CHAPTER 4

Derivatives by the Chain Rule

4.1 The Chain Rule

You remember that the derivative of $f(x)g(x)$ is not $(df/dx)(dg/dx)$. The derivative of $\sin x$ times x^2 is not $\cos x$ times $2x$. The product rule gave two terms, not one term. But there is another way of combining the sine function f and the squaring function g into a single function. The derivative of that new function does involve the cosine times $2x$ (but with a certain twist). We will first explain the new function, and then find the "***chain rule***" for its derivative.

May I say here that the chain rule is important. It is easy to learn, and you will use it often. I see it as the third basic way to find derivatives of new functions from derivatives of old functions. (So far the old functions are x^n, $\sin x$, and $\cos x$. Still ahead are e^x and $\log x$.) When f and g are added and multiplied, derivatives come from the *sum rule* and *product rule*. This section combines f and g in a third way.

The new function is $\sin(x^2)$—***the sine of*** x^2. It is created out of the two original functions: if $x = 3$ then $x^2 = 9$ and $\sin(x^2) = \sin 9$. There is a "chain" of functions, combining $\sin x$ and x^2 into the composite function $\sin(x^2)$. You start with x, *then find* $g(x)$, ***then find*** $f(g(x))$:

The squaring function gives $y = x^2$. This is $g(x)$.
The sine function produces $z = \sin y = \sin(x^2)$. This is $f(g(x))$.

The "***inside function***" $g(x)$ gives y. *This is the input to the* "***outside function***" $f(y)$. That is called ***composition.*** It starts with x and ends with z. The composite function is sometimes written $f \circ g$ (the circle shows the difference from an ordinary product fg). More often you will see $f(g(x))$:

$$z(x) = f \circ g\ (x) = f(g(x)). \tag{1}$$

Other examples are $\cos 2x$ and $(2x)^3$, with $g = 2x$. *On a calculator you input x, then push the "g" button, then push the "f" button:*

***From** x **compute** $y = g(x)$ $\qquad$ **From** y **compute** $z = f(y)$.*

There is not a button for every function! But the squaring function and sine function are on most calculators, and they are used ***in that order***. Figure 4.1a shows how squaring will stretch and squeeze the sine function.

That graph of $\sin x^2$ is a crazy FM signal (the Frequency is Modulated). The wave goes up and down like $\sin x$, but not at the same places. Changing to $\sin g(x)$ moves the peaks left and right. Compare with a product $g(x) \sin x$, which is an AM signal (the Amplitude is Modulated).

Remark $f(g(x))$ is usually different from $g(f(x))$. ***The order of f and g is usually important***. For $f(x) = \sin x$ and $g(x) = x^2$, the chain in the opposite order $g(f(x))$ gives something different:

First apply the sine function: $y = \sin x$
Then apply the squaring function: $z = (\sin x)^2$.

That result is often written $\sin^2 x$, to save on parentheses. It is never written $\sin x^2$, which is totally different. Compare them in Figure 4.1.

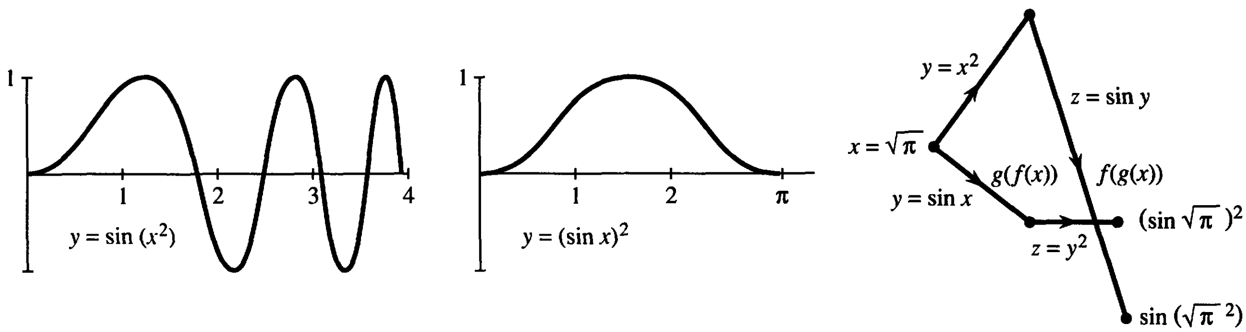


Fig. 4.1 $f(g(x))$ is different from $g(f(x))$. Apply g then f, or f then g.

EXAMPLE 1 The composite function $f \circ g$ can be deceptive. If $g(x) = x^3$ and $f(y) = y^4$, how does $f(g(x))$ differ from the ordinary product $f(x)g(x)$? The ordinary product is x^7. The chain starts with $y = x^3$, and then $z = y^4 = x^{12}$. The composition of x^3 and y^4 gives $f(g(x)) = x^{12}$.

EXAMPLE 2 In Newton's method, $F(x)$ is composed with itself. This is ***iteration***. Every output x_n is fed back as input, to find $x_{n+1} = F(x_n)$. The example $F(x) = \frac{1}{2}x + 4$ has $F(F(x)) = \frac{1}{2}(\frac{1}{2}x + 4) + 4$. That produces $z = \frac{1}{4}x + 6$.

The derivative of $F(x)$ is $\frac{1}{2}$. The derivative of $z = F(F(x))$ is $\frac{1}{4}$, which is $\frac{1}{2}$ times $\frac{1}{2}$. ***We multiply derivatives***. This is a special case of the chain rule.

