$. ***Linear approximation, quadratic error.***

DIFFERENTIALS

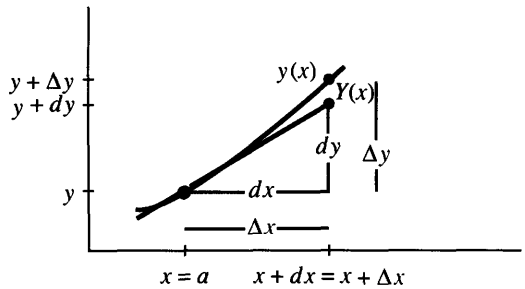
There is one more notation for this linear approximation. It has to be presented, because it is often used. The notation is suggestive and confusing at the same time—

it keeps the same symbols dx and dy that appear in the derivative. Earlier we took great pains to emphasize that dy/dx is not an ordinary fraction.† Until this paragraph, dx and dy have had no independent meaning. Now they become separate variables, like x and y but with their own names. These quantities dx and dy are called ***differentials***.

The symbols dx and dy measure changes *along the tangent line*. They do for the approximation $Y(x)$ exactly what Δx and Δy did for $y(x)$. Thus dx and Δx both measure distance across.

Figure 3.2 has $\Delta x = dx$. But the change in y does not equal the change in Y. One is Δy (exact for the function). The other is dy (exact for the tangent line). ***The differential*** dy ***is equal to*** ΔY, ***the change along the tangent line***. Where Δy is the true change, dy is its linear approximation $(dy/dx)dx$.

You often see dy written as $f'(x)dx$.



Δy = change in y (along curve)

dy = change in Y (along tangent)

Fig. 3.2 The linear approximation to Δy is $dy = f'(x)\,dx$.

EXAMPLE 3 $y = x^2$ has $dy/dx = 2x$ so $dy = 2x\,dx$. The table has basepoint $x = 2$. The prediction dy differs from the true Δy by exactly $(\Delta x)^2 = .01$ and $.04$ and $.09$.

	dx	dy	Δx	Δy	
$y = x^2$	.1	0.4	.1	0.41	$\Delta y = (2+\Delta x)^2 - 2^2$
$dy = 4dx$	.2	0.8	.2	0.84	$\Delta y = 4\Delta x + (\Delta x)^2$
	.3	1.2	.3	1.29	

The differential $dy = f'(x)dx$ is consistent with the derivative $dy/dx = f'(x)$. We finally have $dy = (dy/dx)dx$, but this is not as obvious as it seems! It looks like cancellation—it is really a definition. Entirely new symbols could be used, but dx and dy have two advantages: They suggest small steps and they satisfy $dy = f'(x)dx$. Here are three examples and three rules:

$$d(x^n) = nx^{n-1}dx \qquad d(f+g) = df + dg$$

$$d(\sin x) = \cos x\, dx \qquad d(cf) = c\,df$$

$$d(\tan x) = \sec^2 x\, dx \qquad d(fg) = f\,dg + g\,df$$

Science and engineering and virtually all applications of mathematics depend on linear approximation. The true function is "***linearized***," using its slope v:

Increasing the time by Δt increases the distance by $\approx v\Delta t$

Increasing the force by Δf increases the deflection by $\approx v\Delta f$

Increasing the production by Δp increases its value by $\approx v\Delta p$.

†Fraction or not, it is absolutely forbidden to cancel the d's.

The goal of dynamics or statics or economics is to predict this multiplier v—the derivative that equals the slope of the tangent line. The multiplier gives a *local prediction* of the change in the function. The exact law is nonlinear—but Ohm's law and Hooke's law and Newton's law are linear approximations.

ABSOLUTE CHANGE, RELATIVE CHANGE, PERCENTAGE CHANGE

The change Δy or Δf can be measured in three ways. So can Δx:

Absolute change	Δf	Δx
Relative change	$\dfrac{\Delta f}{f(x)}$	$\dfrac{\Delta x}{x}$
Percentage change	$\dfrac{\Delta f}{f(x)} \times 100$	$\dfrac{\Delta x}{x} \times 100$

Relative change is often more realistic than absolute change. If we know the distance to the moon within three miles, that is more impressive than knowing our own height within one inch. Absolutely, one inch is closer than three miles. Relatively, three miles is much closer:

$$\frac{3 \text{ miles}}{300{,}000 \text{ miles}} < \frac{1 \text{ inch}}{70 \text{ inches}} \quad \text{or} \quad .001\% < 1.4\%.$$

EXAMPLE 4 The radius of the Earth is within 80 miles of $r = 4000$ miles.

(a) Find the variation dV in the volume $V = \frac{4}{3}\pi r^3$, using linear approximation.

(b) Compute the relative variations dr/r and dV/V and $\Delta V/V$.

Solution The job of calculus is to produce the derivative. After $dV/dr = 4\pi r^2$, its work is done. The variation in volume is $dV = 4\pi(4000)^2(80)$ cubic miles. A 2% relative variation in r gives a 6% relative variation in V:

$$\frac{dr}{r} = \frac{80}{4000} = 2\% \qquad \frac{dV}{V} = \frac{4\pi(4000)^2(80)}{4\pi(4000)^3/3} = 6\%.$$

Without calculus we need the exact volume at $r = 4000 + 80$ (also at $r = 3920$):

$$\frac{\Delta V}{V} = \frac{4\pi(4080)^3/3 - 4\pi(4000)^3/3}{4\pi(4000)^3/3} \approx 6.1\%$$

One comment on $dV = 4\pi r^2 dr$. This is (area of sphere) times (change in radius). It is the volume of a thin shell around the sphere. The shell is added when the radius grows by dr. The exact $\Delta V/V$ is 3917312/640000%, but calculus just calls it 6%.

3.1 EXERCISES

Read-through questions

On the graph, a linear approximation is given by the __a__ line. At $x = a$, the equation for that line is $Y = f(a) +$ __b__. Near $x = a = 10$, the linear approximation to $y = x^3$ is $Y =$ $1000 +$ __c__. At $x = 11$ the exact value is $(11)^3 =$ __d__. The approximation is $Y =$ __e__. In this case $\Delta y =$ __f__ and $dy =$ __g__. If we know $\sin x$, then to estimate $\sin(x + \Delta x)$ we add __h__.

In terms of x and Δx, linear approximation is $f(x + \Delta x) \approx f(x) +$ __i__. The error is of order $(\Delta x)^p$ or $(x - a)^p$ with $p =$ __j__. The differential dy equals __k__ times the differential __l__. Those movements are along the __m__ line, where Δy is along the __n__.

Find the linear approximation Y to $y = f(x)$ near $x = a$:

1 $f(x) = x + x^4$, $a = 0$

2 $f(x) = 1/x$, $a = 2$

3 $f(x) = \tan x$, $a = \pi/4$

4 $f(x) = \sin x$, $a = \pi/2$

5 $f(x) = x \sin x$, $a = 2\pi$

6 $f(x) = \sin^2 x$, $a = 0$

Compute 7–12 within .01 by deciding on $f(x)$, choosing the basepoint a, and evaluating $f(a) + f'(a)(x - a)$. A calculator shows the error.

7 $(2.001)^6$

8 $\sin(.02)$

9 $\cos(.03)$

10 $(15.99)^{1/4}$

11 $1/.98$

12 $\sin(3.14)$

Calculate the numerical error in these linear approximations and compare with $\frac{1}{2}(\Delta x)^2 f''(x)$:

13 $(1.01)^3 \approx 1 + 3(.01)$

14 $\cos(.01) \approx 1 + 0(.01)$

15 $(\sin .01)^2 \approx 0 + 0(.01)$

16 $(1.01)^{-3} \approx 1 - 3(.01)$

17 $(1 + \frac{1}{10})^{10} \approx 2$

18 $\sqrt{8.99} \approx 3 + \frac{1}{6}(-.01)$

Confirm the approximations 19–21 by computing $f'(0)$:

19 $\sqrt{1 - x} \approx 1 - \frac{1}{2}x$

20 $1/\sqrt{1 - x^2} \approx 1 + \frac{1}{2}x^2$ (use $f = 1/\sqrt{1 - u}$, then put $u = x^2$)

21 $\sqrt{c^2 + x^2} \approx c + \frac{1}{2}\frac{x^2}{c}$ (use $f(u) = \sqrt{c^2 + u}$, then put $u = x^2$)

22 Write down the differentials df for $f(x) = \cos x$ and $(x + 1)/(x - 1)$ and $(x^2 + 1)^2$.

In 23–27 find the linear change dV in the volume or dA in the surface area.

23 dV if the sides of a cube change from 10 to 10.1.

24 dA if the sides of a cube change from x to $x + dx$.

25 dA if the radius of a sphere changes by dr.

26 dV if a circular cylinder with $r = 2$ changes height from 3 to 3.05 (recall $V = \pi r^2 h$).

27 dV if a cylinder of height 3 changes from $r = 2$ to $r = 1.9$. **Extra credit**: *What is dV if r and h both change (dr and dh)?*

28 In relativity the mass is $m_0/\sqrt{1 - (v/c)^2}$ at velocity v. By Problem 20 this is near $m_0 +$ ________ for small v. Show that the kinetic energy $\frac{1}{2}mv^2$ and the change in mass satisfy Einstein's equation $e = (\Delta m)c^2$.

29 Enter 1.1 on your calculator. Press the square root key 5 times (slowly). What happens each time to the number after the decimal point? This is because $\sqrt{1 + x} \approx$ ________.

30 In Problem 29 the numbers you see are less than 1.05, 1.025, ... The second derivative of $\sqrt{1 + x}$ is ________ so the linear approximation is higher than the curve.

31 Enter 0.9 on your calculator and press the square root key 4 times. Predict what will appear the fifth time and press again. You now have the ________ root of 0.9. How many decimals agree with $1 - \frac{1}{32}(0.1)$?

3.2 Maximum and Minimum Problems

Our goal is to learn about $f(x)$ from df/dx. We begin with two quick questions. If df/dx is positive, what does that say about f? If the slope is negative, how is that reflected in the function? Then the third question is the critical one:

How do you identify a ***maximum*** or ***minimum***? Normal answer: ***The slope is zero.***

This may be the most important application of calculus, to reach $df/dx = 0$.

Take the easy questions first. Suppose df/dx is *positive* for every x between a and b. All tangent lines slope upward. *The function $f(x)$ is **increasing** as x goes from a to b.*

> **3B** If $df/dx > 0$ then $f(x)$ is *increasing*. If $df/dx < 0$ then $f(x)$ is *decreasing*.

To define increasing and decreasing, look at any two points $x < X$. "Increasing" requires $f(x) < f(X)$. "Decreasing" requires $f(x) > f(X)$. ***A positive slope does not mean a positive function.*** The function itself can be positive or negative.

EXAMPLE 1 $f(x) = x^2 - 2x$ has slope $2x - 2$. This slope is positive when $x > 1$ and negative when $x < 1$. The function increases after $x = 1$ and decreases before $x = 1$.

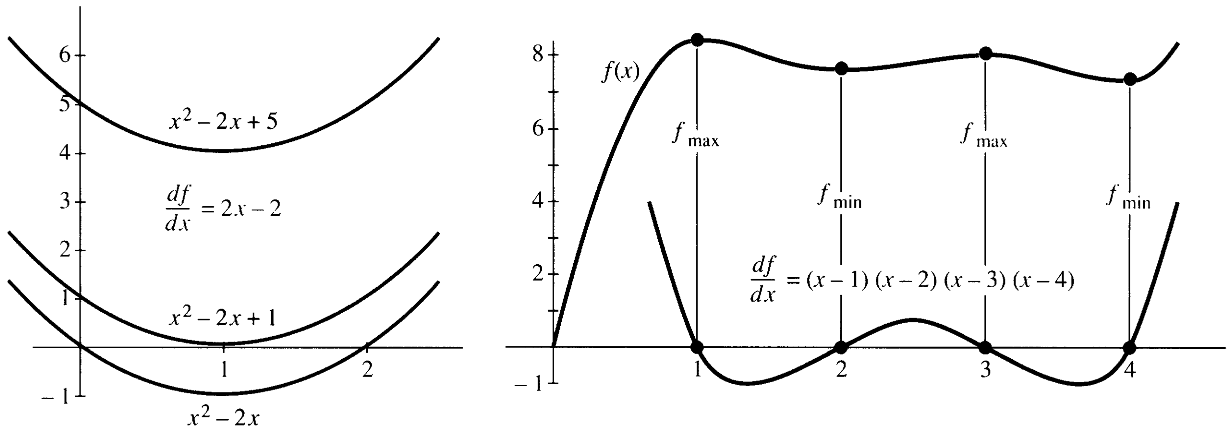


Fig. 3.3 Slopes are $-\ +$. Slope is $+\ -\ +\ -\ +$ so f is up-down-up-down-up.

We say that without computing $f(x)$ at any point! The parabola in Figure 3.3 goes down to its minimum at $x = 1$ and up again.

EXAMPLE 2 $x^2 - 2x + 5$ has the same slope. Its graph is shifted up by 5, a number that disappears in df/dx. All functions with slope $2x - 2$ are parabolas $x^2 - 2x + C$, shifted up or down according to C. Some parabolas cross the x axis (those crossings are solutions to $f(x) = 0$). Other parabolas stay above the axis. The solutions to $x^2 - 2x + 5 = 0$ are complex numbers and we don't see them. The special parabola $x^2 - 2x + 1 = (x - 1)^2$ grazes the axis at $x = 1$. It has a "double zero," where $f(x) = df/dx = 0$.

EXAMPLE 3 Suppose $df/dx = (x - 1)(x - 2)(x - 3)(x - 4)$. This slope is positive beyond $x = 4$ and up to $x = 1$ ($df/dx = 24$ at $x = 0$). And df/dx is positive again between 2 and 3. At $x = 1, 2, 3, 4$, this slope is zero and $f(x)$ changes direction.

