5.5 The Definite Integral

The integral of $v(x)$ is an antiderivative $f(x)$ plus a constant C. This section takes two steps. First, we choose C. Second, we construct $f(x)$. The object is ***to define the integral***—in the most frequent case when a suitable $f(x)$ is not directly known.

The indefinite integral contains "$+C$." The constant is not settled because $f(x)+C$ has the same slope for every C. When we care only about the derivative, C makes no difference. When the goal is a number—a ***definite integral***—C can be assigned a definite value at the starting point.

For mileage traveled, ***we subtract the reading at the start***. This section does the same for area. Distance is $f(t)$ and area is $f(x)$—while the definite integral is $f(b)-f(a)$. Don't pay attention to t or x, pay attention to the great formula of integral calculus:

$$\int_a^b v(t)\,dt = \int_a^b v(x)\,dx = f(b)-f(a). \tag{1}$$

Viewpoint 1: When f is known, the equation gives the area from a to b.
Viewpoint 2: When f is *not* known, the equation defines f from the area.

For a typical $v(x)$, we can't find $f(x)$ by guessing or substitution. But still $v(x)$ has an "area" under its graph—and this yields the desired integral $f(x)$.

Most of this section is theoretical, leading to the definition of the integral. You may think we should have defined integrals before computing them—which is logically true. But the idea of area (and the use of rectangles) was already pretty clear in our first examples. Now we go much further. ***Every continuous function*** $v(x)$ ***has an integral*** (also some discontinuous functions). Then the Fundamental Theorem completes the circle: The integral leads back to $df/dx = v(x)$. The area up to x is the antiderivative that we couldn't otherwise discover.

THE CONSTANT OF INTEGRATION

Our goal is to turn $f(x)+C$ into a definite integral—the area between a and b. The first requirement is to have *area = zero* at the start:

$$f(a)+C = \text{starting area} = 0 \quad \text{so} \quad C = -f(a). \tag{2}$$

For the area up to x (moving endpoint, indefinite integral), use t as the dummy variable:

the area from a to x is $\int_a^x v(t)\,dt = f(x)-f(a)$ (***indefinite integral***)

the area from a to b is $\int_a^b v(x)\,dx = f(b)-f(a)$ (***definite integral***)

EXAMPLE 1 The area under the graph of $5(x+1)^4$ from a to b has $f(x)=(x+1)^5$:

$$\int_a^b 5(x+1)^4\,dx = (x+1)^5\Big]_a^b = (b+1)^5-(a+1)^5.$$

The calculation has two separate steps—first find $f(x)$, then substitute b and a. After the first step, check that df/dx is v. The upper limit in the second step gives *plus* $f(b)$, the lower limit gives *minus* $f(a)$. Notice the brackets (or the vertical bar):

$$f(x)\Big]_a^b = f(b)-f(a) \qquad x^3\big|_1^2 = 8-1 \qquad \Big[\cos x\Big]_0^{2t} = \cos 2t - 1.$$

Changing the example to $f(x)=(x+1)^5-1$ gives an equally good antiderivative—

and now $f(0)=0$. But $f(b)-f(a)$ stays the same, because the -1 disappears:

$$\left[(x+1)^5-1\right]_a^b=((b+1)^5-1)-((a+1)^5-1)=(b+1)^5-(a+1)^5.$$

EXAMPLE 2 When $v=2x\sin x^2$ we recognize $f=-\cos x^2$. The area from 0 to 3 is

$$\int_0^3 2x\sin x^2\,dx=-\cos x^2\Big]_0^3=-\cos 9+\cos 0.$$

The upper limit copies the minus sign. The lower limit gives $-(-\cos 0)$, which is $+\cos 0$. ***That example shows the right form for solving exercises on definite integrals.***

Example 2 jumped directly to $f(x)=-\cos x^2$. But most problems involving the chain rule go more slowly—by *substitution*. Set $u=x^2$, with $du/dx=2x$:

$$\int_0^3 2x\sin x^2\,dx=\int_0^3 \sin u\,\frac{du}{dx}\,dx=\int_?^? \sin u\,du. \tag{3}$$

We need new limits when u replaces x^2. Those limits on u are a^2 and b^2. (In this case $a^2=0^2$ and $b^2=3^2=9$.) ***If x goes from a to b, then u goes from*** $u(a)$ ***to*** $u(b)$.

$$\int_a^b v(u(x))\,\frac{du}{dx}\,dx=\int_{u(a)}^{u(b)} v(u)\,du=f(u(b))-f(u(a)). \tag{4}$$