An extremely special case is $f(x) = x$ and $g(x) = x$. The ordinary product is x^2. The chain $f(g(x))$ produces only x! The output from the "*identity function*" is $g(x) = x$.† When the second identity function operates on x it produces x again. The derivative is 1 times 1. I can give more composite functions in a table:

$y = g(x)$	$z = f(y)$	$z = f(g(x))$
$x^2 - 1$	$\sqrt{y}$	$\sqrt{x^2 - 1}$
$\cos x$	y^3	$(\cos x)^3$
2^x	2^y	2^{2^x}
$x + 5$	$y - 5$	x

The last one adds 5 to get y. Then it subtracts 5 to reach z. So $z = x$. Here output

†A calculator has no button for the identity function. It wouldn't do anything.

equals input: $f(g(x)) = x$. These "***inverse functions***" are in Section 4.3. The other examples create new functions $z(x)$ and we want their derivatives.

THE DERIVATIVE OF $f(g(x))$

What is the derivative of $z = \sin x^2$? It is the limit of $\Delta z/\Delta x$. Therefore we look at a nearby point $x + \Delta x$. That change in x produces a change in $y = x^2$—which moves to $y + \Delta y = (x + \Delta x)^2$. From this change in y, there is a change in $z = f(y)$. It is a "domino effect," in which each changed input yields a changed output: Δx ***produces*** Δy ***produces*** Δz. We have to connect the final Δz to the original Δx.

The key is to write $\Delta z/\Delta x$ ***as*** $\Delta z/\Delta y$ ***times*** $\Delta y/\Delta x$. Then let Δx approach zero. In the limit, dz/dx is given by the "chain rule":

$$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y}\frac{\Delta y}{\Delta x} \ \textit{becomes the chain rule}\ \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}. \qquad (2)$$

As Δx goes to zero, the ratio $\Delta y/\Delta x$ approaches dy/dx. Therefore Δy must be going to zero, and $\Delta z/\Delta y$ approaches dz/dy. The limit of a product is the product of the separate limits (end of quick proof). *We multiply derivatives*:

4A ***Chain Rule*** Suppose $g(x)$ has a derivative at x and $f(y)$ has a derivative at $y = g(x)$. Then the derivative of $z = f(g(x))$ is

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = f'(g(x))\,g'(x). \qquad (3)$$

The slope at x is df/dy (at y) times dg/dx (at x).

Caution The chain rule does *not* say that the derivative of $\sin x^2$ is $(\cos x)(2x)$. True, $\cos y$ is the derivative of $\sin y$. The point is that $\cos y$ *must be evaluated at* y (not at x). We do not want df/dx at x, we want df/dy at $y = x^2$:

$$\textit{The derivative of}\ \sin x^2\ \textit{is}\ (\cos x^2)\ \textit{times}\ (2x). \qquad (4)$$

EXAMPLE 3 If $z = (\sin x)^2$ then $dz/dx = (2\sin x)(\cos x)$. Here $y = \sin x$ is *inside*.

In this order, $z = y^2$ leads to $dz/dy = 2y$. *It does not lead to* $2x$. The inside function $\sin x$ produces $dy/dx = \cos x$. The answer is $2y \cos x$. We have not yet found the function whose derivative is $2x \cos x$.

EXAMPLE 4 The derivative of $z = \sin 3x$ is $\dfrac{dz}{dx} = \dfrac{dz}{dy}\dfrac{dy}{dx} = 3\cos 3x$.

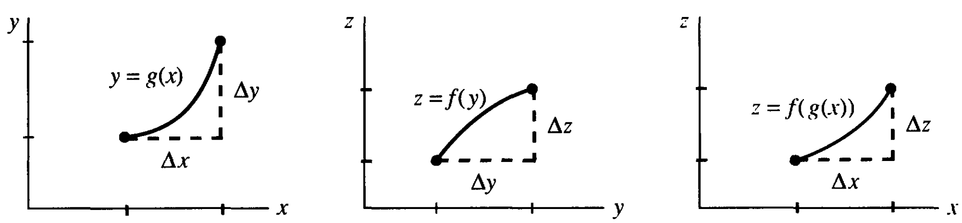


Fig. 4.2 The chain rule: $\dfrac{\Delta z}{\Delta x} = \dfrac{\Delta z}{\Delta y}\dfrac{\Delta y}{\Delta x}$ approaches $\dfrac{dz}{dx} = \dfrac{dz}{dy}\dfrac{dy}{dx}$.

The outside function is $z = \sin y$. The inside function is $y = 3x$. Then $dz/dy = \cos y$—this is $\cos 3x$, not $\cos x$. Remember the other factor $dy/dx = 3$.

I can explain that factor 3, especially if x is switched to t. The distance is $z = \sin 3t$. That oscillates like $\sin t$ except *three times as fast*. The speeded-up function $\sin 3t$ completes a wave at time $2\pi/3$ (instead of 2π). Naturally the velocity contains the extra factor 3 from the chain rule.

EXAMPLE 5 Let $z = f(y) = y^n$. Find the derivative of $f(g(x)) = [g(x)]^n$.