Here $f(x)$ is a fifth-degree polynomial, because $f'(x)$ is fourth-degree. The graph of f goes up-down-up-down-up. It might cross the x axis five times. *It must cross at least once* (like this one). When complex numbers are allowed, every fifth-degree polynomial has five roots.

You may feel that "*positive slope implies increasing function*" is obvious—perhaps it is. But there is still something delicate. Starting from $df/dx > 0$ at every *single* point, we have to deduce $f(X) > f(x)$ at *pairs* of points. That is a "local to global" question, to be handled by the Mean Value Theorem. It could also wait for the Fundamental Theorem of Calculus: ***The difference $f(X) - f(x)$ equals the area under the graph of df/dx***. That area is positive, so $f(X)$ exceeds $f(x)$.

MAXIMA AND MINIMA

Which x makes $f(x)$ as large as possible? Where is the smallest $f(x)$? Without calculus we are reduced to computing values of $f(x)$ and comparing. With calculus, the information is in df/dx.

Suppose the maximum or minimum is at a particular point x. It is possible that the graph has a corner—and no derivative. *But if df/dx exists, it must be zero.* The tangent line is level. The parabolas in Figure 3.3 change from decreasing to increasing. The slope changes from negative to positive. At this crucial point *the slope is zero.*

3C ***Local Maximum or Minimum*** Suppose the maximum or minimum occurs at a point x inside an interval where $f(x)$ and df/dx are defined. Then $f'(x) = 0$.

The word "*local*" allows the possibility that in other intervals, $f(x)$ goes higher or lower. *We only look near* x, and we use the definition of df/dx.

Start with $f(x + \Delta x) - f(x)$. If $f(x)$ is the maximum, this difference is negative or zero. The step Δx can be forward or backward:

$$\text{if } \Delta x > 0: \quad \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\text{negative}}{\text{positive}} \leqslant 0 \quad \text{and in the limit} \quad \frac{df}{dx} \leqslant 0.$$

$$\text{if } \Delta x < 0: \quad \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\text{negative}}{\text{negative}} \geqslant 0 \quad \text{and in the limit} \quad \frac{df}{dx} \geqslant 0.$$

Both arguments apply. Both conclusions $df/dx \leqslant 0$ and $df/dx \geqslant 0$ are correct. Thus $df/dx = 0$.

Maybe Richard Feynman said it best. He showed his friends a plastic curve that was made in a special way — ***"no matter how you turn it, the tangent at the lowest point is horizontal."*** They checked it out. It was true.

Surely You're Joking, Mr. Feynman! is a good book (but rough on mathematicians).

EXAMPLE 3 (continued) Look back at Figure 3.3b. The points that stand out are not the "ups" or "downs" but the "turns." Those are ***stationary points***, where $df/dx = 0$. We see two maxima and two minima. None of them are absolute maxima or minima, because $f(x)$ starts at $-\infty$ and ends at $+\infty$.

EXAMPLE 4 $f(x) = 4x^3 - 3x^4$ has slope $12x^2 - 12x^3$. That derivative is zero when x^2 equals x^3, at the two points $x = 0$ and $x = 1$. To decide between minimum and maximum (local or absolute), the first step is to evaluate $f(x)$ at these *stationary points*. We find $f(0) = 0$ and $f(1) = 1$.

Now look at large x. The function goes down to $-\infty$ in both directions. (*You can mentally substitute* $x = 1000$ *and* $x = -1000$). For large x, $-3x^4$ dominates $4x^3$.

Conclusion $f = 1$ is an absolute maximum. $f = 0$ is not a maximum or minimum (local or absolute). We have to recognize this exceptional possibility, that a curve (or a car) can pause for an instant ($f' = 0$) and continue in the same direction. The reason is the "double zero" in $12x^2 - 12x^3$, from its double factor x^2.

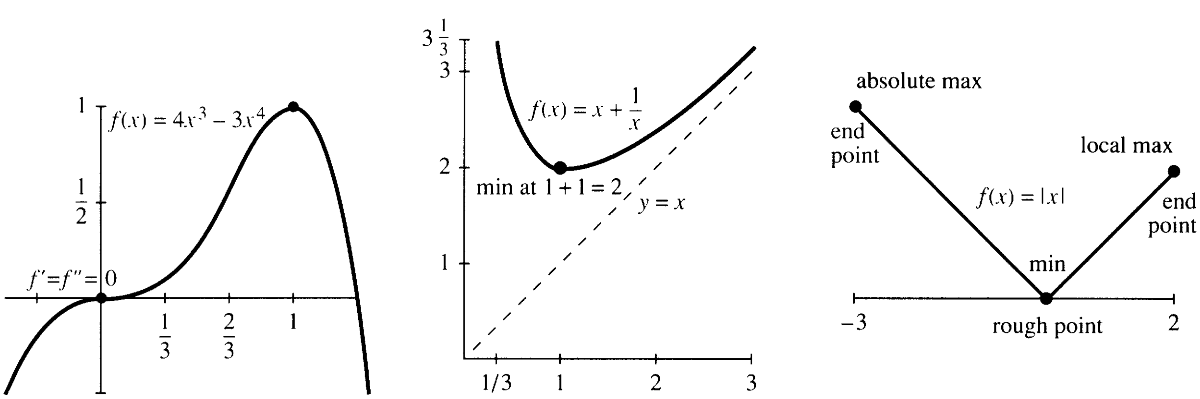


Fig. 3.4 The graphs of $4x^3 - 3x^4$ and $x + x^{-1}$. Check rough points and endpoints.

EXAMPLE 5 Define $f(x) = x + x^{-1}$ for $x > 0$. Its derivative $1 - 1/x^2$ is zero at $x = 1$. At that point $f(1) = 2$ is the minimum value. Every combination like $\frac{1}{3} + 3$ or $\frac{2}{3} + \frac{3}{2}$ is larger than $f_{min} = 2$. Figure 3.4 shows that *the maximum of* $x + x^{-1}$ *is* $+\infty$.†

Important The maximum always occurs at a ***stationary point*** (where $df/dx = 0$) or a ***rough point*** (no derivative) or an ***endpoint*** of the domain. These are the three types of ***critical points***. All maxima and minima occur at critical points! At every other point $df/dx > 0$ or $df/dx < 0$. Here is the procedure:

1. Solve $df/dx = 0$ to find the stationary points $f(x)$.
2. Compute $f(x)$ at every critical point—*stationary point, rough point, endpoint.*
3. Take the maximum and minimum of those critical values of $f(x)$.

EXAMPLE 6 (***Absolute value*** $f(x) = |x|$) The minimum is zero at a rough point. The maximum is at an endpoint. There are no stationary points.

The derivative of $y = |x|$ is never zero. Figure 3.4 shows the maximum and minimum on the interval $[-3, 2]$. This is typical of piecewise linear functions.

Question Could the minimum be zero when the function never reaches $f(x) = 0$?
Answer *Yes,* $f(x) = 1/(1 + x)^2$ approaches but never reaches zero as $x \to \infty$.

Remark* 1** $x \to \pm\infty$ and $f(x) \to \pm\infty$ are avoided when f is ***continuous on a closed interval $a \leqslant x \leqslant b$. Then $f(x)$ reaches its maximum and its minimum (*Extreme Value Theorem*). But $x \to \infty$ and $f(x) \to \infty$ are too important to rule out. You test $x \to \infty$ by considering large x. You recognize $f(x) \to \infty$ by going above every finite value.

***Remark* 2** Note the difference between critical *points* (specified by x) and critical *values* (specified by $f(x)$). The example $x + x^{-1}$ had the minimum *point* $x = 1$ and the minimum *value* $f(1) = 2$.

MAXIMUM AND MINIMUM IN APPLICATIONS

To find a maximum or minimum, solve $f'(x) = 0$. The slope is zero at the top and bottom of the graph. The idea is clear—and then check rough points and endpoints. But to be honest, that is not where the problem starts.

In a real application, the first step (often the hardest) is to choose the unknown and *find the function*. It is we ourselves who decide on x and $f(x)$. The equation $df/dx = 0$ comes in the middle of the problem, not at the beginning. I will start on a new example, with a question instead of a function.

EXAMPLE 7 Where should you get onto an expressway for minimum driving time, if the expressway speed is 60 mph and ordinary driving speed is 30 mph?

I know this problem well—it comes up every morning. The Mass Pike goes to MIT and I have to join it somewhere. There is an entrance near Route 128 and another entrance further in. I used to take the second one, now I take the first. Mathematics should decide which is faster—some mornings I think they are maxima.

Most models are simplified, to focus on the key idea. We will allow the expressway to be entered at *any point* x (Figure 3.5). Instead of two entrances (a discrete problem)

†A good word is *approach* when $f(x) \to \infty$. Infinity is not reached. But I still say "the maximum is ∞."

we have a continuous choice (a calculus problem). The trip has two parts, at speeds 30 and 60:

a distance $\sqrt{a^2+x^2}$ up to the expressway, in $\sqrt{a^2+x^2}/30$ hours

a distance $b-x$ on the expressway, in $(b-x)/60$ hours

Problem Minimize $f(x) = \text{total time} = \frac{1}{30}\sqrt{a^2+x^2} + \frac{1}{60}(b-x)$.

We have the function $f(x)$. Now comes calculus. The first term uses the power rule: *The derivative of $u^{1/2}$ is $\frac{1}{2}u^{-1/2}du/dx$*. Here $u = a^2+x^2$ has $du/dx = 2x$:

$$f'(x) = \frac{1}{30}\frac{1}{2}(a^2+x^2)^{-1/2}(2x) - \frac{1}{60}. \tag{1}$$

To solve $f'(x)=0$, multiply by 60 and square both sides:

$$(a^2+x^2)^{-1/2}(2x) = 1 \quad \text{gives} \quad 2x = (a^2+x^2)^{1/2} \quad \text{and} \quad 4x^2 = a^2+x^2. \tag{2}$$

Thus $3x^2 = a^2$. This yields two candidates, $x = a/\sqrt{3}$ and $x = -a/\sqrt{3}$. But a negative x would mean useless driving on the expressway. In fact f' is *not zero* at $x = -a/\sqrt{3}$. That false root entered when we squared $2x$.

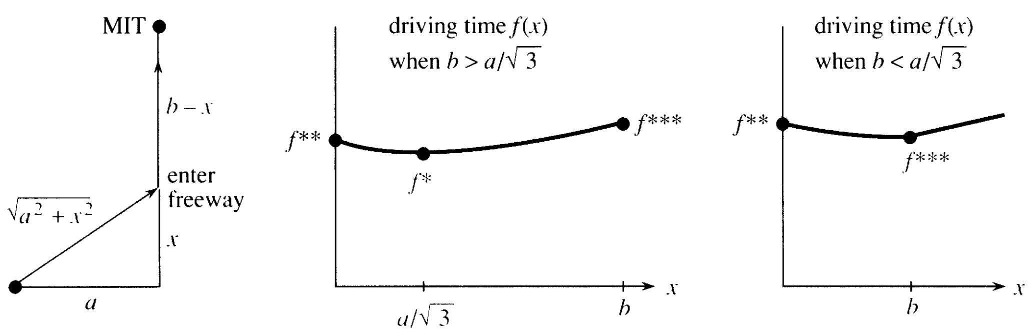


Fig. 3.5 Join the freeway at x—minimize the driving time $f(x)$.

I notice something surprising. The stationary point $x = a/\sqrt{3}$ does not depend on b. The total time includes the constant $b/60$, which disappeared in df/dx. Somehow b must enter the answer, and this is a warning to go carefully. The minimum might occur at a rough point or an endpoint. Those are the other critical points of f, and our drawing may not be realistic. Certainly we expect $x \leqslant b$, or we are entering the expressway beyond MIT.

Continue with calculus. Compute the driving time $f(x)$ for an entrance at $x^* = a/\sqrt{3}$:

$$f(x) = \frac{1}{30}\sqrt{a^2+(a^2/3)} + \frac{1}{60}\left(b - \frac{a}{\sqrt{3}}\right) = \frac{\sqrt{3}a}{60} + \frac{b}{60} = f^*.$$

The square root of $4a^2/3$ is $2a/\sqrt{3}$. We combined $2/30 - 1/60 = 3/60$ and divided by $\sqrt{3}$. ***Is this stationary value f^* a minimum***? You must look also at *endpoints*:

enter at $x=0$: travel time is $a/30 + b/60 = f^{**}$

enter at $x=b$: travel time is $\sqrt{a^2+b^2}/30 = f^{***}$.

The comparison $f^* < f^{**}$ should be automatic. Entering at $x = 0$ was a candidate and calculus didn't choose it. The derivative is not zero at $x = 0$. It is not smart to go perpendicular to the expressway.

The second comparison has $x = b$. We drive directly to MIT at speed 30. This option has to be taken seriously. In fact it is optimal when b is small or a is large.

This choice $x = b$ can arise mathematically in two ways. If all entrances are between 0 and b, then b is an ***endpoint***. If we can enter beyond MIT, then b is a ***rough point***. The graph in Figure 3.5c has a corner at $x = b$, where the derivative jumps. The reason is that distance on the expressway is the *absolute value* $|b - x|$—never negative.

Either way $x = b$ is a critical point. ***The optimal** x **is the smaller of** $a/\sqrt{3}$ **and** b.*

if $a/\sqrt{3} \leqslant b$: stationary point wins, enter at $x = a/\sqrt{3}$, total time f^*

if $a/\sqrt{3} \geqslant b$: no stationary point, drive directly to MIT, time f^{***}

The heart of this subject is in "word problems." All the calculus is in a few lines, computing f' and solving $f'(x) = 0$. The formulation took longer. Step 1 usually does:

1. Express the quantity to be minimized or maximized as a function $f(x)$. *The variable x has to be selected.*
2. Compute $f'(x)$, solve $f'(x) = 0$, check critical points for $f_{\min}$ and $f_{\max}$.

A picture of the problem (and the graph of $f(x)$) makes all the difference.