EXAMPLE 3 $\displaystyle\int_{x=0}^{1}(x^2+5)^3x\,dx=\int_{u=5}^{6}u^3\,\frac{du}{2}=\frac{u^4}{8}\Big]_5^6=\frac{6^4}{8}-\frac{5^4}{8}.$

In this case $u=x^2+5$. Therefore $du/dx=2x$ (or $du=2x\,dx$ for differentials). We have to account for the missing 2. The integral is $\frac{1}{8}u^4$. The limits on $u=x^2+5$ are $u(0)=0^2+5$ and $u(1)=1^2+5$. That is why the u-integral goes from 5 to 6. The alternative is to find $f(x)=\frac{1}{8}(x^2+5)^4$ in one jump (*and check it*).

EXAMPLE 4 $\int_0^1 \sin x^2\,dx=\,??$ (***no elementary function gives this integral***).

If we try $\cos x^2$, the chain rule produces an extra $2x$—no adjustment will work. Does $\sin x^2$ still have an antiderivative? *Yes*! Every continuous $v(x)$ has an $f(x)$. Whether $f(x)$ has an algebraic formula or not, we can write it as $\int v(x)\,dx$. To define that integral, we now take the limit of rectangular areas.

INTEGRALS AS LIMITS OF "RIEMANN SUMS"

We have come to the ***definition of the integral***. The chapter started with the integrals of x and x^2, from formulas for $1+\cdots+n$ and $1^2+\cdots+n^2$. We will not go back to those formulas. But for other functions, too irregular to find exact sums, the rectangular areas also approach a limit.

That limit is the integral. This definition is a major step in the theory of calculus. It can be studied in detail, or understood in principle. The truth is that the definition is not so painful—you virtually know it already.

Problem Integrate the continuous function $v(x)$ ***over the interval*** $[a,b]$.
Step 1 Split $[a,b]$ into n subintervals $[a,x_1],[x_1,x_2],\ldots,[x_{n-1},b]$.

The "meshpoints" $x_1,x_2,\ldots$ divide up the interval from a to b. The endpoints are $x_0=a$ and $x_n=b$. The length of subinterval k is $\Delta x_k=x_k-x_{k-1}$. In that smaller interval, the minimum of $v(x)$ is m_k. The maximum is M_k.

Now construct rectangles. The "*lower rectangle*" over interval k has height m_k. The "*upper rectangle*" reaches to M_k. Since v is continuous, there are points x_{min} and x_{max} where $v = m_k$ and $v = M_k$ (extreme value theorem). ***The graph of*** $v(x)$ ***is in between.***

Important: The area under $v(x)$ contains the area "s" of the lower rectangles:

$$\int_a^b v(x)\,dx \geqslant m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_n\Delta x_n = s. \tag{5}$$

The area under $v(x)$ is contained in the area "S" of the upper rectangles:

$$\int_a^b v(x)\,dx \leqslant M_1\Delta x_1 + M_2\Delta x_2 + \cdots + M_n\Delta x_n = S. \tag{6}$$

The ***lower sum*** s and the ***upper sum*** S were computed earlier in special cases—when v was x or x^2 and the spacings Δx were equal. Figure 5.9a shows why $s \leqslant$ *area* $\leqslant S$.

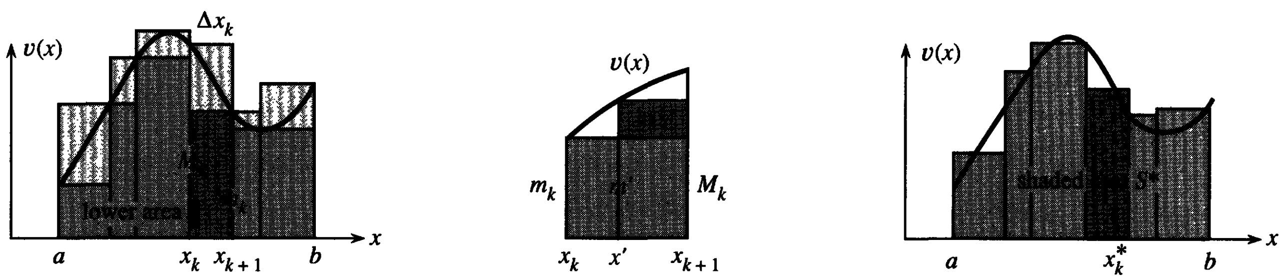


Fig. 5.9 Area of lower rectangles $= s$. Upper sum S includes top pieces. Riemann sum S^* is in between.