In this case dz/dy is ny^{n-1}. The chain rule multiplies by dy/dx:

$$\frac{dz}{dx} = ny^{n-1}\frac{dy}{dx} \quad \text{or} \quad \frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}\frac{dg}{dx}. \tag{5}$$

This is the ***power rule***! It was already discovered in Section 2.5. Square roots (when $n = 1/2$) are frequent and important. Suppose $y = x^2 - 1$:

$$\frac{d}{dx}\sqrt{x^2 - 1} = \frac{1}{2}(x^2 - 1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 - 1}}. \tag{6}$$

Question A Buick uses 1/20 of a gallon of gas per mile. You drive at 60 miles per hour. How many gallons per hour?

Answer (***Gallons/hour***) = (***gallons/mile***)(***miles/hour***). The chain rule is $(dy/dt) = (dy/dx)(dx/dt)$. The answer is $(1/20)(60) = 3$ gallons/hour.

Proof of the chain rule The discussion above was correctly based on

$$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y}\frac{\Delta y}{\Delta x} \quad \text{and} \quad \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}. \tag{7}$$

It was here, over the chain rule, that the "battle of notation" was won by Leibniz. His notation practically tells you what to do: Take the limit of each term. (I have to mention that when Δx is approaching zero, it is theoretically possible that Δy might *hit* zero. If that happens, $\Delta z/\Delta y$ becomes 0/0. We have to assign it the correct meaning, which is dz/dy.) As $\Delta x \to 0$,

$$\frac{\Delta y}{\Delta x} \to g'(x) \quad \text{and} \quad \frac{\Delta z}{\Delta y} \to f'(y) = f'(g(x)).$$

Then $\Delta z/\Delta x$ approaches $f'(y)$ times $g'(x)$, which is the chain rule $(dz/dy)(dy/dx)$. In the table below, the derivative of $(\sin x)^3$ is $3(\sin x)^2 \cos x$. That extra factor $\cos x$ is easy to forget. It is even easier to forget the -1 in the last example.

$z = (x^3 + 1)^5$	$dz/dx = 5(x^3 + 1)^4$	times $3x^2$
$z = (\sin x)^3$	$dz/dx = 3 \sin^2 x$	times $\cos x$
$z = (1 - x)^2$	$dz/dx = 2(1 - x)$	times -1

Important All kinds of letters are used for the chain rule. We named the output z. Very often it is called y, and the inside function is called u:

The derivative of $y = \sin u(x)$ ***is*** $\dfrac{dy}{dx} = \cos u \dfrac{du}{dx}$.

Examples with du/dx are extremely common. I have to ask you to accept whatever letters may come. What never changes is the key idea—*derivative of outside function times derivative of inside function.*

EXAMPLE 6 The chain rule is barely needed for $\sin(x-1)$. Strictly speaking the inside function is $u = x - 1$. Then du/dx is just 1 (not -1). ***If*** $y = \sin(x-1)$ ***then*** $dy/dx = \cos(x-1)$. The graph is shifted and the slope shifts too.

Notice especially: The cosine is computed at $x - 1$ and not at the unshifted x.

RECOGNIZING $f(y)$ AND $g(x)$

A big part of the chain rule is *recognizing the chain.* The table started with $(x^3+1)^5$. You look at it for a second. Then you see it as u^5. The inside function is $u = x^3 + 1$. With practice this decomposition (the opposite of composition) gets easy:

$$\cos(2x+1) \text{ is } \cos u \qquad \sqrt{1+\sin t} \text{ is } \sqrt{u} \qquad x \sin x \text{ is ... (product rule!)}$$

In calculations, the careful way is to write down all the functions:

$$z = \cos u \quad u = 2x + 1 \quad dz/dx = (-\sin u)(2) = -2\sin(2x+1).$$

The quick way is to keep in your mind "the derivative of what's inside." The slope of $\cos(2x+1)$ is $-\sin(2x+1)$, *times* 2 *from the chain rule.* The derivative of $2x+1$ is remembered—without z or u or f or g.

EXAMPLE 7 $\sin\sqrt{1-x}$ is a chain of $z = \sin y$, $y = \sqrt{u}$, $u = 1 - x$ (*three functions*).

With that triple chain you will have the hang of the chain rule:

$$\text{The derivative of } \sin\sqrt{1-x} \text{ is } (\cos\sqrt{1-x})\left(\frac{1}{2\sqrt{1-x}}\right)(-1).$$

This is $(dz/dy)(dy/du)(du/dx)$. Evaluate them at the right places y, u, x.

Finally there is the question of *second derivatives.* The chain rule gives dz/dx as a product, so d^2z/dx^2 needs the product rule:

$$\underset{}{\frac{dz}{dx}} = \underbrace{\frac{dz}{dy}}_{u}\underbrace{\frac{dy}{dx}}_{v} \quad \text{leads to} \quad \frac{d^2z}{dx^2} = \underbrace{\frac{dz}{dy}}_{u}\underbrace{\frac{d^2y}{dx^2}}_{v'} + \underbrace{\frac{d}{dx}\left(\frac{dz}{dy}\right)}_{u'}\underbrace{\frac{dy}{dx}}_{v}. \tag{8}$$

That last term needs the chain rule again. It becomes d^2z/dy^2 times $(dy/dx)^2$.

EXAMPLE 8 The derivative of $\sin x^2$ is $2x\cos x^2$. Then the product rule gives $d^2z/dx^2 = 2\cos x^2 - 4x^2 \sin x^2$. In this case $y'' = 2$ and $(y')^2 = 4x^2$.