EXAMPLE 7 (continued) Choose x as an *angle* instead of a distance. Figure 3.6 shows the triangle with angle x and side a. The driving distance to the expressway is $a \sec x$. The distance on the expressway is $b - a \tan x$. Dividing by the speeds 30 and 60, the driving time has a nice form:

$$f(x) = \text{total time} = \frac{a \sec x}{30} + \frac{b - a \tan x}{60}. \tag{3}$$

The derivatives of $\sec x$ and $\tan x$ go into df/dx:

$$\frac{df}{dx} = \frac{a}{30} \sec x \tan x - \frac{a}{60} \sec^2 x. \tag{4}$$

Now set $df/dx = 0$, divide by a, and multiply by $30 \cos^2 x$:

$$\sin x = \tfrac{1}{2}. \tag{5}$$

This answer is beautiful. The angle x is 30°! That optimal angle ($\pi/6$ radians) has $\sin x = \frac{1}{2}$. The triangle with side a and hypotenuse $a/\sqrt{3}$ is a 30–60–90 right triangle.

I don't know whether you prefer $\sqrt{a^2 + x^2}$ or trigonometry. The minimum is exactly as before—either at 30° or going directly to MIT.

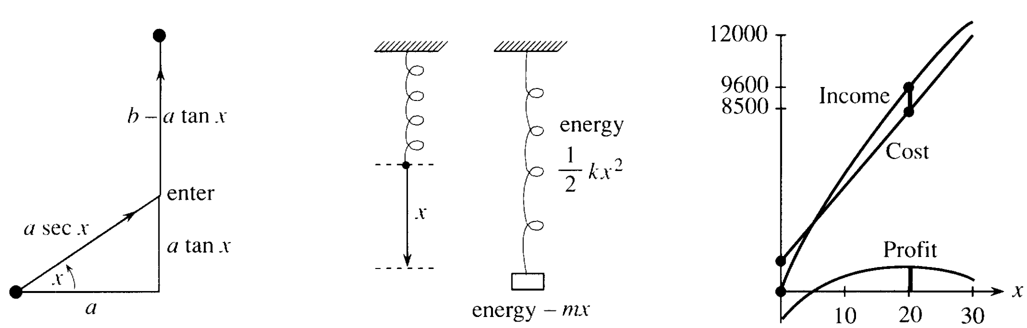


Fig. 3.6 (a) Driving at angle x. (b) Energies of spring and mass. (c) Profit = income − cost.

EXAMPLE 8 In mechanics, *nature chooses minimum energy*. A spring is pulled down by a mass, the energy is $f(x)$, and $df/dx = 0$ gives equilibrium. It is a philosophical question why so many laws of physics involve minimum energy or minimum time—which makes the mathematics easy.

The energy has two terms—for the spring and the mass. The spring energy is $\frac{1}{2}kx^2$—positive in stretching ($x > 0$ is downward) and also positive in compression ($x < 0$). The potential energy of the mass is taken as $-mx$—decreasing as the mass goes down. The balance is at the minimum of $f(x) = \frac{1}{2}kx^2 - mx$.

I apologize for giving you such a small problem, but it makes a crucial point. *When $f(x)$ is quadratic, the equilibrium equation $df/dx = 0$ is linear.*

$$df/dx = kx - m = 0.$$

Graphically, $x = m/k$ is at the bottom of the parabola. Physically, $kx = m$ is a balance of forces—the spring force against the weight. *Hooke's law* for the spring force is elastic constant k times displacement x.

EXAMPLE 9 ***Derivative of cost = marginal cost*** (our first management example).

The paper to print x copies of this book might cost $C = 1000 + 3x$ dollars. The derivative is $dC/dx = 3$. This is the *marginal cost* of paper for each additional book. If x increases by one book, the cost C increases by \$3. The marginal cost is like the velocity and the total cost is like the distance.

Marginal cost is in dollars per book. Total cost is in dollars. On the plus side, the income is $I(x)$ and the marginal income is dI/dx. To apply calculus, we overlook the restriction to whole numbers.

Suppose the number of books increases by dx.† The cost goes up by $(dC/dx)\,dx$. The income goes up by $(dI/dx)\,dx$. If we skip all other costs, then ***profit*** $P(x) =$ ***income*** $I(x) -$ ***cost*** $C(x)$. In most cases P increases to a maximum and falls back.

At the high point on the profit curve, ***the marginal profit is zero***:

$$dP/dx = 0 \quad \text{or} \quad dI/dx = dC/dx. \tag{6}$$

Profit is maximized when marginal income I' equals marginal cost C'.

This basic rule of economics comes directly from calculus, and we give an example:

$C(x) = \textit{cost of } x \textit{ advertisements} = 900 + 400x - x^2$

setup cost 900, print cost $400x$, volume savings x^2

$I(x) = \textit{income due to } x \textit{ advertisements} = 600x - 6x^2$

sales 600 per advertisement, subtract $6x^2$ for diminishing returns

$\textit{optimal decision } dC/dx = dI/dx \quad \text{or} \quad 400 - 2x = 600 - 12x \quad \text{or} \quad x = 20$

$$\textit{profit} = \textit{income} - \textit{cost} = 9600 - 8500 = 1100.$$

The next section shows how to verify that this profit is a maximum not a minimum.

The first exercises ask you to solve $df/dx = 0$. Later exercises also look for $f(x)$.

†Maybe dx is a differential calculus book. I apologize for that.

3.2 EXERCISES

Read-through questions

If $df/dx > 0$ in an interval then $f(x)$ is ____a____. If a maximum or minimum occurs at x then $f'(x) =$ ____b____. Points where $f'(x) = 0$ are called ____c____ points. The function $f(x) = 3x^2 - x$ has a (minimum)(maximum) at $x =$ ____d____. A stationary point that is not a maximum or minimum occurs for $f(x) =$ ____e____.

Extreme values can also occur where ____f____ is not defined or at the ____g____ of the domain. The minima of $|x|$ and $5x$ for $-2 \leqslant x \leqslant 2$ are at $x =$ ____h____ and $x =$ ____i____, even though df/dx is not zero. x^* is an absolute ____j____ when $f(x^*) \geqslant f(x)$ for all x. A ____k____ minimum occurs when $f(x^*) \leqslant f(x)$ for all x near x^*.

The minimum of $\frac{1}{2}ax^2 - bx$ is ____l____ at $x =$ ____m____.

Find the stationary points and rough points and endpoints. Decide whether each point is a local or absolute minimum or maximum.

1 $f(x) = x^2 + 4x + 5,\ -\infty < x < \infty$

2 $f(x) = x^3 - 12x,\ -\infty < x < \infty$

3 $f(x) = x^2 + 3,\ -1 \leqslant x \leqslant 4$

4 $f(x) = x^2 + (2/x),\ 1 \leqslant x \leqslant 4$

5 $f(x) = (x - x^2)^2,\ -1 \leqslant x \leqslant 1$

6 $f(x) = 1/(x - x^2),\ 0 < x < 1$

7 $f(x) = 3x^4 + 8x^3 - 18x^2,\ -\infty < x < \infty$

8 $f(x) = \{x^2 - 4x \text{ for } 0 \leqslant x \leqslant 1,\ x^2 - 4 \text{ for } 1 \leqslant x \leqslant 2\}$

9 $f(x) = \sqrt{x-1} + \sqrt{9-x},\ 1 \leqslant x \leqslant 9$

10 $f(x) = x + \sin x,\ 0 \leqslant x \leqslant 2\pi$

11 $f(x) = x^3(1 - x)^6,\ -\infty < x < \infty$

12 $f(x) = x/(1 + x),\ 0 \leqslant x \leqslant 100$

13 $f(x) =$ distance from $x \geqslant 0$ to nearest whole number

14 $f(x) =$ distance from $x \geqslant 0$ to nearest prime number

15 $f(x) = |x + 1| + |x - 1|,\ -3 \leqslant x \leqslant 2$

16 $f(x) = x\sqrt{1 - x^2},\ 0 \leqslant x \leqslant 1$

17 $f(x) = x^{1/2} - x^{3/2},\ 0 \leqslant x \leqslant 4$

18 $f(x) = \sin x + \cos x,\ 0 \leqslant x \leqslant 2\pi$

19 $f(x) = x + \sin x,\ 0 \leqslant x \leqslant 2\pi$

20 $f(\theta) = \cos^2\theta \sin\theta,\ -\pi \leqslant \theta \leqslant \pi$

21 $f(\theta) = 4\sin\theta - 3\cos\theta,\ 0 \leqslant \theta \leqslant 2\pi$

22 $f(x) = \{x^2 + 1 \text{ for } x \leqslant 1,\ x^2 - 4x + 5 \text{ for } x \geqslant 1\}$.

In applied problems, choose metric units if you prefer.

23 The airlines accept a box if length + width + height $= l + w + h \leqslant 62''$ or 158 cm. If h is fixed show that the maximum volume $(62-w-h)wh$ is $V = h(31 - \frac{1}{2}h)^2$. Choose h to maximize V. The box with greatest volume is a ________.

24 If a patient's pulse measures 70, then 80, then 120, what least squares value minimizes $(x - 70)^2 + (x - 80)^2 + (x - 120)^2$? If the patient got nervous, assign 120 a lower weight and minimize $(x - 70)^2 + (x - 80)^2 + \frac{1}{2}(x - 120)^2$.

25 At speed v, a truck uses $av + (b/v)$ gallons of fuel per mile. How many miles per gallon at speed v? Minimize the fuel consumption. Maximize the number of miles per gallon.

26 A limousine gets $(120 - 2v)/5$ miles per gallon. The chauffeur costs \$10/hour, the gas costs \$1/gallon.

(a) Find the cost per mile at speed v.

(b) Find the cheapest driving speed.

27 You should shoot a basketball at the angle θ requiring minimum speed. Avoid line drives and rainbows. Shooting from $(0, 0)$ with the basket at (a, b), minimize $f(\theta) = 1/(a\sin\theta\cos\theta - b\cos^2\theta)$.

(a) If $b = 0$ you are level with the basket. Show that $\theta = 45°$ is best (Jabbar sky hook).

(b) Reduce $df/d\theta = 0$ to $\tan 2\theta = -a/b$. Solve when $a = b$.

(c) Estimate the best angle for a free throw.

The same angle allows the largest margin of error (*Sports Science* by Peter Brancazio). Section 12.2 gives the flight path.

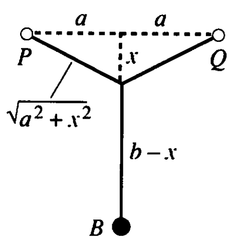
28 On the longest and shortest days, in June and December, why does the length of day change the least?

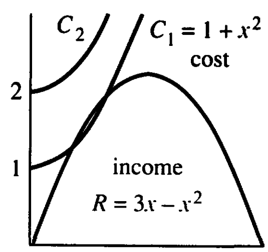
29 Find the shortest **Y** connecting P, Q, and B in the figure. Originally B was a birdfeeder. The length of **Y** is $L(x) = (b - x) + 2\sqrt{a^2 + x^2}$.

(a) Choose x to minimize L (not allowing $x > b$).

(b) Show that the center of the **Y** has 120° angles.

(c) The best **Y** becomes a **V** when $a/b =$ ________.





30 If the distance function is $f(t) = (1 + 3t)/(1 + 3t^2)$, when does the forward motion end? How far have you traveled? Extra credit: Graph $f(t)$ and df/dt.

In 31–34, we make and sell x pizzas. The income is $R(x) = ax + bx^2$ and the cost is $C(x) = c + dx + ex^2$.

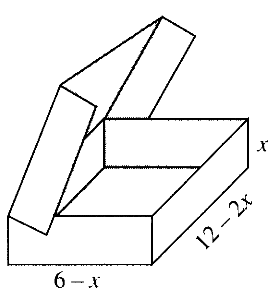
31 The profit is $\Pi(x) =$ ________. The average profit per pizza is $=$ ________. The marginal profit per additional pizza is $d\Pi/dx =$ ________. We should maximize the (profit) (average profit) (marginal profit).

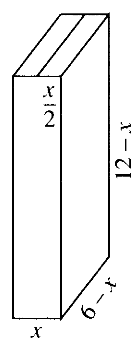
32 We receive $R(x) = ax + bx^2$ when the price per pizza is $p(x) =$ ________. In reverse: When the price is p we sell $x =$ ________ pizzas (a function of p). We expect $b < 0$ because ________.

33 Find x to maximize the profit $\Pi(x)$. At that x the marginal profit is $d\Pi/dx =$ ________.

34 Figure B shows $R(x) = 3x - x^2$ and $C_1(x) = 1 + x^2$ and $C_2(x) = 2 + x^2$. With cost C_1, which sales x makes a profit? Which x makes the most profit? With higher fixed cost in C_2, the best plan is ________.

The cookie box and popcorn box were created by Kay Dundas from a 12″ × 12″ square. A box with no top is a calculus classic.





35 Choose x to find the maximum volume of the cookie box.

36 Choose x to maximize the volume of the popcorn box.

37 A high-class chocolate box adds a strip of width x down across the front of the cookie box. Find the new volume $V(x)$ and the x that maximizes it. Extra credit: Show that $V_{\max}$ is reduced by more than 20%.

38 For a box with no top, cut four squares of side x from the corners of the 12″ square. Fold up the sides so the height is x. Maximize the volume.

Geometry provides many problems, more applied than they seem.

39 A wire four feet long is cut in two pieces. One piece forms a circle of radius r, the other forms a square of side x. Choose r to minimize the sum of their areas. Then choose r to maximize.

40 A fixed wall makes one side of a rectangle. We have 200 feet of fence for the other three sides. Maximize the area A in 4 steps:

1 Draw a picture of the situation.
2 Select one unknown quantity as x (but not A!).
3 Find all other quantities in terms of x.
4 Solve $dA/dx = 0$ and check endpoints.