Notice an important fact. When a new dividing point x' is added, *the lower sum increases.* The minimum in one piece can be greater (see second figure) than the original m_k. Similarly *the upper sum decreases.* The maximum in one piece can be below the overall maximum. ***As new points are added,*** s ***goes up and*** S ***comes down.*** So the sums come closer together:

$$s \leqslant s' \leqslant \qquad \leqslant S' \leqslant S. \tag{7}$$

I have left space in between for the curved area—the integral of $v(x)$.

Now add more and more meshpoints in such a way that $\Delta x_{max} \to 0$. The lower sums increase and the upper sums decrease. They never pass each other. ***If*** $v(x)$ ***is continuous, those sums close in on a single number*** A. That number is the definite integral—the area under the graph.

DEFINITION The area A is the common limit of the lower and upper sums:

$$s \to A \text{ and } S \to A \text{ as } \Delta x_{max} \to 0. \tag{8}$$

This limit A exists for all continuous $v(x)$, and also for some discontinuous functions. When it exists, A is the "***Riemann integral***" of $v(x)$ from a to b.

REMARKS ON THE INTEGRAL

As for derivatives, so for integrals: The definition involves a limit. Calculus is built on limits, and we always add "if the limit exists." That is the delicate point. I hope the next five remarks (increasingly technical) will help to distinguish functions that are *Riemann integrable* from functions that are not.

Remark 1 The sums s and S may fail to approach the same limit. A standard example has $V(x) = 1$ at all fractions $x = p/q$, and $V(x) = 0$ at all other points. Every interval contains rational points (fractions) and irrational points (nonrepeating decimals). *Therefore* $m_k = 0$ *and* $M_k = 1$. The lower sum is always $s = 0$. The upper sum is always $S = b - a$ (the sum of 1's times Δx's). ***The gap in equation* (7) *stays open*.** This function $V(x)$ is not Riemann integrable. The area under its graph is not defined (at least by Riemann—see Remark 5).

Remark 2 The ***step function*** $U(x)$ is discontinuous but still integrable. On every interval the minimum m_k equals the maximum M_k—except on the interval containing the jump. That jump interval has $m_k = 0$ and $M_k = 1$. But when we multiply by Δx_k, and require $\Delta x_{max} \to 0$, the difference between s and S goes to zero. The area under a step function is clear—the rectangles fit exactly.

Remark 3 With patience another key step could be proved: ***If*** $s \to A$ ***and*** $S \to A$ ***for one sequence of meshpoints, then this limit A is approached by every choice of meshpoints with*** $\Delta x_{max} \to 0$. The integral is the lower bound of all upper sums S, and it is the upper bound of all possible s—provided those bounds are equal. The gap must close, to define the integral.

The same limit A is approached by "in-between rectangles." The height $v(x_k^*)$ can be computed at any point x_k^* in subinterval k. See Figures 5.9c and 5.10. Then the total rectangular area is a "***Riemann sum***" between s and S:

$$S^* = v(x_1^*)\Delta x_1 + v(x_2^*)\Delta x_2 + \cdots + v(x_n^*)\Delta x_n. \tag{9}$$

We cannot tell whether the true area is above or below S^*. Very often A is closer to S^* than to s or S. The *midpoint rule* takes x_k^* in the middle of its interval (Figure 5.10), and Section 5.8 will establish its extra accuracy. The extreme sums s and S are used in the definition while S^* is used in computation.

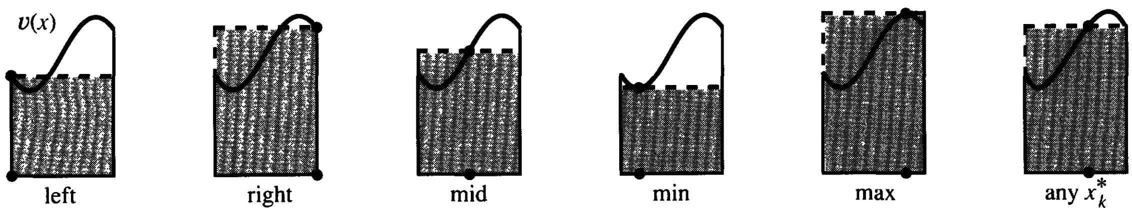


Fig. 5.10 Various positions for x_k^* in the base. The rectangles have height $v(x_k^*)$.