4.1 EXERCISES

Read-through questions

$z = f(g(x))$ comes from $z = f(y)$ and $y =$ __a__. At $x = 2$, the chain $(x^2-1)^3$ equals __b__. Its inside function is $y =$ __c__, its outside function is $z =$ __d__. Then dz/dx equals __e__. The first factor is evaluated at $y =$ __f__ (not at $y = x$). For $z = \sin(x^4 - 1)$ the derivative is __g__. The triple chain $z = \cos(x+1)^2$ has a shift and a __h__ and a cosine. Then $dz/dx =$ __i__.

The proof of the chain rule begins with $\Delta z/\Delta x =$ (__j__)(__k__) and ends with __l__. Changing letters, $y = \cos u(x)$ has $dy/dx =$ __m__. The power rule for $y = [u(x)]^n$ is the chain rule $dy/dx =$ __n__. The slope of $5g(x)$ is __o__ and the slope of $g(5x)$ is __p__. When $f =$ cosine and $g =$ sine and $x = 0$, the numbers $f(g(x))$ and $g(f(x))$ and $f(x)g(x)$ are __q__.

In 1–10 identify $f(y)$ and $g(x)$. From their derivatives find $\dfrac{dz}{dx}$.

1 $z = (x^2-3)^3$

2 $z = (x^3-3)^2$

3 $z = \cos(x^3)$

4 $z = \tan 2x$

5 $z = \sqrt{\sin x}$

6 $z = \sin\sqrt{x}$

7 $z = \tan(1/x) + 1/\tan x$ **8** $z = \sin(\cos x)$

9 $z = \cos(x^2 + x + 1)$ **10** $z = \sqrt{x^2}$

In 11–16 write down dz/dx. Don't write down f and g.

11 $z = \sin(17x)$ **12** $z = \tan(x + 1)$

13 $z = \cos(\cos x)$ **14** $z = (x^2)^{3/2}$

15 $z = x^2 \sin x$ **16** $z = (9x + 4)^{3/2}$

Problems 17–22 involve three functions $z(y)$, $y(u)$, and $u(x)$. Find dz/dx from $(dz/dy)(dy/du)(du/dx)$.

17 $z = \sin \sqrt{x + 1}$ **18** $z = \sqrt{\sin(x + 1)}$

19 $z = \sqrt{1 + \sin x}$ **20** $z = \sin(\sqrt{x} + 1)$

21 $z = \sin(1/\sin x)$ **22** $z = (\sin x^2)^2$

In 23–26 find dz/dx by the chain rule and also by rewriting z.

23 $z = ((x^2)^2)^2$ **24** $z = (3x)^3$

25 $z = (x + 1)^2 + \sin(x + \pi)$ **26** $z = \sqrt{1 - \cos^2 x}$

27 If $f(x) = x^2 + 1$ what is $f(f(x))$? If $U(x)$ is the unit step function (from 0 to 1 at $x = 0$) draw the graphs of $\sin U(x)$ and $U(\sin x)$. If $R(x)$ is the *ramp function* $\frac{1}{2}(x + |x|)$, draw the graphs of $R(x)$ and $R(\sin x)$.

28 (Recommended) If $g(x) = x^3$ find $f(y)$ so that $f(g(x)) = x^3 + 1$. Then find $h(y)$ so that $h(g(x)) = x$. Then find $k(y)$ so that $k(g(x)) = 1$.

29 If $f(y) = y - 2$ find $g(x)$ so that $f(g(x)) = x$. Then find $h(x)$ so that $f(h(x)) = x^2$. Then find $k(x)$ so that $f(k(x)) = 1$.

30 Find two different pairs $f(y)$, $g(x)$ so that $f(g(x)) = \sqrt{1 - x^2}$.

31 The derivative of $f(f(x))$ is ________. Is it $(df/dx)^2$? Test your formula on $f(x) = 1/x$.

32 If $f(3) = 3$ and $g(3) = 5$ and $f'(3) = 2$ and $g'(3) = 4$, find the derivative at $x = 3$ if possible for

(a) $f(x)g(x)$ (b) $f(g(x))$ (c) $g(f(x))$ (d) $f(f(x))$

33 For $F(x) = \frac{1}{2}x + 8$, show how iteration gives $F(F(x)) = \frac{1}{4}x + 12$. Find $F(F(F(x)))$—also called $F^{(3)}(x)$. The derivative of $F^{(4)}(x)$ is ________.

34 In Problem 33 the limit of $F^{(n)}(x)$ is a constant $C =$ ________. From any start (try $x = 0$) the iterations $x_{n+1} = F(x_n)$ converge to C.

35 Suppose $g(x) = 3x + 1$ and $f(y) = \frac{1}{3}(y - 1)$. Then $f(g(x)) =$ ________ and $g(f(y)) =$ ________. These are *inverse functions.*

36 Suppose $g(x)$ is continuous at $x = 4$, say $g(4) = 7$. Suppose $f(y)$ is continuous at $y = 7$, say $f(7) = 9$. Then $f(g(x))$ is continuous at $x = 4$ and $f(g(4)) = 9$.

Proof ε is given. Because ________ is continuous, there is a δ such that $|f(g(x)) - 9| < \varepsilon$ whenever $|g(x) - 7| < \delta$. Then because ________ is continuous, there is a θ such that $|g(x) - 7| < \delta$ whenever $|x - 4| < \theta$. Conclusion: *If* $|x - 4| < \theta$ *then* ________. This shows that $f(g(x))$ approaches $f(g(4))$.

37 Only six functions can be constructed by compositions (in any sequence) of $g(x) = 1 - x$ and $f(x) = 1/x$. Starting with g and f, find the other four.