41 With no fixed wall, the sides of the rectangle satisfy $2x + 2y = 200$. Maximize the area. Compare with the area of a circle using the same fencing.

42 Add 200 meters of fence to an existing straight 100-meter fence, to make a rectangle of maximum area (invented by Professor Klee).

43 How large a rectangle fits into the triangle with sides $x = 0$, $y = 0$, and $x/4 + y/6 = 1$? Find the point on this third side that maximizes the area xy.

44 The largest rectangle in Problem 43 may not sit straight up. Put one side along $x/4 + y/6 = 1$ and maximize the area.

45 The distance around the rectangle in Problem 43 is $P = 2x + 2y$. Substitute for y to find $P(x)$. Which rectangle has $P_{\max} = 12$?

46 Find the right circular cylinder of largest volume that fits in a sphere of radius 1.

47 How large a cylinder fits in a cone that has base radius R and height H? For the cylinder, choose r and h on the sloping surface $r/R + h/H = 1$ to maximize the volume $V = \pi r^2 h$.

48 The cylinder in Problem 47 has side area $A = 2\pi rh$. Maximize A instead of V.

49 Including top and bottom, the cylinder has area

$$A = 2\pi rh + 2\pi r^2 = 2\pi rH(1 - (r/R)) + 2\pi r^2.$$

Maximize A when $H > R$. Maximize A when $R > H$.

***50** A wall 8 feet high is 1 foot from a house. Find the length L of the shortest ladder over the wall to the house. Draw a triangle with height y, base $1 + x$, and hypotenuse L.

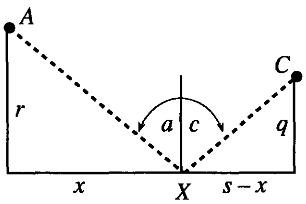
51 Find the closed cylinder of volume $V = \pi r^2 h = 16\pi$ that has the least surface area.

52 Draw a kite that has a triangle with sides 1, 1, $2x$ next to a triangle with sides $2x$, 2, 2. Find the area A and the x that maximizes it. *Hint*: In dA/dx simplify $\sqrt{1 - x^2} - x^2/\sqrt{1 - x^2}$ to $(1 - 2x^2)/\sqrt{1 - x^2}$.

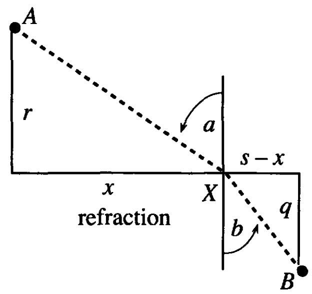
In 53–56, x and y are nonnegative numbers with $x + y = 10$. Maximize and minimize:

53 xy **54** $x^2 + y^2$ **55** $y - (1/x)$ **56** $\sin x \sin y$

57 Find the total distance $f(x)$ from A to X to C. Show that $df/dx = 0$ leads to $\sin a = \sin c$. Light reflects at an equal angle to minimize travel time.



reflection



refraction

58 Fermat's principle says that light travels from A to B on the quickest path. Its velocity above the x axis is v and below the x axis is w.

(a) Find the time $T(x)$ from A to X to B. On AX, time = distance/velocity = $\sqrt{r^2+x^2}/v$.

(b) Find the equation for the minimizing x.

(c) Deduce *Snell's law* $(\sin a)/v = (\sin b)/w$.

"Closest point problems" are models for many applications.

59 Where is the parabola $y = x^2$ closest to $x = 0$, $y = 2$?

60 Where is the line $y = 5 - 2x$ closest to (0, 0)?

61 What point on $y = -x^2$ is closest to what point on $y = 5 - 2x$? At the nearest points, the graphs have the same slope. Sketch the graphs.

62 Where is $y = x^2$ closest to $(0, \frac{1}{3})$? Minimizing $x^2 + (y - \frac{1}{3})^2 = y + (y - \frac{1}{3})^2$ gives $y < 0$. What went wrong?

63 Draw the line $y = mx$ passing near (2, 3), (1, 1), and (−1, 1). For a least squares fit, minimize

$$(3-2m)^2 + (1-m)^2 + (1+m)^2.$$

64 A triangle has corners (−1, 1), (x, x^2), and (3, 9) on the parabola $y = x^2$. Find its maximum area for x between −1 and 3. *Hint*: The distance from (X, Y) to the line $y = mx + b$ is $|Y - mX - b|/\sqrt{1+m^2}$.

65 Submarines are located at (2, 0) and (1, 1). Choose the slope m so the line $y = mx$ goes between the submarines but stays as far as possible from the nearest one.

Problems 66–72 go back to the theory.

66 To find where the graph of $y(x)$ has greatest slope, solve ________. For $y = 1/(1+x^2)$ this point is ________.

67 When the difference between $f(x)$ and $g(x)$ is smallest, their slopes are ________. Show this point on the graphs of $f = 2 + x^2$ and $g = 2x - x^2$.

68 Suppose y is fixed. The minimum of $x^2 + xy - y^2$ (a function of x) is $m(y) =$ ________. Find the maximum of $m(y)$.

Now x is fixed. The maximum of $x^2 + xy - y^2$ (a function of y) is $M(x) =$ ________. Find the minimum of $M(x)$.

69 For each m the minimum value of $f(x) - mx$ occurs at $x = m$. What is $f(x)$?

70 $y = x + 2x^2 \sin(1/x)$ has slope 1 at $x = 0$. But show that y is not increasing on an *interval* around $x = 0$, by finding points where $dy/dx = 1 - 2\cos(1/x) + 4x \sin(1/x)$ is negative.

71 *True or false*, with a reason: Between two local minima of a smooth function $f(x)$ there is a local maximum.

72 Create a function $y(x)$ that has its maximum at a rough point and its minimum at an endpoint.

73 Draw a circular pool with a lifeguard on one side and a drowner on the opposite side. The lifeguard swims with velocity v and runs around the rest of the pool with velocity $w = 10v$. If the swim direction is at angle θ with the direct line, choose θ to minimize and maximize the arrival time.

3.3 Second Derivatives: Bending and Acceleration

When $f'(x)$ is positive, $f(x)$ is increasing. When dy/dx is negative, $y(x)$ is decreasing. That is clear, but what about the *second* derivative? From looking at the curve, can you decide the sign of $f''(x)$ or d^2y/dx^2? The answer is *yes* and the key is in the ***bending***.

A straight line doesn't bend. The slope of $y = mx + b$ is m (a constant). The second derivative is zero. We have to go to curves, to see a changing slope. Changes in the derivative show up in $f''(x)$:

$f = x^2$ has $f' = 2x$ and $f'' = 2$ (this parabola bends *up*)

$y = \sin x$ has $dy/dx = \cos x$ and $d^2y/dx^2 = -\sin x$ (the sine bends *down*)

The slope $2x$ gets larger *even when the parabola is falling*. The sign of f or f' is not revealed by f''. The second derivative tells about ***change in slope***.

A function with $f''(x) > 0$ is ***concave up***. It bends upward as the slope increases. It is also called *convex*. A function with decreasing slope—this means $f''(x) < 0$—is ***concave down***. Note how $\cos x$ and $1 + \cos x$ and even $1 + \frac{1}{2}x + \cos x$ change from concave down to concave up at $x = \pi/2$. At that point $f'' = -\cos x$ changes from negative to positive. The extra $1 + \frac{1}{2}x$ tilts the graph but the bending is the same.

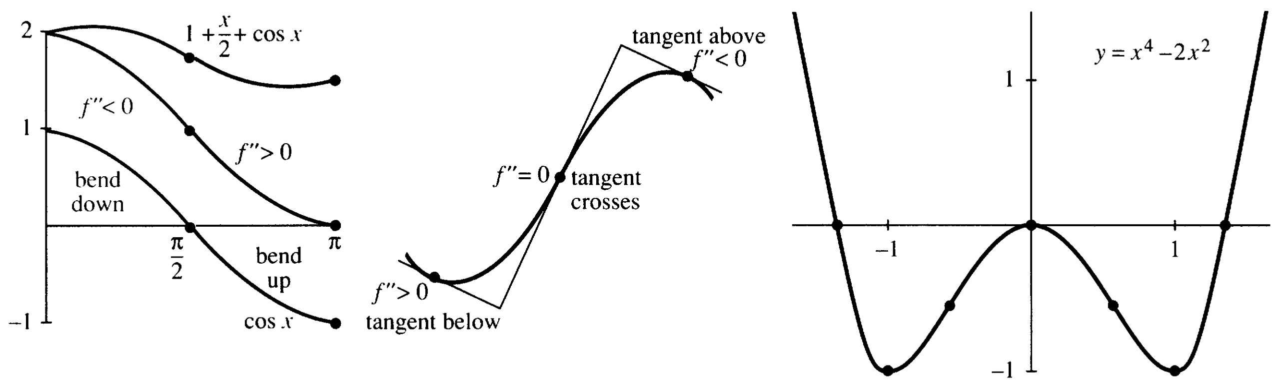


Fig. 3.7 Increasing slope = concave up ($f'' > 0$). Concave down is $f'' < 0$. Inflection point $f'' = 0$.

Here is another way to see the sign of f''. ***Watch the tangent lines***. When the curve is concave up, the tangent stays below it. A linear approximation is too low. This section computes a *quadratic* approximation—which includes the term with $f'' > 0$. When the curve bends down ($f'' < 0$), the opposite happens—the tangent lines are above the curve. The linear approximation is too high, and f'' lowers it.

In physical motion, $f''(t)$ is the *acceleration*—in units of distance/(time)^2. Acceleration is rate of change of velocity. The oscillation $\sin 2t$ has $v = 2 \cos 2t$ (maximum speed 2) and $a = -4 \sin 2t$ (maximum acceleration 4).

An increasing population means $f' > 0$. ***An increasing growth rate means*** $f'' > 0$. Those are different. The rate can slow down while the growth continues.

MAXIMUM VS. MINIMUM

Remember that $f'(x) = 0$ locates a stationary point. That may be a *minimum* or a *maximum*. ***The second derivative decides***! Instead of computing $f(x)$ at many points, we compute $f''(x)$ at one point—the stationary point. It is a minimum if $f''(x) > 0$.

> **3D** When $f'(x) = 0$ and $f''(x) > 0$, there is a ***local minimum*** at x.
> When $f'(x) = 0$ and $f''(x) < 0$, there is a ***local maximum*** at x.

To the left of a minimum, the curve is falling. After the minimum, the curve rises. The slope has changed from negative to positive. The graph bends upward and $f''(x) > 0$.

At a maximum the slope drops from positive to negative. In the exceptional case, when $f'(x) = 0$ and also $f''(x) = 0$, anything can happen. An example is x^3, which pauses at $x = 0$ and continues up (its slope is $3x^2 \geq 0$). However x^4 pauses and goes down (with a very flat graph).

We emphasize that the information from $f'(x)$ and $f''(x)$ is only "*local*." To be certain of an *absolute* minimum or maximum, we need information over the whole domain.

EXAMPLE 1 $f(x) = x^3 - x^2$ has $f'(x) = 3x^2 - 2x$ and $f''(x) = 6x - 2$.

To find the maximum and/or minimum, solve $3x^2 - 2x = 0$. The stationary points are $x = 0$ and $x = \frac{2}{3}$. At those points we need the second derivative. It is $f''(0) = -2$ (local maximum) and $f''(\frac{2}{3}) = +2$ (local minimum).

Between the maximum and minimum is the ***inflection point***. *That is where* $f''(x) = 0$. The curve changes from concave down to concave up. This example has $f''(x) = 6x - 2$, so the inflection point is at $x = \frac{1}{3}$.

INFLECTION POINTS

In mathematics it is a special event when a function passes through zero. When the function is f, its graph crosses the axis. When the function is f', the tangent line is horizontal. When f'' goes through zero, we have an *inflection point.*

The direction of bending changes at an inflection point. Your eye picks that out in a graph. For an instant the graph is straight (straight lines have $f'' = 0$). It is easy to see crossing points and stationary points and inflection points. Very few people can recognize where $f''' = 0$ or $f'''' = 0$. I am not sure if those points have names.

There is a genuine maximum or minimum when $f'(x)$ changes sign. Similarly, there is a genuine inflection point when $f''(x)$ changes sign. ***The graph is concave down on one side of an inflection point and concave up on the other side.***† The tangents are above the curve on one side and below it on the other side. At an inflection point, *the tangent line crosses the curve* (Figure 3.7b).

Notice that a parabola $y = ax^2 + bx + c$ has no inflection points: y'' is constant. A cubic curve has one inflection point, because f'' is linear. A fourth-degree curve might or might not have inflection points—the quadratic $f''(x)$ might or might not cross the axis.

EXAMPLE 2 $x^4 - 2x^2$ is W-shaped, $4x^3 - 4x$ has two bumps, $12x^2 - 4$ is U-shaped. The table shows the signs at the important values of x:

x	$-\sqrt{2}$		-1		$-1/\sqrt{3}$		0		$1/\sqrt{3}$		1	$\sqrt{2}$
$f(x)$	0		$-$		$-$		$0, 0$		$-$		$-$	0
$f'(x)$			0		$+$		0		$-$		0	
$f''(x)$					0		$-$		0			

Between zeros of $f(x)$ come zeros of $f'(x)$ (stationary points). Between zeros of $f'(x)$ come zeros of $f''(x)$ (inflection points). In this example $f(x)$ has a double zero at the origin, so a single zero of f' is caught there. It is a local maximum, since $f''(0) < 0$.

Inflection points are important—not just for mathematics. We know the world population will keep rising. We don't know if the *rate* of growth will slow down. Remember: ***The rate of growth stops growing at the inflection point***. Here is the 1990 report of the UN Population Fund.

> The next ten years will decide whether the world population trebles or merely doubles before it finally stops growing. This may decide the future of the earth as a habitation for humans. The population, now 5.3 billion, is increasing by a quarter of a million every day. Between 90 and 100 million people will be added every year

†That rules out $f(x) = x^4$, which has $f'' = 12x^2 > 0$ on both sides of zero. Its tangent line is the x axis. The line stays below the graph—so no inflection point.