Remark 4 ***Every continuous function is Riemann integrable.*** The proof is optional (in my class), but it belongs here for reference. It starts with continuity at x^*: "*For any* ε *there is a* δ ...." When the rectangles sit between $x^* - \delta$ and $x^* + \delta$, the bounds M_k and m_k differ by less than 2ε. Multiplying by the base Δx_k, the areas differ by less than $2\varepsilon(\Delta x_k)$. Combining all rectangles, the upper and lower sums differ by less than $2\varepsilon(\Delta x_1 + \Delta x_2 + \cdots + \Delta x_n) = 2\varepsilon(b - a)$.

As $\varepsilon \to 0$ we conclude that S comes arbitrarily close to s. They squeeze in on a single number A. The Riemann sums approach the Riemann integral, *if v is continuous.*

Two problems are hidden by that reasoning. One is at the end, where S and s come together. We have to know that the line of real numbers has no "holes," so there is a number A to which these sequences converge. That is true.

Any increasing sequence, if it is bounded above, approaches a limit.

The decreasing sequence S, bounded below, converges to the same limit. So A exists.

The other problem is about continuity. We assumed without saying so that the width 2δ is the same around every point x^*. We did not allow for the possibility that δ might approach zero where $v(x)$ is rapidly changing—in which case an infinite number of rectangles could be needed. Our reasoning requires that

*$v(x)$ is **uniformly continuous**: δ depends on ε **but not on the position of** x^*.*

For each ε there is a δ that works at all points in the interval. *A continuous function on a closed interval is **uniformly** continuous.* This fact (proof omitted) makes the reasoning correct, and $v(x)$ is integrable.

On an infinite interval, even $v = x^2$ is not uniformly continuous. It changes across a subinterval by $(x^* + \delta)^2 - (x^* - \delta)^2 = 4x^*\delta$. As x^* gets larger, δ must get smaller—to keep $4x^*\delta$ below ε. No single δ succeeds at all x^*. But on a finite interval $[0, b]$, the choice $\delta = \varepsilon/4b$ works everywhere—so $v = x^2$ is uniformly continuous.

Remark 5 If those four remarks were fairly optional, this one is totally at your discretion. Modern mathematics needs to integrate the zero-one function $V(x)$ in the first remark. Somehow V has more 0's than 1's. The fractions (where $V(x) = 1$) can be put in a list, but the irrational numbers (where $V(x) = 0$) are "uncountable." The integral ought to be zero, but Riemann's upper sums all involve $M_k = 1$.

Lebesgue discovered a major improvement. He allowed *infinitely many subintervals* (smaller and smaller). Then all fractions can be covered with intervals of total width ε. (Amazing, when the fractions are packed so densely.) The idea is to cover $1/q, 2/q, \ldots, q/q$ by narrow intervals of total width $\varepsilon/2^q$. Combining all $q = 1, 2, 3, \ldots$, the total width to cover all fractions is no more than $\varepsilon(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots) = \varepsilon$. Since $V(x) = 0$ everywhere else, the upper sum S is only ε. And since ε was arbitrary, the "*Lebesgue integral*" is zero as desired.

That completes a fair amount of theory, possibly more than you want or need—but it is satisfying to get things straight. The definition of the integral is still being studied by experts (and so is the derivative, again to allow more functions). By contrast, the *properties* of the integral are used by everybody. Therefore the next section turns from definition to properties, collecting the rules that are needed in applications. They are very straightforward.

5.5 EXERCISES

Read-through questions

In $\int_a^x v(t)\,dt = f(x) + C$, the constant C equals __a__. Then at $x = a$ the integral is __b__. At $x = b$ the integral becomes __c__. The notation $f(x)]_a^b$ means __d__. Thus $\cos x]_0^\pi$ equals __e__. Also $[\cos x + 3]_0^\pi$ equals __f__, which shows why the antiderivative includes an arbitrary __g__. Substituting $u = 2x - 1$ changes $\int_1^3 \sqrt{2x-1}\,dx$ into __h__ (with limits on u).

The integral $\int_a^b v(x)\,dx$ can be defined for any __i__ function $v(x)$, even if we can't find a simple __j__. First the meshpoints $x_1, x_2, \ldots$ divide $[a, b]$ into subintervals of length $\Delta x_k =$ __k__. The upper rectangle with base Δx_k has height $M_k =$ __l__. The upper sum S is equal to __m__. The lower sum s is __n__. The __o__ is between s and S. As more meshpoints are added, S __p__ and s __q__. If S and s approach the same __r__, that defines the integral. The intermediate sums S^*, named after __s__, use rectangles of height $v(x_k^*)$. Here x_k^* is any point between __t__, and $S^* =$ __u__ approaches the area.