38 If $g(x) = 1 - x$ then $g(g(x)) = 1 - (1 - x) = x$. If $g(x) = 1/x$ then $g(g(x)) = 1/(1/x) = x$. Draw graphs of those g's and explain from the graphs why $g(g(x)) = x$. Find two more g's with this special property.

39 Construct functions so that $f(g(x))$ is always zero, but $f(y)$ is not always zero.

40 True or false

(a) If $f(x) = f(-x)$ then $f'(x) = f'(-x)$.

(b) The derivative of the identity function is zero.

(c) The derivative of $f(1/x)$ is $-1/(f(x))^2$.

(d) The derivative of $f(1 + x)$ is $f'(1 + x)$.

(e) The second derivative of $f(g(x))$ is $f''(g(x))g''(x)$.

41 On the same graph draw the parabola $y = x^2$ and the curve $z = \sin y$ (keep y upwards, with x and z across). Starting at $x = 3$ find your way to $z = \sin 9$.

42 On the same graph draw $y = \sin x$ and $z = y^2$ (y upwards for both). Starting at $x = \pi/4$ find $z = (\sin x)^2$ on the graph.

43 Find the second derivative of

(a) $\sin(x^2 + 1)$ (b) $\sqrt{x^2 - 1}$ (c) $\cos \sqrt{x}$

44 Explain why $\dfrac{d}{dx}\left(\dfrac{dz}{dy}\right) = \left(\dfrac{d^2z}{dy^2}\right)\left(\dfrac{dy}{dx}\right)$ in equation (8). Check this when $z = y^2$, $y = x^3$.

Final practice with the chain rule and other rules (and other letters!). Find the x or t derivative of z or y.

45 $z = f(u(t))$ **46** $z = u^3$, $u = x^3$

47 $y = \sin u(x) \cos u(x)$ **48** $y = \sqrt{u(t)}$

49 $y = x^2 u(x)$ **50** $y = f(x^2) + (f(x))^2$

51 $z = \sqrt{1 - u}$, $u = \sqrt{1 - x}$ **52** $z = 1/u^n(t)$

53 $z = f(u)$, $u = v^2$, $v = \sqrt{t}$ **54** $y = u$, $u = x$, $x = 1/t$

55 If $f = x^4$ and $g = x^3$ then $f' = 4x^3$ and $g' = 3x^2$. The chain rule multiplies derivatives to get $12x^5$. But $f(g(x)) = x^{12}$ and its derivative is not $12x^5$. Where is the flaw?

56 The derivative of $y = \sin(\sin x)$ is $dy/dx =$

$\cos(\cos x)$ $\sin(\cos x)\cos x$ $\cos(\sin x)\cos x$ $\cos(\cos x)\cos x$.

57 (a) A book has 400 words per page. There are 9 pages per section. So there are ________ words per section.

(b) You read 200 words per minute. So you read ________ pages per minute. How many minutes per section?

58 (a) You walk in a train at 3 miles per hour. The train moves at 50 miles per hour. Your ground speed is ________ miles per hour.

(b) You walk in a train at 3 miles per hour. The train is shown on TV (1 mile train = 20 inches on TV screen). Your speed across the screen is ________ inches per hour.

59 Coke costs 1/3 dollar per bottle. The buyer gets ________ bottles per dollar. If $dy/dx = 1/3$ then $dx/dy =$ ________.

60 (Computer) Graph $F(x) = \sin x$ and $G(x) = \sin(\sin x)$—not much difference. Do the same for $F'(x)$ and $G'(x)$. Then plot $F''(x)$ and $G''(x)$ to see where the difference shows up.

4.2 Implicit Differentiation and Related Rates

We start with the equations $xy = 2$ and $y^5 + xy = 3$. As x changes, these y's will change—to keep (x, y) on the curve. ***We want to know*** dy/dx ***at a typical point.*** For $xy = 2$ that is no trouble, but the slope of $y^5 + xy = 3$ requires a new idea.

In the first case, solve for $y = 2/x$ and take its derivative: $dy/dx = -2/x^2$. The curve is a hyperbola. At $x = 2$ the slope is $-2/4 = -1/2$.

The problem with $y^5 + xy = 3$ is that it can't be solved for y. Galois proved that there is no solution formula for fifth-degree equations.† ***The function*** $y(x)$ ***cannot be given explicitly***. All we have is the *implicit* definition of y, as a solution to $y^5 + xy = 3$. The point $x = 2$, $y = 1$ satisfies the equation and lies on the curve, but how to find dy/dx?

This section answers that question. It is a situation that often occurs. Equations like $\sin y + \sin x = 1$ or $y \sin y = x$ (maybe even $\sin y = x$) are difficult or impossible to solve directly for y. Nevertheless we can find dy/dx at any point.

The way out is ***implicit differentiation***. Work with the equation as it stands. ***Find the*** x ***derivative of every term in*** $y^5 + xy = 3$. That includes the constant term 3, whose derivative is zero.

EXAMPLE 1 The power rule for y^5 and the product rule for xy yield

$$5y^4 \frac{dy}{dx} + x\frac{dy}{dx} + y = 0. \tag{1}$$

Now substitute the typical point $x = 2$ and $y = 1$, and solve for dy/dx:

$$5\frac{dy}{dx} + 2\frac{dy}{dx} + 1 = 0 \quad \text{produces} \quad \frac{dy}{dx} = -\frac{1}{7}. \tag{2}$$

This is implicit differentiation (**ID**), and you see the idea: Include dy/dx from the chain rule, even if y is not known explicitly as a function of x.