> during the 1990s; a billion people—a whole China—over the decade. The fastest growth will come in the poorest countries.
>
> A few years ago it seemed as if the rate of population growth was slowing† everywhere except in Africa and parts of South Asia. The world's population seemed set to stabilize around 10.2 billion towards the end of the next century.
>
> Today, the situation looks less promising. The world has overshot the marker points of the 1984 "most likely" medium projection. It is now on course for an eventual total that will be closer to 11 billion than to 10 billion.
>
> If fertility reductions continue to be slower than projected, the mark could be missed again. In that case the world could be headed towards a total of up to 14 billion people.

Starting with a census, the UN follows each age group in each country. They estimate the death rate and fertility rate—the medium estimates are published. This report is saying that we are not on track with the estimate.

Section 6.5 will come back to population, with an equation that predicts 10 billion. It assumes we are now at the inflection point. But China's second census just started on July 1, 1990. When it's finished we will know if the inflection point is still ahead.

You now understand the meaning of $f''(x)$. Its sign gives the direction of bending—the change in the slope. *The rest of this section computes* ***how much*** *the curve bends*—using the *size* of f'' and not just its sign. We find quadratic approximations based on $f''(x)$. In some courses they are optional—the main points are highlighted.

CENTERED DIFFERENCES AND SECOND DIFFERENCES

Calculus begins with average velocities, computed on either side of x:

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} \quad\text{and}\quad \frac{f(x)-f(x-\Delta x)}{\Delta x} \quad \text{are close to} \quad f'(x). \tag{1}$$

We never mentioned it, but a better approximation to $f'(x)$ comes from *averaging those two averages*. This produces a ***centered difference***, which is based on $x+\Delta x$ and $x-\Delta x$. It divides by $2\Delta x$:

$$f'(x) \approx \frac{1}{2}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}+\frac{f(x)-f(x-\Delta x)}{\Delta x}\right]=\frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}. \tag{2}$$

We claim this is better. The test is to try it on powers of x.

For $f(x)=x$ these ratios all give $f'=1$ (exactly). For $f(x)=x^2$, only the centered difference correctly gives $f'=2x$. The one-sided ratio gave $2x+\Delta x$ (in Chapter 1 it was $2t+h$). It is only "first-order accurate." But centering leaves no error. We are averaging $2x+\Delta x$ with $2x-\Delta x$. Thus the centered difference is "second-order accurate."

I ask now: *What ratio converges to the second derivative*? One answer is to take differences of the first derivative. Certainly $\Delta f'/\Delta x$ approaches f''. But we want a ratio involving f itself. A natural idea is to take *differences of differences*, which brings us to "***second differences***":

$$\frac{\dfrac{f(x+\Delta x)-f(x)}{\Delta x}-\dfrac{f(x)-f(x-\Delta x)}{\Delta x}}{\Delta x}=\frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{(\Delta x)^2}\rightarrow\frac{d^2f}{dx^2}. \tag{3}$$

†The United Nations watches the second derivative!

On the top, the difference of the difference is $\Delta(\Delta f) = \Delta^2 f$. It corresponds to d^2f. On the bottom, $(\Delta x)^2$ corresponds to dx^2. This explains the way we place the 2's in d^2f/dx^2. To say it differently: dx is squared, df is not squared—as in distance/(time)^2.

Note that $(\Delta x)^2$ becomes much smaller than Δx. If we divide Δf by $(\Delta x)^2$, the ratio blows up. It is the extra cancellation in the second difference $\Delta^2 f$ that allows the limit to exist. That limit is $f''(x)$.

Application The great majority of differential equations can't be solved exactly. A typical case is $f''(x) = -\sin f(x)$ (the pendulum equation). To compute a solution, I would replace $f''(x)$ by the second difference in equation (3). Approximations at points spaced by Δx are a very large part of scientific computing.

To test the accuracy of these differences, here is an experiment on $f(x) = \sin x + \cos x$. The table shows the errors at $x = 0$ from formulas (1), (2), (3):

step length Δx	one-sided errors	centered errors	second difference errors
1/4	.1347	.0104	$-.0052$
1/8	.0650	.0026	$-.0013$
1/16	.0319	.0007	$-.0003$
1/32	.0158	.0002	$-.0001$

The one-sided errors are cut in half when Δx is cut in half. The other columns decrease like $(\Delta x)^2$. Each reduction divides those errors by 4. ***The errors from one-sided differences are*** $O(\Delta x)$ ***and the errors from centered differences are*** $O(\Delta x)^2$.

The "big O" notation When the errors are of order Δx, we write $E = O(\Delta x)$. This means that $E \leqslant C\Delta x$ for some constant C. We don't compute C—in fact we don't want to deal with it. The statement "one-sided errors are Oh of delta x" captures what is important. The main point of the other columns is $E = O(\Delta x)^2$.

LINEAR APPROXIMATION VS. QUADRATIC APPROXIMATION

The second derivative gives a tremendous improvement over linear approximation $f(a) + f'(a)(x-a)$. A tangent line starts out close to the curve, but *the line has no way to bend.* After a while it overshoots or undershoots the true function (see Figure 3.8). That is especially clear for the model $f(x) = x^2$, when the tangent is the x axis and the parabola curves upward.

You can almost guess the term with bending. *It should involve* f'', and also $(\Delta x)^2$. It might be exactly $f''(x)$ times $(\Delta x)^2$ but it is not. The model function x^2 has $f'' = 2$. There must be a factor $\frac{1}{2}$ to cancel that 2:

3E The ***quadratic approximation*** to a smooth function $f(x)$ near $x = a$ is

$$f(x) \approx f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2. \quad (4)$$

At the basepoint this is $f(a) = f(a)$. The derivatives also agree at $x = a$. Furthermore *the second derivatives agree.* On both sides of (4), the second derivative at $x = a$ is $f''(a)$.

The quadratic approximation bends with the function. It is not the absolutely final word, because there is a cubic term $\frac{1}{6}f'''(a)(x-a)^3$ and a fourth-degree term $\frac{1}{24}f''''(a)(x-a)^4$ and so on. The whole infinite sum is a "Taylor series." Equation (4) carries that series through the quadratic term—which for practical purposes gives a terrific approximation. You will see that in numerical experiments.

Two things to mention. First, equation (4) shows why $f'' > 0$ brings the curve above the tangent line. The linear part gives the line, while the quadratic part is positive and bends upward. Second, equation (4) comes from (2) and (3). Where one-sided differences give $f(x + \Delta x) \approx f(x) + f'(x)\Delta x$, centered differences give the quadratic:

$$\text{from (2):}\quad f(x+\Delta x) \approx f(x-\Delta x) + 2f'(x)\,\Delta x$$

$$\text{from (3):}\quad f(x+\Delta x) \approx 2f(x) - f(x-\Delta x) + f''(x)(\Delta x)^2.$$

Add and divide by 2. The result is $f(x+\Delta x) \approx f(x) + f'(x)\Delta x + \frac{1}{2}f''(x)(\Delta x)^2$. This is correct through $(\Delta x)^2$ and misses by $(\Delta x)^3$, as examples show:

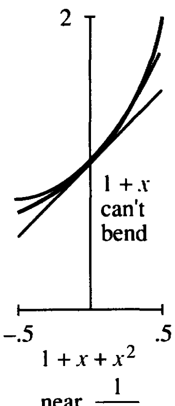


Fig. 3.8

EXAMPLE 3 $(x+\Delta x)^3 \approx (x^3) + (3x^2)(\Delta x) + \frac{1}{2}(6x)(\Delta x)^2 + \text{error } (\Delta x)^3.$

EXAMPLE 4 $(1+x)^n \approx 1 + nx + \frac{1}{2}n(n-1)x^2.$

The first derivative at $x = 0$ is n. The second derivative is $n(n-1)$. The cubic term would be $\frac{1}{6}n(n-1)(n-2)x^3$. We are just producing the binomial expansion!

EXAMPLE 5 $\dfrac{1}{1-x} \approx 1 + x + x^2 = \text{start of a geometric series.}$

$1/(1-x)$ has derivative $1/(1-x)^2$. Its second derivative is $2/(1-x)^3$. At $x = 0$ those equal 1,1,2. The factor $\frac{1}{2}$ cancels the 2, which leaves 1,1,1. This explains $1 + x + x^2$.

The next terms are x^3 and x^4. The whole series is $1/(1-x) = 1 + x + x^2 + x^3 + \cdots$.

Numerical experiment $1/\sqrt{1+x} \approx 1 - \frac{1}{2}x + \frac{3}{8}x^2$ is tested for accuracy. Dividing x by 2 almost divides the error by 8. If we only keep the linear part $1 - \frac{1}{2}x$, the error is only divided by 4. Here are the errors at $x = \frac{1}{4}, \frac{1}{8}$, and $\frac{1}{16}$:

linear approximation $\left(\text{error} \approx \frac{3}{8}x^2\right)$:	.0194	.0053	.0014
quadratic approximation $\left(\text{error} \approx \frac{-5}{16}x^3\right)$:	$-.00401$	$-.00055$	$-.00007$

3.3 EXERCISES

Read-through questions

The direction of bending is given by the sign of __a__. If the second derivative is __b__ in an interval, the function is concave up (or convex). The graph bends __c__. The tangent lines are __d__ the graph. If $f''(x) < 0$ then the graph is concave __e__, and the slope is __f__.

At a point where $f'(x) = 0$ and $f''(x) > 0$, the function has a __g__. At a point where __h__, the function has a maximum. A point where $f''(x) = 0$ is an __i__ point, provided f'' changes sign. The tangent line __j__ the graph.

The centered approximation to $f'(x)$ *is* [__k__]/$2\Delta x$. The 3-point approximation to $f''(x)$ is [__l__]/$(\Delta x)^2$. The second-order approximation to $f(x+\Delta x)$ is $f(x) + f'(x)\Delta x +$ __m__. Without that extra term this is just the __n__ approximation. With that term the error is O(__o__).

1 A graph that is concave upward is inaccurately said to "hold water." Sketch a graph with $f''(x) > 0$ that would not hold water.

2 Find a function that is concave down for $x < 0$ and concave up for $0 < x < 1$ and concave down for $x > 1$.

3 Can a function be always concave down and never cross zero? Can it be always concave down and positive? Explain.

4 Find a function with $f''(2) = 0$ and no other inflection point.

True or false, when $f(x)$ is a 9th degree polynomial with $f'(1) = 0$ and $f'(3) = 0$. Give (or draw) a reason.

5 $f(x) = 0$ somewhere between $x = 1$ and $x = 3$.

6 $f''(x) = 0$ somewhere between $x = 1$ and $x = 3$.

7 There is no absolute maximum at $x = 3$.

8 There are seven points of inflection.

9 If $f(x)$ has nine zeros, it has seven inflection points.

10 If $f(x)$ has seven inflection points, it has nine zeros.

In 11–16 decide which stationary points are maxima or minima.

11 $f(x) = x^2 - 6x$ **12** $f(x) = x^3 - 6x^2$

13 $f(x) = x^4 - 6x^3$ **14** $f(x) = x^{11} - 6x^{10}$

15 $f(x) = \sin x - \cos x$ **16** $f(x) = x + \sin 2x$

Locate the inflection points and the regions where $f(x)$ is concave up or down.

17 $f(x) = x + x^2 - x^3$ **18** $f(x) = \sin x + \tan x$

19 $f(x) = (x-2)^2(x-4)^2$ **20** $f(x) = \sin x + (\sin x)^3$

21 If $f(x)$ is an even function, the centered difference $[f(\Delta x) - f(-\Delta x)]/2\Delta x$ exactly equals $f'(0) = 0$. Why?

22 If $f(x)$ is an odd function, the second difference $[f(\Delta x) - 2f(0) + f(-\Delta x)]/(\Delta x)^2$ exactly equals $f''(0) = 0$. Why?

Write down the quadratic $f(0) + f'(0)x + \frac{1}{2}f''(0)x^2$ in 23–26.

23 $f(x) = \cos x + \sin x$ **24** $f(x) = \tan x$

25 $f(x) = (\sin x)/x$ **26** $f(x) = 1 + x + x^2$

In 26, find $f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2$ around $a = 1$.

27 Find A and B in $\sqrt{1-x} \approx 1 + Ax + Bx^2$.

28 Find A and B in $1/(1-x)^2 \approx 1 + Ax + Bx^2$.

29 Substitute the quadratic approximation into $[f(x+\Delta x) - f(x)]/\Delta x$, to estimate the error in this one-sided approximation to $f'(x)$.

30 What is the quadratic approximation at $x = 0$ to $f(-\Delta x)$?

31 Substitute for $f(x+\Delta x)$ and $f(x-\Delta x)$ in the centered approximation $[f(x+\Delta x) - f(x-\Delta x)]/2\Delta x$, to get $f'(x)$ + error. Find the Δx and $(\Delta x)^2$ terms in this error. Test on $f(x) = x^3$ at $x = 0$.

32 Guess a third-order approximation $f(\Delta x) \approx f(0) + f'(0)\Delta x + \frac{1}{2}f''(0)(\Delta x)^2 +$ ________. Test it on $f(x) = x^3$.

Construct a table as in the text, showing the actual errors at $x = 0$ in one-sided differences, centered differences, second differences, and quadratic approximations. By hand take two values of Δx, by calculator take three, by computer take four.

33 $f(x) = x^3 + x^4$ **34** $f(x) = 1/(1-x)$

35 $f(x) = x^2 + \sin x$

36 Example 5 was $1/(1-x) \approx 1 + x + x^2$. What is the error at $x = 0.1$? What is the error at $x = 2$?

37 Substitute $x = .01$ and $x = -0.1$ in the geometric series $1/(1-x) = 1 + x + x^2 + \cdots$ to find $1/.99$ and $1/1.1$—first to four decimals and then to all decimals.