If $v(x) = df/dx$, what constants C make 1–10 true?

1 $\int_2^b v(x)\,dx = f(b) + C$

2 $\int_1^4 v(x)\,dx = f(4) + C$

3 $\int_x^3 v(t)\,dt = -f(x) + C$

4 $\int_{\pi/2}^b v(\sin x)\cos x\,dx = f(\sin b) + C$

5 $\int_1^x v(t)\,dt = f(t) + C$ (careful)

6 $df/dx = v(x) + C$

7 $\int_0^1 (x^2 - 1)^3\, 2x\,dx = \int_{-1}^C u^3\,du$.

8 $\int_0^{x^2} v(t)\,dt = f(x^2) + C$

9 $\int_a^b v(-x)\,dx = C$ (change $-x$ to t; also dx and limits)

10 $\int_0^2 v(x)\,dx = C\int_0^1 v(2t)\,dt$.

Choose $u(x)$ in 11–18 and *change limits*. Compute the integral in 11–16.

11 $\int_0^1 (x^2+1)^{10}x\,dx$

12 $\int_0^{\pi/2} \sin^8 x \cos x\,dx$

13 $\int_0^{\pi/4} \tan x \sec^2 x\,dx$

14 $\int_0^2 x^{2n+1}\,dx$ (take $u = x^2$)

15 $\int_0^{\pi/4} \sec^2 x \tan x\,dx$

16 $\int_0^1 x\,dx/\sqrt{1-x^2}$

17 $\int_1^2 dx/x$ (take $u = 1/x$)

18 $\int_0^1 x^3(1-x)^3\,dx$ $(u = 1-x)$

With $\Delta x = \frac{1}{2}$ in 19–22, find the maximum M_k and minimum m_k and upper and lower sums S and s.

19 $\int_0^1 (x^2+1)^4\,dx$

20 $\int_0^1 \sin 2\pi x\,dx$

21 $\int_0^2 x^3\,dx$

22 $\int_{-1}^1 x\,dx$.

23 Repeat 19 and 20 with $\Delta x = \frac{1}{4}$ and compare with the correct answer.

24 The difference $S - s$ in 21 is the area $2^3\,\Delta x$ of the far right rectangle. Find Δx so that $S < 4.001$.

25 If $v(x)$ is *increasing* for $a \leqslant x \leqslant b$, the difference $S - s$ is the area of the ________ rectangle minus the area of the ________ rectangle. Those areas approach zero. *So every increasing function on $[a, b]$ is Riemann integrable.*

26 Find the Riemann sum S^* for $V(x)$ in Remark 1, when $\Delta x = 1/n$ and each x_k^* is the midpoint. This S^* is well-behaved but still $V(x)$ is not Riemann integrable.

27 $W(x)$ equals 1 at $x = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$, and elsewhere $W(x) = 0$. For $\Delta x = .01$ find the upper sum S. Is $W(x)$ integrable?

28 Suppose $M(x)$ is a *multistep* function with jumps of $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ at the points $x = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$. Draw a rough graph with $M(0) = 0$ and $M(1) = 1$. With $\Delta x = \frac{1}{3}$ find S and s.

29 For $M(x)$ in Problem 28 find the difference $S - s$ (which approaches zero as $\Delta x \to 0$). What is the area under the graph?

30 If $df/dx = -v(x)$ and $f(1) = 0$, explain $f(x) = \int_x^1 v(t)\,dt$.

31 (a) If $df/dx = +v(x)$ and $f(0) = 3$, find $f(x)$.
(b) If $df/dx = +v(x)$ and $f(3) = 0$, find $f(x)$.

32 In your own words define the integral of $v(x)$ from a to b.

33 ***True or false***, with reason or example.
(a) Every continuous $v(x)$ has an antiderivative $f(x)$.
(b) If $v(x)$ is not continuous, S and s approach different limits.
(c) If S and s approach A as $\Delta x \to 0$, then all Riemann sums S^* in equation (9) also approach A.
(d) If $v_1(x) + v_2(x) = v_3(x)$, their upper sums satisfy $S_1 + S_2 = S_3$.
(e) If $v_1(x) + v_2(x) = v_3(x)$, their Riemann sums at the midpoints x_k^* satisfy $S_1^* + S_2^* = S_3^*$.
(f) The midpoint sum is the average of S and s.
(g) One x_k^* in Figure 5.10 gives the exact area.