EXAMPLE 2 $\sin y + \sin x = 1$ leads to $\cos y \dfrac{dy}{dx} + \cos x = 0$

EXAMPLE 3 $y \sin y = x$ leads to $y \cos y \dfrac{dy}{dx} + \sin y \dfrac{dy}{dx} = 1$

Knowing the slope makes it easier to draw the curve. We still need points (x, y) that satisfy the equation. Sometimes we can solve for x. Dividing $y^5 + xy = 3$ by y

†That was before he went to the famous duel, and met his end. Fourth-degree equations do have a solution formula, but it is practically never used.

gives $x = 3/y - y^4$. Now the derivative (the x derivative!) is

$$1 = \left(-\frac{3}{y^2} - 4y^3\right)\frac{dy}{dx} = -7\frac{dy}{dx} \text{ at } y = 1. \tag{3}$$

Again $dy/dx = -1/7$. All these examples confirm the main point of the section:

> **4B** (*Implicit differentiation*) An equation $F(x, y) = 0$ can be differentiated directly by the chain rule, without solving for y in terms of x.

The example $xy = 2$, done implicitly, gives $x\,dy/dx + y = 0$. The slope dy/dx is $-y/x$. That agrees with the explicit slope $-2/x^2$.

ID is explained better by examples than theory (maybe everything is). The essential theory can be boiled down to one idea: "***Go ahead and differentiate.***"

EXAMPLE 4 Find the tangent direction to the circle $x^2 + y^2 = 25$.

We can solve for $y = \pm\sqrt{25 - x^2}$, or operate directly on $x^2 + y^2 = 25$:

$$2x + 2y\frac{dy}{dx} = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{x}{y}. \tag{4}$$

Compare with the radius, which has slope y/x. The radius goes across x and up y. The tangent goes across $-y$ and up x. The slopes multiply to give $(-x/y)(y/x) = -1$.

To emphasize implicit differentiation, go on to the *second derivative*. The top of the circle is concave down, so d^2y/dx^2 is negative. Use the quotient rule on $-x/y$:

$$\frac{dy}{dx} = -\frac{x}{y} \quad \text{so} \quad \frac{d^2y}{dx^2} = -\frac{y\,dx/dx - x\,dy/dx}{y^2} = -\frac{y + (x^2/y)}{y^2} = -\frac{y^2 + x^2}{y^3}. \tag{5}$$

RELATED RATES

There is a group of problems that has never found a perfect place in calculus. They seem to fit here—as applications of the chain rule. The problem is to compute df/dt, but the odd thing is that *we are given another derivative dg/dt*. To find df/dt, we need a relation between f and g.

The chain rule is $df/dt = (df/dg)(dg/dt)$. Here the variable is t because that is typical in applications. From the rate of change of g we find *the rate of change of f*. This is the problem of ***related rates***, and examples will make the point.

EXAMPLE 5 The radius of a circle is growing by $dr/dt = 7$. How fast is the circumference growing? Remember that $C = 2\pi r$ (this relates C to r).

Solution
$$\frac{dC}{dt} = \frac{dC}{dr}\frac{dr}{dt} = (2\pi)(7) = 14\pi.$$

That is pretty basic, but its implications are amazing. Suppose you want to put a rope around the earth that any 7-footer can walk under. If the distance is 24,000 miles, what is the additional length of the rope? Answer: Only 14π feet.

More realistically, if two lanes on a circular track are separated by 5 feet, how much head start should the outside runner get? Only 10π feet. If your speed around a turn is 55 and the car in the next lane goes 56, who wins? See Problem 14.

Examples 6–8 *are from the* 1988 *Advanced Placement Exams* (copyright 1989 by the College Entrance Examination Board). Their questions are carefully prepared.

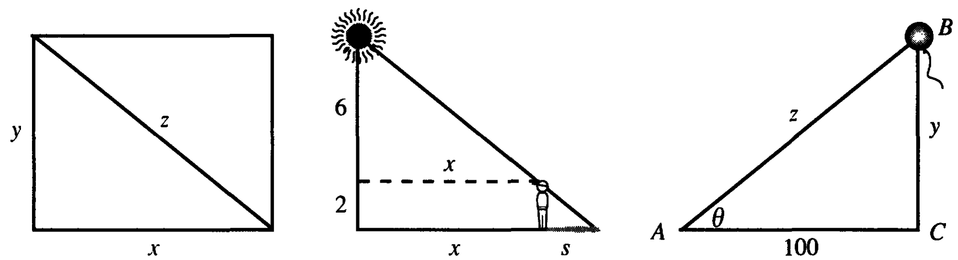


Fig. 4.3 Rectangle for Example 6, shadow for Example 7, balloon for Example 8.

EXAMPLE 6 The sides of the rectangle increase in such a way that $dz/dt = 1$ and $dx/dt = 3dy/dt$. At the instant when $x = 4$ and $y = 3$, what is the value of dx/dt?

Solution The key relation is $x^2 + y^2 = z^2$. Take its derivative (*implicitly*):

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt} \quad \text{produces} \quad 8\frac{dx}{dt} + 6\frac{dy}{dt} = 10.$$

We used all information, including $z = 5$, except for $dx/dt = 3dy/dt$. The term $6dy/dt$ equals $2dx/dt$, so we have $10dx/dt = 10$. Answer: $dx/dt = 1$.