38 Compute $\cos 1°$ by equation (4) with $a = 0$. OK to check on a calculator. Also compute $\cos 1$. Why so far off?

39 Why is $\sin x \approx x$ not only a linear approximation but also a quadratic approximation? $x = 0$ is an ________ point.

40 If $f(x)$ is an even function, find its quadratic approximation at $x = 0$. What is the equation of the tangent line?

41 For $f(x) = x + x^2 + x^3$, what is the centered difference $[f(3) - f(1)]/2$, and what is the true slope $f'(2)$?

42 For $f(x) = x + x^2 + x^3$, what is the second difference $[f(3) - 2f(2) + f(1)]/1^2$, and what is the exact $f''(2)$?

43 The error in $f(a) + f'(a)(x-a)$ is approximately $\frac{1}{2}f''(a)(x-a)^2$. This error is positive when the function is ________. Then the tangent line is ________ the curve.

44 Draw a piecewise linear $y(x)$ that is concave up. Define "concave up" without using the test $d^2y/dx^2 \geqslant 0$. If derivatives don't exist, a new definition is needed.

45 What do these sentences say about f or f' or f'' or f'''?

1. The population is growing more slowly.
2. The plane is landing smoothly.
3. The economy is picking up speed.
4. The tax rate is constant.
5. A bike accelerates faster but a car goes faster.
6. Stock prices have peaked.
7. The rate of acceleration is slowing down.
8. This course is going downhill.

46 (Recommended) Draw a curve that goes up-down-up. Below it draw its derivative. Then draw its second derivative. *Mark the same points on all curves*—the maximum, minimum, and inflection points of the first curve.

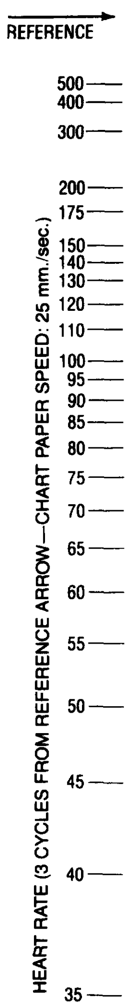
47 Repeat Problem 46 on a printout showing $y(x) = x^3 - 4x^2 + x + 2$ and dy/dx and d^2y/dx^2 on the same graph.

3.4 Graphs

Reading a graph is like appreciating a painting. Everything is there, but you have to know what to look for. One way to learn is by sketching graphs yourself, and in the past that was almost the only way. Now it is obsolete to spend weeks drawing curves—a computer or graphing calculator does it faster and better. That doesn't remove the need to appreciate a graph (or a painting), since a curve displays a tremendous amount of information.

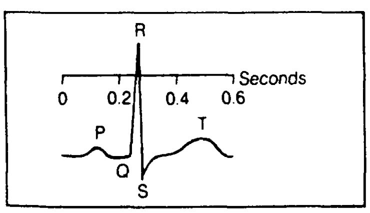
This section combines two approaches. One is to study actual machine-produced graphs (especially electrocardiograms). The other is to understand the mathematics of graphs—slope, concavity, asymptotes, shifts, and scaling. We introduce the ***centering transform*** and ***zoom transform***. These two approaches are like the rest of calculus, where special derivatives and integrals are done by hand and day-to-day applications are by computer. Both are essential—the machine can do experiments that we could never do. But without the mathematics our instructions miss the point. To create good graphs you have to know a few of them personally.

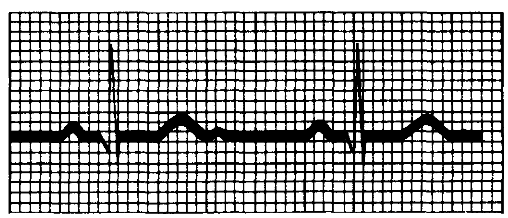
READING AN ELECTROCARDIOGRAM (ECG or EKG)



The graphs of an ECG show the electrical potential during a heartbeat. There are twelve graphs—six from leads attached to the chest, and six from leads to the arms and left leg. (It doesn't hurt, but everybody is nervous. You have to lie still, because contraction of other muscles will mask the reading from the heart.) The graphs record electrical impulses, as the cells depolarize and the heart contracts.

What can I explain in two pages? The graph shows the fundamental pattern of the ECG. ***Note the*** P ***wave***, ***the*** QRS ***complex***, ***and the*** T ***wave***. Those patterns, seen differently in the twelve graphs, tell whether the heart is normal or out of rhythm—or suffering an infarction (a heart attack).





First of all the graphs show the ***heart rate***. The dark vertical lines are by convention $\frac{1}{5}$ second apart. The light lines are $\frac{1}{25}$ second apart. If the heart beats every $\frac{1}{5}$ second (one dark line) the rate is 5 beats per second or 300 per minute. That is extreme *tachycardia*—not compatible with life. The normal rate is between three dark lines per beat ($\frac{3}{5}$ second, or 100 beats per minute) and five dark lines (one second between beats, 60 per minute). A baby has a faster rate, over 100 per minute. In this figure the rate is _____. A rate below 60 is *bradycardia*, not in itself dangerous. For a resting athlete that is normal.

Doctors memorize the six rates 300, 150, 100, 75, 60, 50. Those correspond to 1, 2, 3, 4, 5, 6 dark lines between heartbeats. The distance is easiest to measure between spikes (the peaks of the R wave). Many doctors put a printed scale next to the chart. One textbook emphasizes that "Where the next wave falls determines the rate. No mathematical computation is necessary." But you see where those numbers come from.

The next thing to look for is ***heart rhythm***. The regular rhythm is set by the pacemaker, which produces the P wave. A constant distance between waves is good—and then each beat is examined. When there is a block in the pathway, it shows as a delay in the graph. Sometimes the pacemaker fires irregularly. Figure 3.10 shows *sinus arrythmia* (fairly normal). The time between peaks is changing. In disease or emergency, there are potential pacemakers in all parts of the heart.

I should have pointed out the main parts. We have four chambers, an atrium-ventricle pair on the left and right. The SA node should be the pacemaker. The stimulus spreads from the atria to the ventricles—from the small chambers that "prime the pump" to the powerful chambers that drive blood through the body. The P wave comes with contraction of the atria. There is a pause of $\frac{1}{10}$ second at the AV node. Then the big QRS wave starts contraction of the ventricles, and the T wave is when the ventricles relax. The cells switch back to negative charge and the heart cycle is complete.

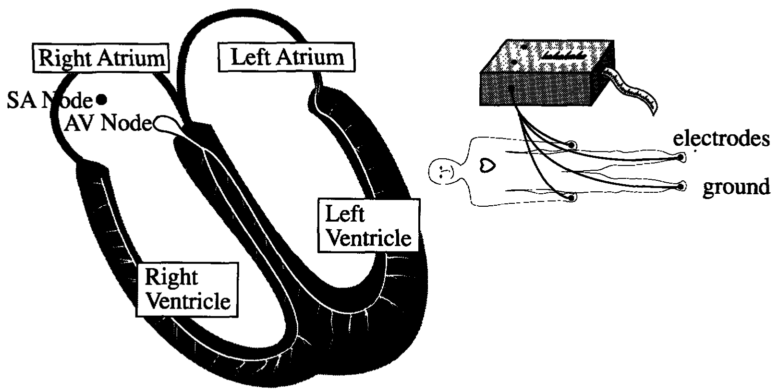


Fig. 3.9 Happy person with a heart and a normal electrocardiogram.

The ECG shows when the pacemaker goes wrong. Other pacemakers take over—the AV node will pace at 60/minute. An early firing in the ventricle can give a wide spike in the QRS complex, followed by a long pause. The impulses travel by a slow path. Also the pacemaker can suddenly speed up (paroxysmal tachycardia is 150–250/minute). But the most critical danger is *fibrillation*.

Figure 3.10b shows a dying heart. The ECG indicates irregular contractions—no normal PQRST sequence at all. What kind of heart would generate such a rhythm? The muscles are quivering or "fibrillating" independently. The pumping action is nearly gone, which means emergency care. The patient needs immediate CPR—someone to do the pumping that the heart can't do. Cardio-pulmonary resuscitation is a combination of chest pressure and air pressure (hand and mouth) to restart the rhythm. CPR can be done on the street. A hospital applies a defibrillator, which shocks the heart back to life. It depolarizes *all* the heart cells, so the timing can be reset. Then the charge spreads normally from SA node to atria to AV node to ventricles.

This discussion has not used all twelve graphs to locate the problem. That needs *vectors*. Look ahead at Section 11.1 for the heart vector, and especially at Section 11.2 for its *twelve projections*. Those readings distinguish between atrium and ventricle, left and right, forward and back. This information is of vital importance in the event of a heart attack. A "heart attack" is a *myocardial infarction* (MI).

An MI occurs when part of an artery to the heart is blocked (a coronary occlusion).

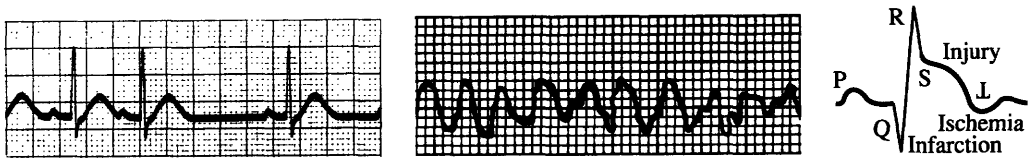


Fig. 3.10 Doubtful rhythm. Serious fibrillation. Signals of a heart attack.

An area is without blood supply—therefore without oxygen or glucose. Often the attack is in the thick left ventricle, which needs the most blood. The cells are first ischemic, then injured, and finally infarcted (dead). The classical ECG signals involve those three I's:

> I*schemia*: Reduced blood supply, upside-down T wave in the chest leads.
> I*njury*: An elevated segment between S and T means a recent attack.
> I*nfarction*: The Q wave, normally a tiny dip or absent, is as wide as a small square ($\frac{1}{25}$ second). It may occupy a third of the entire QRS complex.

The Q wave gives the diagnosis. You can find all three I's in Figure 3.10c.

It is absolutely amazing how much a good graph can do.

THE MECHANICS OF GRAPHS

From the meaning of graphs we descend to the mechanics. A formula is now given for $f(x)$. The problem is *to create the graph.* It would be too old-fashioned to evaluate $f(x)$ by hand and draw a curve through a dozen points. A computer has a much better idea of a parabola than an artist (who tends to make it asymptotic to a straight line). There are some things a computer knows, and other things an artist knows, and still others that you and I know—because we understand derivatives.

Our job is to apply calculus. We extract information from f' and f'' as well as f. Small movements in the graph may go unnoticed, but the important properties come through. Here are the main tests:

1. The sign of $f(x)$ (above or below axis: $f = 0$ at *crossing point*)
2. The sign of $f'(x)$ (increasing or decreasing: $f' = 0$ at *stationary point*)
3. The sign of $f''(x)$ (concave up or down: $f'' = 0$ at *inflection point*)
4. The behavior of $f(x)$ as $x \to \infty$ and $x \to -\infty$
5. The points at which $f(x) \to \infty$ or $f(x) \to -\infty$
6. Even or odd? Periodic? Jumps in f or f'? Endpoints? $f(0)$?

EXAMPLE 1 $\quad f(x) = \dfrac{x^2}{1 - x^2} \qquad f'(x) = \dfrac{2x}{(1 - x^2)^2} \qquad f''(x) = \dfrac{2 + 6x^2}{(1 - x^2)^3}$

The sign of $f(x)$ depends on $1 - x^2$. Thus $f(x) > 0$ in the inner interval where $x^2 < 1$. The graph bends upwards ($f''(x) > 0$) in that same interval. There are no inflection points, since f'' is never zero. The stationary point where f' vanishes is $x = 0$. We have a *local minimum* at $x = 0$.

The guidelines (or ***asymptotes***) meet the graph at infinity. For large x the important terms are x^2 and $-x^2$. Their ratio is $+x^2/-x^2 = -1$—which is the limit as $x \to \infty$ and $x \to -\infty$. ***The horizontal asymptote is the line*** $y = -1$.

The other infinities, where f blows up, occur when $1 - x^2$ is zero. That happens at $x = 1$ and $x = -1$. ***The vertical asymptotes are the lines*** $x = 1$ ***and*** $x = -1$. The graph in Figure 3.11a approaches those lines.

if $f(x) \to b$ as $x \to +\infty$ *or* $-\infty$, the line $y = b$ is a *horizontal asymptote*
if $f(x) \to +\infty$ or $-\infty$ as $x \to a$, the line $x = a$ is a *vertical asymptote*
if $f(x) - (mx + b) \to 0$ as $x \to +\infty$ or $-\infty$, the line $y = mx + b$ is a *sloping asymptote.*

Finally comes the vital fact that this function is ***even***: $f(x) = f(-x)$ because squaring x obliterates the sign. The graph is symmetric across the y axis.

To summarize the effect of dividing by $1 - x^2$: No effect near $x = 0$. Blowup at 1 and -1 from zero in the denominator. The function approaches -1 as $|x| \to \infty$.

EXAMPLE 2 $\quad f(x) = \dfrac{x^2}{x-1} \qquad f'(x) = \dfrac{x^2 - 2x}{(x-1)^2} \qquad f''(x) = \dfrac{2}{(x-1)^3}$

This example divides by $x - 1$. Therefore $x = 1$ is a vertical asymptote, where $f(x)$ becomes infinite. Vertical asymptotes come mostly from *zero denominators.*

Look beyond $x = 1$. Both $f(x)$ and $f''(x)$ are positive for $x > 1$. The slope is zero at $x = 2$. That must be a local minimum.

What happens as $x \to \infty$? Dividing x^2 by $x - 1$, the leading term is x. The function becomes large. It grows linearly—we expect a ***sloping asymptote***. To find it, do the division properly:

$$\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}. \tag{1}$$

The last term goes to zero. The function approaches $y = x + 1$ as the asymptote.