5.6 Properties of the Integral and Average Value

The previous section reached the definition of $\int_a^b v(x)\,dx$. But the subject cannot stop there. The integral was defined in order to be used. Its properties are important, and its applications are even more important. The definition was chosen so that the integral has properties that make the applications possible.

One direct application is to the ***average value*** of $v(x)$. The average of n numbers is clear, and the integral extends that idea—it produces the average of a whole continuum of numbers $v(x)$. This develops from the last rule in the following list (Property 7). We now collect together ***seven basic properties of definite integrals.***

The addition rule for $\int [v(x) + w(x)]\,dx$ will not be repeated—even though this property of linearity is the most fundamental. We start instead with a different kind of addition. There is only one function $v(x)$, but now there are two intervals.

The integral from a to b is added to its neighbor from b to c. Their sum is the integral from a to c. That is the first (not surprising) property in the list.

Property **1** Areas over neighboring intervals add to the area over the combined interval:

$$\int_a^b v(x)\,dx + \int_b^c v(x)\,dx = \int_a^c v(x)\,dx. \tag{1}$$

This sum of areas is graphically obvious (Figure 5.11a). It also comes from the formal definition of the integral. Rectangular areas obey (1)—with a meshpoint at $x = b$ to make sure. When Δx_{max} approaches zero, their limits also obey (1). *All the normal rules for rectangular areas are obeyed in the limit by integrals.*

Property **1** is worth pursuing. It indicates how to define the integral when $a = b$. The integral "*from b to b*" is the area over a point, which we expect to be zero. It is.

Property **2** $\int_b^b v(x)\,dx = 0.$

That comes from Property **1** when $c = b$. Equation (1) has two identical integrals, so the one from b to b must be zero. Next we see what happens if $c = a$—which makes the second integral go from b to a.

What happens when ***an integral goes backward***? The "lower limit" is now the larger number b. The "upper limit" a is smaller. Going backward reverses the sign:

Property **3** $\int_b^a v(x)\,dx = -\int_a^b v(x)\,dx = f(a) - f(b).$

Proof When $c = a$ the right side of (1) is zero. Then the integrals on the left side must cancel, which is Property **3**. *In going from b to a the steps* Δx *are negative*. That justifies a minus sign on the rectangular areas, and a minus sign on the integral (Figure 5.11b). *Conclusion*: Property **1** holds for any ordering of a, b, c.

EXAMPLES $\displaystyle\int_x^0 t^2\,dt = -\frac{x^3}{3}$ $\displaystyle\int_1^0 dt = -1$ $\displaystyle\int_2^2 \frac{dt}{t} = 0$

Property **4** For odd functions $\int_{-a}^a v(x)\,dx = 0$. "***Odd***" means that $v(-x) = -v(x)$. For even functions $\int_{-a}^a v(x)\,dx = 2\int_0^a v(x)\,dx$. "***Even***" means that $v(-x) = +v(x)$.

The functions $x, x^3, x^5, \ldots$ are odd. If x changes sign, these powers change sign. The functions $\sin x$ and $\tan x$ are also odd, together with their inverses. This is an important family of functions, and ***the integral of an odd function from*** $-a$ ***to*** a ***equals zero.*** Areas cancel:

$$\int_{-a}^a 6x^5\,dx = x^6\Big]_{-a}^a = a^6 - (-a)^6 = 0.$$

If $v(x)$ is odd then $f(x)$ is even! All powers $1, x^2, x^4, \ldots$ are even functions. ***Curious fact***: Odd function times even function is *odd*, but odd number times even number is *even*.

For even functions, areas add: $\int_{-a}^a \cos x\,dx = \sin a - \sin(-a) = 2\sin a$.

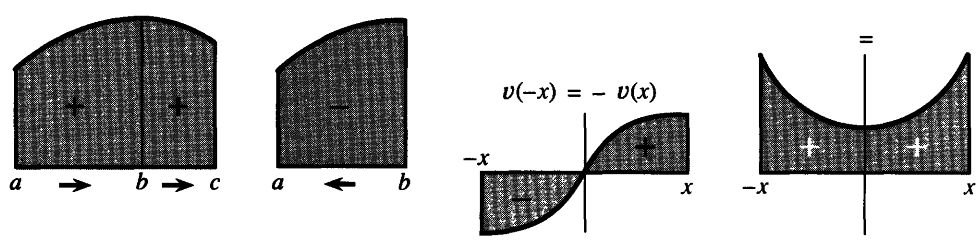


Fig. 5.11 Properties 1–4: Add areas, change sign to go backward, odd cancels, even adds.