EXAMPLE 7 A person 2 meters tall walks directly away from a streetlight that is 8 meters above the ground. If the person's shadow is lengthening at the rate of 4/9 meters per second, at what rate in meters per second is the person walking?

Solution Draw a figure! You must relate the shadow length s to the distance x from the streetlight. The problem gives $ds/dt = 4/9$ and asks for dx/dt:

$$\text{By similar triangles } \frac{x}{6} = \frac{s}{2} \quad \text{so} \quad \frac{dx}{dt} = \frac{6}{2}\frac{ds}{dt} = (3)\left(\frac{4}{9}\right) = \frac{4}{3}.$$

Note This problem was hard. I drew three figures before catching on to x and s. It is interesting that *we never knew x or s or the angle.*

EXAMPLE 8 An observer at point A is watching balloon B as it rises from point C. (*The figure is given.*) The balloon is rising at a constant rate of 3 meters per second (*this means* $dy/dt = 3$) and the observer is 100 meters from point C.

(a) Find the rate of change in z at the instant when $y = 50$. (*They want* dz/dt.)

$$z^2 = y^2 + 100^2 \Rightarrow 2z\frac{dz}{dt} = 2y\frac{dy}{dt}$$

$$z = \sqrt{50^2 + 100^2} = 50\sqrt{5} \Rightarrow \frac{dz}{dt} = \frac{2 \cdot 50 \cdot 3}{2 \cdot 50\sqrt{5}} = \frac{3\sqrt{5}}{5}.$$

(b) Find the rate of change in the area of right triangle BCA when $y = 50$.

$$A = \frac{1}{2}(100)(y) = 50y \qquad \frac{dA}{dt} = 50\frac{dy}{dt} = 50 \cdot 3 = 150.$$

(c) Find the rate of change in θ when $y = 50$. (*They want* $d\theta/dt$.)

$$y = 50 \Rightarrow \cos\theta = \frac{100}{50\sqrt{5}} = \frac{2}{\sqrt{5}}$$

$$\tan\theta = \frac{y}{100} \Rightarrow \sec^2\theta\frac{d\theta}{dt} = \frac{1}{100}\frac{dy}{dt} \Rightarrow \frac{d\theta}{dt} = \left(\frac{2}{\sqrt{5}}\right)^2\frac{3}{100} = \frac{3}{125}$$

In all problems I first wrote down a relation from the figure. Then I took its derivative. Then I substituted known information. (The substitution is *after* taking the derivative of $\tan\theta = y/100$. If we substitute $y = 50$ too soon, the derivative of 50/100 is useless.)

"Candidates are advised to show their work in order to minimize the risk of not receiving credit for it." 50% solved Example 6 and 21% solved Example 7. From 12,000 candidates, the average on Example 8 (free response) was 6.1 out of 9.

EXAMPLE 9 A is a lighthouse and BC is the shoreline (same figure as the balloon). The light at A turns once a second ($d\theta/dt = 2\pi$ radians/second). How quickly does the receiving point B move up the shoreline?

Solution The figure shows $y = 100\tan\theta$. The speed dy/dt is $100\sec^2\theta\, d\theta/dt$. This is $200\pi\sec^2\theta$, so B speeds up as $\sec\theta$ increases.

Paradox When θ approaches a right angle, $\sec\theta$ approaches infinity. So does dy/dt. ***B moves faster than light***! This contradicts Einstein's theory of relativity. The paradox is resolved (I hope) in Problem 18.

If you walk around a light at A, your shadow at B seems to go faster than light. Same problem. This speed is impossible—something has been forgotten.

Smaller paradox (not destroying the theory of relativity). The figure shows $y = z\sin\theta$. Apparently $dy/dt = (dz/dt)\sin\theta$. ***This is totally wrong***. Not only is it wrong, the exact opposite is true: $dz/dt = (dy/dt)\sin\theta$. If you can explain that (Problem 15), then **ID** and related rates hold no terrors.

4.2 EXERCISES

Read-through questions

For $x^3 + y^3 = 2$ the derivative dy/dx comes from __a__ differentiation. We don't have to solve for __b__. Term by term the derivative is $3x^2 +$ __c__ $= 0$. Solving for dy/dx gives __d__. At $x = y = 1$ this slope is __e__. The equation of the tangent line is $y - 1 =$ __f__.

A second example is $y^2 = x$. The x derivative of this equation is __g__. Therefore $dy/dx =$ __h__. Replacing y by $\sqrt{x}$, this is $dy/dx =$ __i__.

In related rates, we are given dg/dt and we want df/dt. We need a relation between f and __j__. If $f = g^2$, then $(df/dt) =$ __k__ (dg/dt). If $f^2 + g^2 = 1$, then $df/dt =$ __l__. If the sides of a cube grow by $ds/dt = 2$, then its volume grows by $dV/dt =$ __m__. To find a number (8 is wrong), you also need to know __n__.

By implicit differentiation find dy/dx in 1–10.

1 $y^n + x^n = 1$

2 $x^2y + y^2x = 1$

3 $(x - y)^2 = 4$

4 $\sqrt{x} + \sqrt{y} = 3$ at $x = 4$

5 $x = F(y)$

6 $f(x) + F(y) = xy$

7 $x^2y = y^2x$

8 $x = \sin y$

9 $x = \tan y$

10 $y^n = x$ at $x = 1$

11 Show that the hyperbolas $xy = C$ are perpendicular to the hyperbolas $x^2 - y^2 = D$. (Perpendicular means that the product of slopes is -1.)