This function is not odd or even. Its graph is in Figure 3.11b. With ***zoom out*** you see the asymptotes. ***Zoom in*** for $f = 0$ or $f' = 0$ or $f'' = 0$.

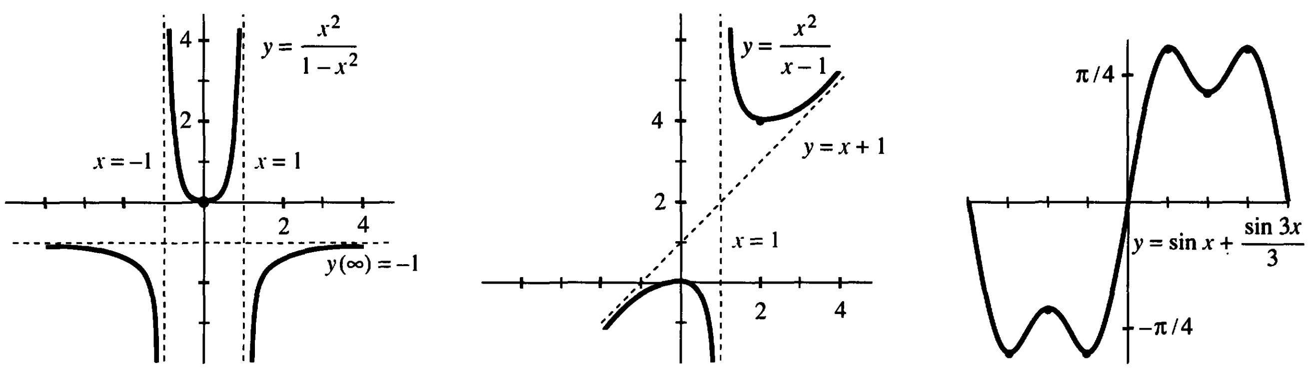


Fig. 3.11 The graphs of $x^2/(1 - x^2)$ and $x^2/(x - 1)$ and $\sin x + \frac{1}{3} \sin 3x$.

EXAMPLE 3 $\quad f(x) = \sin x + \frac{1}{3} \sin 3x \quad$ has the slope $\quad f'(x) = \cos x + \cos 3x$.

Above all these functions are ***periodic***. If x increases by 2π, nothing changes. The graphs from 2π to 4π are repetitions of the graphs from 0 to 2π. Thus $f(x + 2\pi) = f(x)$ and the period is 2π. Any interval of length 2π will show a complete picture, and Figure 3.11c picks the interval from $-\pi$ to π.

The second outstanding property is that f is ***odd***. The sine functions satisfy $f(-x) = -f(x)$. The graph is symmetric through the origin. By reflecting the right half through the origin, you get the left half. In contrast, the cosines in $f'(x)$ are even.

To find the zeros of $f(x)$ and $f'(x)$ and $f''(x)$, rewrite those functions as

$$f(x) = 2 \sin x - \tfrac{4}{3} \sin^3 x \quad f'(x) = -2 \cos x + 4 \cos^3 x \quad f''(x) = -10 \sin x + 12 \sin^3 x.$$

We changed $\sin 3x$ to $3\sin x - 4\sin^3 x$. For the derivatives use $\sin^2 x = 1 - \cos^2 x$. Now find the zeros—the *crossing points*, *stationary points*, and *inflection points*:

$f = 0 \quad 2\sin x = \frac{4}{3}\sin^3 x \;\Rightarrow\; \sin x = 0$ or $\sin^2 x = \frac{3}{2} \;\Rightarrow\; x = 0, \pm\pi$

$f' = 0 \quad 2\cos x = 4\cos^3 x \;\Rightarrow\; \cos x = 0$ or $\cos^2 x = \frac{1}{2} \;\Rightarrow\; x = \pm\pi/4, \pm\pi/2, \pm 3\pi/4$

$f'' = 0 \quad 5\sin x = 6\sin^3 x \;\Rightarrow\; \sin x = 0$ or $\sin^2 x = \frac{5}{6} \;\Rightarrow\; x = 0, \pm 66°, \pm 114°, \pm\pi$

That is more than enough information to sketch the graph. The stationary points $\pi/4, \pi/2, 3\pi/4$ are evenly spaced. At those points $f(x)$ is $\sqrt{8}/3$ (maximum), 2/3 (local minimum), $\sqrt{8}/3$ (maximum). Figure 3.11c shows the graph.

I would like to mention a beautiful continuation of this same pattern:

$f(x) = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots \qquad f'(x) = \cos x + \cos 3x + \cos 5x + \cdots$

If we stop after ten terms, $f(x)$ is extremely close to a ***step function***. If we don't stop, *the exact step function contains infinitely many sines.* It jumps from $-\pi/4$ to $+\pi/4$ as x goes past zero. More precisely it is a "***square wave***," because the graph jumps back down at π and repeats. The slope $\cos x + \cos 3x + \cdots$ also has period 2π. ***Infinitely many cosines add up to a delta function***! (The slope at the jump is an infinite spike.) These sums of sines and cosines are ***Fourier series***.

GRAPHS BY COMPUTERS AND CALCULATORS

We have come to a topic of prime importance. If you have ***graphing software*** for a computer, or if you have a ***graphing calculator***, you can bring calculus to life. A graph presents $y(x)$ in a new way—different from the formula. Information that is buried in the formula is clear on the graph. *But don't throw away $y(x)$ and dy/dx.* The derivative is far from obsolete.

These pages discuss how calculus and graphs go together. We work on a crucial problem of applied mathematics—to find where $y(x)$ reaches its minimum. There is no need to tell you a hundred applications. Begin with the formula. How do you find the point x^* where $y(x)$ is smallest?

First, draw the graph. That shows the main features. We should see (roughly) where x^* lies. There may be several minima, or possibly none. But what we see depends on a decision that is ours to make—*the range of x and y in the viewing window.*

If nothing is known about $y(x)$, the range is hard to choose. We can accept a default range, and zoom in or out. We can use the autoscaling program in Section 1.7. Somehow x^* can be observed on the screen. Then the problem is to compute it.

I would like to work with a specific example. We solved it by calculus—to find the best point x^* to enter an expressway. The speeds in Section 3.2 were 30 and 60. The length of the fast road will be $b = 6$. *The range of reasonable values for the entering point is* $0 \leqslant x \leqslant 6$. The distance to the road in Figure 3.12 is $a = 3$. We drive a distance $\sqrt{3^2 + x^2}$ at speed 30 and the remaining distance $6 - x$ at speed 60:

$$\textit{driving time} \quad y(x) = \frac{1}{30}\sqrt{3^2 + x^2} + \frac{1}{60}(6 - x). \tag{2}$$

This is the function to be minimized. Its graph is extremely flat.

It may seem unusual for the graph to be so level. On the contrary, it is common. *A flat graph is the whole point of $dy/dx = 0$.*

The graph near the minimum looks like $y = Cx^2$. It is a parabola sitting on a horizontal tangent. At a distance of $\Delta x = .01$, we only go up by $C(\Delta x)^2 = .0001\,C$. Unless C is a large number, this Δy can hardly be seen.

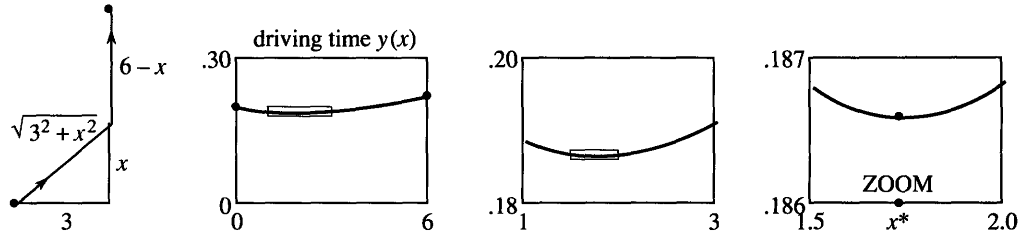


Fig. 3.12 Enter at x. The graph of driving time $y(x)$. Zoom boxes locate x^*.

The solution is to change scale. ***Zoom in on*** x^*. The tangent line stays flat, since dy/dx is still zero. But the bending from C is increased. Figure 3.12 shows the *zoom box* blown up into a new graph of $y(x)$.

A calculator has one or more ways to find x^*. With a TRACE mode, you direct a cursor along the graph. From the display of y values, read y_{max} and x^* to the nearest pixel. A zoom gives better accuracy, because it stretches the axes—each pixel represents a smaller Δx and Δy. The TI-81 stretches by 4 as default. Even better, let the whole process be graphical—*draw the actual* ZOOM BOX *on the screen.* Pick two opposite corners, press ENTER, and the box becomes the new viewing window (Figure 3.12).

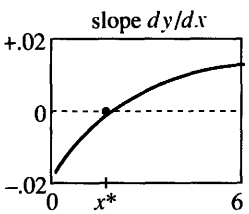
The first zoom narrows the search for x^*. It lies between $x = 1$ and $x = 3$. We build a new ZOOM BOX and zoom in again. Now $1.5 \leqslant x^* \leqslant 2$. Reasonable accuracy comes quickly. High accuracy does not come quickly. It takes time to create the box and execute the zoom.

Question 1 What happens as we zoom in, if all boxes are square (equal scaling)?
Answer The picture gets flatter and flatter. We are zooming in to the tangent line. Changing x to $X/4$ and y to $Y/4$, the parabola $y = x^2$ flattens to $Y = X^2/4$. To see any bending, *we must use a long thin zoom box.*

I want to change to a totally different approach. Suppose we have a formula for dy/dx. *That derivative was produced by an infinite zoom!* The limit of $\Delta y/\Delta x$ came by brainpower alone:

$$\frac{dy}{dx} = \frac{x}{30\sqrt{3^2+x^2}} - \frac{1}{60}. \qquad \text{Call this } f(x).$$

This function is zero at x^*. The computing problem is completely changed: Solve $f(x) = 0$. *It is easier to find a root of $f(x)$ than a minimum of $y(x)$.* The graph of $f(x)$ crosses the x axis. The graph of $y(x)$ goes flat—this is harder to pinpoint.



zero slope at minimum

Fig. 3.13

Take the model function $y = x^2$ for $|x| < .01$. The slope $f = 2x$ changes from $-.02$ to $+.02$. The value of x^2 moves only by .0001—its minimum point is hard to see.

To repeat: Minimization is easier with dy/dx. The screen shows an order of magnitude improvement, when we trace or zoom on $f(x) = 0$. In calculus, we have been taking the derivative for granted. It is natural to get blasé about $dy/dx = 0$. We forget how intelligent it is, to work with the slope instead of the function.

Question 2 How do you get another order of magnitude improvement?
Answer Use the next derivative! With a formula for df/dx, which is d^2y/dx^2, the convergence is even faster. In two steps the error goes from .01 to .0001 to .00000001. Another infinite zoom went into the formula for df/dx, and *Newton's method* takes account of it. Sections 3.6 and 3.7 study $f(x) = 0$.

The expressway example allows perfect accuracy. We can solve $dy/dx = 0$ by algebra. The equation simplifies to $60x = 30\sqrt{3^2 + x^2}$. Dividing by 30 and squaring yields $4x^2 = 3^2 + x^2$. Then $3x^2 = 3^2$. The exact solution is $x^* = \sqrt{3} = 1.73205\ldots$

A model like this is a benchmark, to test competing methods. It also displays what we never appreciated—the extreme flatness of the graph. The difference in driving time between entering at $x^* = \sqrt{3}$ and $x = 2$ is *one second.*

THE CENTERING TRANSFORM AND ZOOM TRANSFORM

For a photograph we do two things—point the right way and stand at the right distance. Then take the picture. Those steps are the same for a graph. First we pick the new center point. The graph is *shifted*, to move that point from (a, b) to $(0, 0)$. Then we decide how far the graph should reach. It fits in a rectangle, just like the photograph. *Rescaling* to x/c and y/d puts the desired section of the curve into the rectangle.

A good photographer does more (like an artist). The subjects are placed and the camera is focused. For good graphs those are necessary too. But an everyday calculator or computer or camera is built to operate without an artist—just aim and shoot. I want to explain how to aim at $y = f(x)$.

We are doing exactly what a calculator does, with one big difference. *It doesn't change coordinates. We do.* When $x = 1$, $y = -2$ moves to the center of the viewing window, the calculator still shows that point as $(1, -2)$. When the ***centering transform*** acts on $y + 2 = m(x - 1)$, those numbers disappear. This will be confusing unless x and y also change. *The new coordinates are $X = x - 1$ and $Y = y + 2$. Then the new equation is $Y = mX$.*

The main point (for humans) is to make the algebra simpler. The computer has no preference for $Y = mX$ over $y - y_0 = m(x - x_0)$. It accepts $2x^2 - 4x$ as easily as x^2. But we do prefer $Y = mX$ and $y = x^2$, partly because their graphs go through $(0, 0)$. Ever since zero was invented, mathematicians have liked that number best.

3F A ***centering transform*** shifts left by a and down by b:

$$X = x - a \text{ and } Y = y - b \text{ change } y = f(x) \text{ into } Y + b = f(X + a).$$

EXAMPLE 4 The parabola $y = 2x^2 - 4x$ has its minimum when $dy/dx = 4x - 4 = 0$. Thus $x = 1$ and $y = -2$. Move this bottom point to the center: $y = 2x^2 - 4x$ is

$$Y + 2 = 2(X - 1)^2 - 4(X - 1) \qquad \text{or} \qquad Y = 2X^2.$$

The new parabola $Y = 2X^2$ has its bottom at $(0, 0)$. It is the same curve, shifted across and up. The only simpler parabola is $y = x^2$. This final step is the job of the zoom.

Next comes scaling. We may want more detail (zoom in to see the tangent line). We may want a big picture (zoom out to check asymptotes). We might stretch one axis more than the other, if the picture looks like a pancake or a skyscraper.