The next properties involve inequalities. If $v(x)$ is positive, the area under its graph is positive (not surprising). Now we have a proof: The lower sums s are positive and they increase toward the area integral. So the integral is positive:

***Property* 5** If $v(x) > 0$ for $a < x < b$ then $\int_a^b v(x)\,dx > 0$.

A positive velocity means a positive distance. A positive v lies above a positive area. A more general statement is true. Suppose $v(x)$ stays between a lower function $l(x)$ and an upper function $u(x)$. Then the rectangles for v stay between the rectangles for l and u. In the limit, the area under v (Figure 5.12) is between the areas under l and u:

***Property* 6** If $l(x) \leqslant v(x) \leqslant u(x)$ for $a \leqslant x \leqslant b$ then

$$\int_a^b l(x)\,dx \leqslant \int_a^b v(x)\,dx \leqslant \int_a^b u(x)\,dx. \tag{2}$$

EXAMPLE 1 $\cos t \leqslant 1 \;\Rightarrow\; \int_0^x \cos t\,dt \leqslant \int_0^x 1\,dt \;\Rightarrow\; \sin x \leqslant x$

EXAMPLE 2 $1 \leqslant \sec^2 t \;\Rightarrow\; \int_0^x 1\,dt \leqslant \int_0^x \sec^2 t\,dt \;\Rightarrow\; x \leqslant \tan x$

EXAMPLE 3 Integrating $\dfrac{1}{1+x^2} \leqslant 1$ leads to $\tan^{-1} x \leqslant x$.

All those examples are for $x > 0$. You may remember that Section 2.4 used geometry to prove $\sin h < h < \tan h$. Examples 1–2 seem to give new and shorter proofs. But I think the reasoning is doubtful. The inequalities were needed to compute the derivatives (therefore the integrals) in the first place.

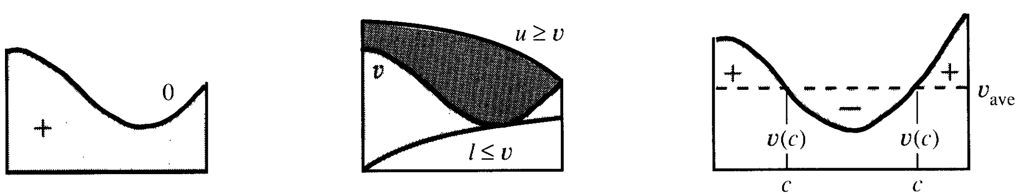


Fig. 5.12 Properties 5–7: v above zero, v between l and u, average value (+ balances −).

Property* 7** (Mean Value Theorem for Integrals***) If $v(x)$ is continuous, there is a point c between a and b where $v(c)$ equals the average value of $v(x)$:

$$v(c) = \frac{1}{b-a}\int_a^b v(x)\,dx = \text{“}\textbf{\textit{average value of}}\ v(x).\text{”} \tag{3}$$

This is the same as the ordinary Mean Value Theorem (for the derivative of $f(x)$):

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \text{“average slope of } f.\text{”} \tag{4}$$

With $f' = v$, (3) and (4) are the same equation. But honesty makes me admit to a flaw in the logic. We need the Fundamental Theorem of Calculus to guarantee that $f(x) = \int_a^x v(t)\,dt$ really gives $f' = v$.

A direct proof of (3) places one rectangle across the interval from a to b. Now raise the top of that rectangle, starting at $v_{\min}$ (the bottom of the curve) and moving up to $v_{\max}$ (the top of the curve). At some height the area will be just right—equal to the area under the curve. Then the rectangular area, which is $(b-a)$ times $v(c)$, equals the curved area $\int_a^b v(x)\,dx$. This is equation (3).

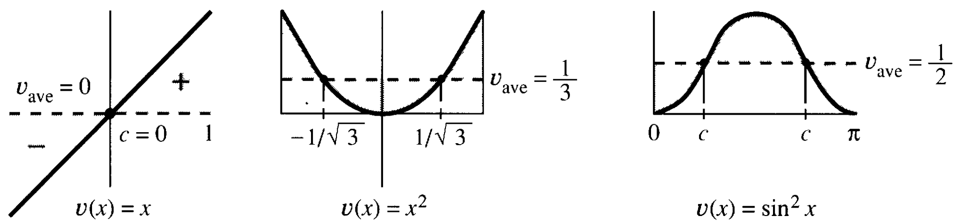


Fig. 5.13 Mean Value Theorem for integrals: area/$(b-a)$ = average height = $v(c)$ at some c.