12 Show that the circles $(x-2)^2 + y^2 = 2$ and $x^2 + (y-2)^2 = 2$ are tangent at the point (1, 1).

13 At 25 meters/second, does your car turn faster or slower than a car traveling 5 meters further out at 26 meters/second? Your radius is (a) 50 meters (b) 100 meters.

14 Equation (4) is $2x + 2y\,dy/dx = 0$ (on a circle). Directly by **ID** reach d^2y/dx^2 in equation (5).

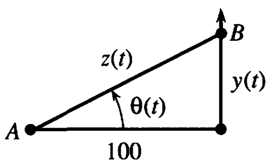
Problems 15–18 resolve the speed of light paradox in Example 9.

15 (Small paradox first) The right triangle has $z^2 = y^2 + 100^2$. Take the t derivative to show that $z' = y'\sin\theta$.

16 (Even smaller paradox) As B moves up the line, why is dy/dt larger than dz/dt? Certainly z is larger than y. But as θ increases they become ________.

17 (Faster than light) The derivative of $y = 100\tan\theta$ in Example 9 is $y' = 100\sec^2\theta\,\theta' = 200\pi\sec^2\theta$. Therefore y'

passes c (the speed of light) when $\sec^2\theta$ passes ________. Such a speed is impossible—we forget that light takes time to reach B.



θ increases by 2π in 1 second

t is arrival time of light

θ is different from $2\pi t$

18 (Explanation by **ID**) Light travels from A to B in time z/c, distance over speed. Its arrival time is $t = \theta/2\pi + z/c$ so $\theta'/2\pi = 1 - z'/c$. Then $z' = y' \sin\theta$ and $y' = 100 \sec^2\theta\, \theta'$ (all these are **ID**) lead to

$$y' = 200\pi c/(c \cos^2\theta + 200\pi \sin\theta)$$

As θ approaches $\pi/2$, this speed approaches ________.
Note: y' still exceeds c for some negative angle. That is for Einstein to explain. See the 1985 *College Math Journal*, page 186, and the 1960 *Scientific American*, "Things that go faster than light."

19 If a plane follows the curve $y = f(x)$, and its ground speed is $dx/dt = 500$ mph, how fast is the plane going up? How fast is the plane going?

20 Why can't we differentiate $x = 7$ and reach $1 = 0$?

Problems 21–29 are applications of related rates.

21 (Calculus classic) The bottom of a 10-foot ladder is going away from the wall at $dx/dt = 2$ feet per second. How fast is the top going down the wall? Draw the right triangle to find dy/dt when the height y is (a) 6 feet (b) 5 feet (c) zero.

22 The top of the 10-foot ladder can go faster than light. At what height y does $dy/dt = -c$?

23 How fast does the level of a Coke go down if you drink a cubic inch a second? The cup is a cylinder of radius 2 inches—first write down the volume.

24 A jet flies at 8 miles up and 560 miles per hour. How fast is it approaching you when (a) it is 16 miles from you; (b) its shadow is 8 miles from you (the sun is overhead); (c) the plane is 8 miles from you (exactly above)?

25 Starting from a 3–4–5 right triangle, the short sides increase by 2 meters/second but the angle between them decreases by 1 radian/second. How fast does the area increase or decrease?

26 A pass receiver is at $x = 4$, $y = 8t$. The ball thrown at $t = 3$ is at $x = c(t-3)$, $y = 10c(t-3)$.

(a) Choose c so the ball meets the receiver.

*(b) At that instant the distance D between them is changing at what rate?

27 A thief is 10 meters away (8 meters ahead of you, across a street 6 meters wide). The thief runs on that side at 7 meters/second, you run at 9 meters/second. How fast are you approaching if (a) you follow on your side; (b) you run toward the thief; (c) you run away on your side?

28 A spherical raindrop evaporates at a rate equal to twice its surface area. Find dr/dt.

29 Starting from $P = V = 5$ and maintaining $PV = T$, find dV/dt if $dP/dt = 2$ and $dT/dt = 3$.

30 (a) The crankshaft AB turns twice a second so $d\theta/dt =$ ________.

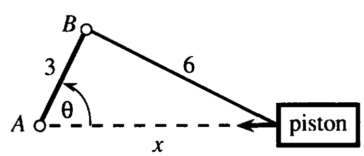
(*b*) Differentiate the cosine law $6^2 = 3^2 + x^2 - 2\,(3x \cos\theta)$ to find the piston speed dx/dt when $\theta = \pi/2$ and $\theta = \pi$.

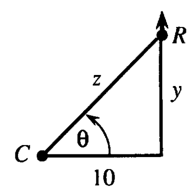
31 A camera turns at C to follow a rocket at R.

(a) Relate dz/dt to dy/dt when $y = 10$.

(b) Relate $d\theta/dt$ to dy/dt based on $y = 10\tan\theta$.

(c) Relate $d^2\theta/dt^2$ to d^2y/dt^2 and dy/dt.





4.3 Inverse Functions and Their Derivatives

There is a remarkable special case of the chain rule. It occurs when $f(y)$ and $g(x)$ are "***inverse functions***." That idea is expressed by a very short and powerful equation: $f(g(x)) = x$. Here is what that means.

Inverse functions: Start with any input, say $x = 5$. *Compute* $y = g(x)$, say $y = 3$. Then compute $f(y)$, and *the answer must be* 5. What one function does, the inverse function