3G A ***zoom transform*** scales the X and Y axes by c and d:

$$\mathbf{x} = cX \quad \text{and} \quad \mathbf{y} = dY \quad \text{change} \quad Y = F(X) \quad \text{to} \quad \mathbf{y} = dF(\mathbf{x}/c).$$

The new **x** and **y** are boldface letters, and the graph is rescaled. Often $c = d$.

EXAMPLE 5 Start with $Y = 2X^2$. Apply a square zoom with $c = d$. In the new $\mathbf{xy}$ coordinates, the equation is $\mathbf{y}/c = 2(\mathbf{x}/c)^2$. The number 2 disappears if $c = d = 2$. With the right centering and the right zoom, every parabola that opens upward is $\mathbf{y} = \mathbf{x}^2$.

Question 3 What happens to the derivatives (*slope and bending*) after a zoom?
Answer The slope (first derivative) is multiplied by d/c. Apply the chain rule to $\mathbf{y} = dF(\mathbf{x}/c)$. A square zoom has $d/c = 1$—*lines keep their slope*. The second derivative is multiplied by d/c^2, which changes the bending. A zoom out divides by small numbers $c = d$, so the big picture is more curved.

Combining the centering and zoom transforms, as we do in practice, gives $\mathbf{y}$ in terms of $\mathbf{x}$:

$$y = f(x) \quad \text{becomes} \quad Y = f(X + a) - b \quad \text{and then} \quad \mathbf{y} = d\left[f\left(\frac{\mathbf{x}}{c} + a\right) - b\right].$$

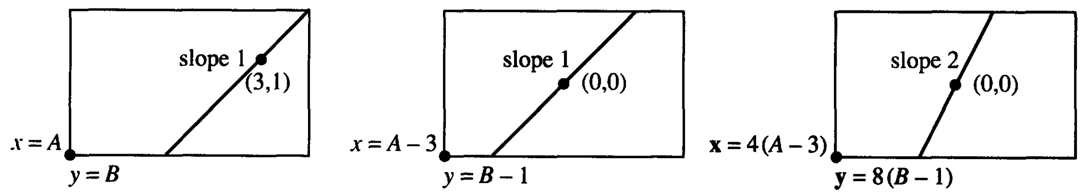


Fig. 3.14 Change of coordinates by centering and zoom. Calculators still show (x, y).

Question 4 Find x and y ranges after two transforms. Start between -1 and 1.
Answer The window after centering is $-1 \leqslant x - a \leqslant 1$ and $-1 \leqslant y - b \leqslant 1$. The window after zoom is $-1 \leqslant c(x - a) \leqslant 1$ and $-1 \leqslant d(y - b) \leqslant 1$. The point (1, 1) was originally in the corner. The point $(c^{-1} + a, d^{-1} + b)$ is now in the corner.

The numbers a, b, c, d are chosen to produce a simpler function (like $\mathbf{y} = \mathbf{x}^2$). Or else—this is important in applied mathematics—they are chosen to make $\mathbf{x}$ and $\mathbf{y}$ "dimensionless." An example is $y = \frac{1}{2}\cos 8t$. The frequency 8 has dimension 1/time. The amplitude $\frac{1}{2}$ is a distance. With $d = 2$ cm and $c = 8$ sec, the units are removed and $\mathbf{y} = \cos \mathbf{t}$.

May I mention one transform that *does* change the slope? It is a ***rotation***. The whole plane is turned. A photographer might use it—but normally people are supposed to be upright. You use rotation when you turn a map or straighten a picture. In the next section, an unrecognizable hyperbola is turned into $Y = 1/X$.

3.4 EXERCISES

Read-through questions

The position, slope, and bending of $y = f(x)$ are decided by __a__, __b__ and __c__. If $|f(x)| \to \infty$ as $x \to a$, the line $x = a$ is a vertical __d__. If $f(x) \to b$ for large x, then $y = b$ is a __e__. If $f(x) - mx \to b$ for large x, then $y = mx + b$ is a __f__. The asymptotes of $y = x^2/(x^2 - 4)$ are __g__. This function is even because $y(-x) =$ __h__. The function $\sin kx$ has period __i__.

Near a point where $dy/dx = 0$, the graph is extremely __j__. For the model $y = Cx^2$, $x = .1$ gives $y =$ __k__. A box around the graph looks long and __l__. We __m__ in to that box for another digit of x^*. But solving $dy/dx = 0$ is more accurate, because its graph __n__ the x axis. The slope of dy/dx is __o__. Each derivative is like an __p__ zoom.

To move (a, b) to (0, 0), shift the variables to $X =$ __q__ and $Y =$ __r__. This __s__ transform changes $y = f(x)$ to $Y =$ __t__. The original slope at (a, b) equals the new slope at __u__. To stretch the axes by c and d, set $\mathbf{x} = cX$ and __v__. The __w__ transform changes $Y = F(X)$ to $\mathbf{y} =$ __x__. Slopes are multiplied by __y__. Second derivatives are multiplied by __z__.

1 Find the pulse rate when heartbeats are $\frac{1}{2}$ second or two dark lines or x seconds apart.

2 Another way to compute the heart rate uses marks for 6-second intervals. Doctors count the cycles in an interval.
(a) How many dark lines in 6 seconds?
(b) With 8 beats per interval, find the rate.
(c) Rule: Heart rate = cycles per interval times ________.

Which functions in 3–18 are even or odd or periodic? Find all asymptotes: $y=b$ or $x=a$ or $y=mx+b$. Draw roughly by hand or smoothly by computer.

3 $f(x)=x-(9/x)$

4 $f(x)=x^n$ (any integer n)

5 $f(x)=\dfrac{1}{1-x^2}$

6 $f(x)=\dfrac{x^3}{4-x^2}$

7 $f(x)=\dfrac{x^2+3}{x^2+1}$

8 $f(x)=\dfrac{x^2+3}{x+1}$

9 $f(x)=(\sin x)(\sin 2x)$

10 $f(x)=\cos x+\cos 3x+\cos 5x$

11 $f(x)=\dfrac{x\sin x}{x^2-1}$

12 $f(x)=\dfrac{x}{\sin x}$

13 $f(x)=\dfrac{1}{x^3+x^2}$

14 $f(x)=\dfrac{1}{x-1}-2x$

15 $f(x)=\dfrac{x^3+1}{x^3-1}$

16 $f(x)=\dfrac{\sin x+\cos x}{\sin x-\cos x}$

17 $f(x)=x-\sin x$

18 $f(x)=(1/x)-\sqrt{x}$

In 19–24 construct $f(x)$ with exactly these asymptotes.

19 $x=1$ and $y=2$

20 $x=1$, $x=2$, $y=0$

21 $y=x$ and $x=4$

22 $y=2x+3$ and $x=0$

23 $y=x\ (x\to\infty)$, $y=-x(x\to-\infty)$

24 $x=1$, $x=3$, $y=x$

25 For $P(x)/Q(x)$ to have $y=2$ as asymptote, the polynomials P and Q must be ________.

26 For $P(x)/Q(x)$ to have a sloping asymptote, the degrees of P and Q must be ________.

27 For $P(x)/Q(x)$ to have the asymptote $y=0$, the degrees of P and Q must ________. The graph of $x^4/(1+x^2)$ has what asymptotes?

28 Both $1/(x-1)$ and $1/(x-1)^2$ have $x=1$ *and* $y=0$ as asymptotes. The most obvious difference in the graphs is ________.

29 If $f'(x)$ has asymptotes $x=1$ and $y=3$ then $f(x)$ has asymptotes ________.

30 **True** (with reason) or **false** (with example).
(a) Every ratio of polynomials has asymptotes
(b) If $f(x)$ is even so is $f''(x)$
(c) If $f''(x)$ is even so is $f(x)$
(d) Between vertical asymptotes, $f'(x)$ touches zero.

31 Construct an $f(x)$ that is "even around $x=3$."

32 Construct $g(x)$ to be "odd around $x=\pi$."

Create graphs of 33–38 on a computer or calculator.

33 $y(x)=(1+1/x)^x,\ -3\leqslant x\leqslant 3$

34 $y(x)=x^{1/x},\ 0.1\leqslant x\leqslant 2$

35 $y(x)=\sin(x/3)+\sin(x/5)$

36 $y(x)=(2-x)/(2+x),\ -3\leqslant x\leqslant 3$

37 $y(x)=2x^3+3x^2-12x+5$ on $[-3,3]$ and $[2.9,3.1]$

38 $100[\sin(x+.1)-2\sin x+\sin(x-.1)]$

In 39–40 show the asymptotes on large-scale computer graphs.

39 (a) $y=\dfrac{x^3+8x-15}{x^2-2}$ (b) $y=\dfrac{x^4-6x^3+1}{2x^4+x^2}$

40 (a) $y=\dfrac{x^2-2}{x^3+8x-15}$ (b) $y=\dfrac{x^2-x+2}{x^2-2x+1}$

41 Rescale $y=\sin x$ so X is in degrees, not radians, and Y changes from meters to centimeters.

Problems 42–46 minimize the driving time $y(x)$ in the text. Some questions may not fit your software.

42 Trace along the graph of $y(x)$ to estimate x^*. Choose an xy range or use the default.

43 Zoom in by $c=d=4$. How many zooms until you reach $x^*=1.73205$ or 1.7320508?

44 Ask your program for the minimum of $y(x)$ and the solution of $dy/dx=0$. Same answer?

45 What are the scaling factors c and d for the two zooms in Figure 3.12? They give the stretching of the x and y axes.

46 Show that $dy/dx=-1/60$ and $d^2y/dx^2=1/90$ at $x=0$. Linear approximation gives $dy/dx\approx-1/60+x/90$. So the slope is zero near $x=$ ________. This is Newton's method, using the next derivative.

Change the function to $y(x)=\sqrt{15+x^2}/30+(10-x)/60$.

47 Find x^* using only the graph of $y(x)$.

48 Find x^* using also the graph of dy/dx.

49 What are the xy and XY and $\mathbf{xy}$ equations for the line in Figure 3.14?

50 Define $f_n(x) = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots$ (n terms). Graph f_5 and f_{10} from $-\pi$ to π. Zoom in and describe the *Gibbs phenomenon* at $x = 0$.

On the graphs of 51–56, zoom in to all maxima and minima (3 significant digits). *Estimate inflection points.*

51 $y = 2x^5 - 16x^4 + 5x^3 - 37x^2 + 21x + 683$

52 $y = x^5 - x^4 - \sqrt{3x+1} - 2$

53 $y = x(x-1)(x-2)(x-4)$

54 $y = 7\sin 2x + 5\cos 3x$

55 $y = (x^3 - 2x + 1)/(x^4 - 3x^2 - 15)$, $-3 \leqslant x \leqslant 5$

56 $y = x\sin(1/x)$, $0.1 \leqslant x \leqslant 1$

57 A 10-digit computer shows $y = 0$ and $dy/dx = .01$ at $x^* = 1$. This root should be correct to about (8 digits) (10 digits) (12 digits). *Hint*: Suppose $y = .01\,(x - 1 + \text{error})$. What errors don't show in 10 digits of y?

58 Which is harder to compute accurately: Maximum point or inflection point? First derivative or second derivative?

3.5 Parabolas, Ellipses, and Hyperbolas

Here is a list of the most important curves in mathematics, so you can tell what is coming. It is not easy to rank the top four:

1. ***straight lines***
2. ***sines and cosines*** (oscillation)
3. ***exponentials*** (growth and decay)
4. ***parabolas, ellipses, and hyperbolas*** (using $1, x, y, x^2, xy, y^2$).

The curves that I wrote last, the Greeks would have written first. It is so natural to go from linear equations to quadratic equations. Straight lines use $1, x, y$. Second degree curves include x^2, xy, y^2. If we go on to x^3 and y^3, the mathematics gets complicated. We now study equations of second degree, and the curves they produce.

It is quite important to see both the ***equations*** and the ***curves***. This section connects two great parts of mathematics—*analysis* of the equation and *geometry* of the curve. Together they produce "***analytic geometry***." You already know about functions and graphs. Even more basic: Numbers correspond to points. We speak about "*the point* (5, 2)." Euclid might not have understood.

Where Euclid drew a 45° line through the origin, Descartes wrote down $y = x$. Analytic geometry has become central to mathematics—we now look at one part of it.

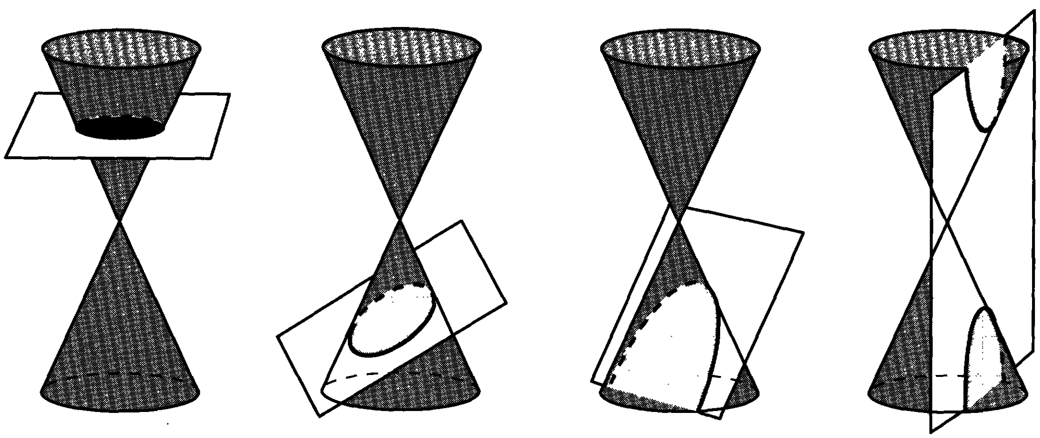


Fig. 3.15 The cutting plane gets steeper: circle to ellipse to parabola to hyperbola.