That direct proof uses the ***intermediate value theorem***: A continuous function $v(x)$ takes on every height between v_{min} and v_{max}. At some point (at two points in Figure 5.12c) the function $v(x)$ equals its own average value.

Figure 5.13 shows equal areas above and below the average height $v(c) = v_{ave}$.

EXAMPLE 4 The average value of an odd function is *zero* (between -1 and 1):

$$\frac{1}{2}\int_{-1}^{1} x\,dx = \frac{x^2}{4}\Big]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0 \qquad \left(\text{note } \frac{1}{b-a} = \frac{1}{2}\right)$$

For once we know c. It is the center point $x = 0$, where $v(c) = v_{ave} = 0$.

EXAMPLE 5 The average value of x^2 is $\frac{1}{3}$ (between 1 and -1):

$$\frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{x^3}{6}\Big]_{-1}^{1} = \frac{1}{6} - \left(-\frac{1}{6}\right) = \frac{1}{3} \qquad \left(\text{note } \frac{1}{b-a} = \frac{1}{2}\right)$$

Where does this function x^2 equal its average value $\frac{1}{3}$? That happens when $c^2 = \frac{1}{3}$, so c can be either of the points $1/\sqrt{3}$ and $-1/\sqrt{3}$ in Figure 5.13b. Those are the ***Gauss points***, which are terrific for numerical integration as Section 5.8 will show.

EXAMPLE 6 The average value of $\sin^2 x$ over a period (zero to π) is $\frac{1}{2}$:

$$\frac{1}{\pi}\int_{0}^{\pi} \sin^2 x\,dx = \frac{x - \sin x \cos x}{2\pi}\Big]_{0}^{\pi} = \frac{1}{2} \qquad \left(\text{note } \frac{1}{b-a} = \frac{1}{\pi}\right)$$

The point c is $\pi/4$ or $3\pi/4$, where $\sin^2 c = \frac{1}{2}$. ***The graph of*** $\sin^2 x$ ***oscillates around its average value*** $\frac{1}{2}$. See the figure or the formula:

$$\sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x. \tag{5}$$

The steady term is $\frac{1}{2}$, the oscillation is $-\frac{1}{2}\cos 2x$. The integral is $f(x) = \frac{1}{2}x - \frac{1}{4}\sin 2x$, which is the same as $\frac{1}{2}x - \frac{1}{2}\sin x \cos x$. *This integral of* $\sin^2 x$ *will be seen again.* Please verify that $df/dx = \sin^2 x$.

THE AVERAGE VALUE AND EXPECTED VALUE

The "average value" from a to b is the integral divided by the length $b - a$. This was computed for x and x^2 and $\sin^2 x$, but not explained. It is a major application of the integral, and it is guided by the ordinary average of n numbers:

$$v_{ave} = \frac{1}{b-a}\int_a^b v(x)\,dx \qquad \text{comes from} \qquad v_{ave} = \frac{1}{n}(v_1 + v_2 + \cdots + v_n).$$

Integration is parallel to summation! Sums approach integrals. Discrete averages approach continuous averages. The average of $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}$ is $\frac{2}{3}$. The average of $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}$ is $\frac{3}{5}$. The average of n numbers from $1/n$ to n/n is

$$v_{\text{ave}} = \frac{1}{n}\left(\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n}\right) = \frac{n+1}{2n}. \tag{7}$$

The middle term gives the average, when n is odd. Or we can do the addition. As $n \to \infty$ the sum approaches an integral (do you see the rectangles?). The ordinary average of numbers becomes the continuous average of $v(x) = x$:

$$\frac{n+1}{2n} \to \frac{1}{2} \quad \text{and} \quad \int_0^1 x\,dx = \frac{1}{2} \quad \left(\text{note } \frac{1}{b-a} = 1\right)$$

In ordinary language: "The average value of the numbers between 0 and 1 is $\frac{1}{2}$." Since a whole continuum of numbers lies between 0 and 1, that statement is meaningless until we have integration.

The average value of the *squares* of those numbers is $(x^2)_{\text{ave}} = \int x^2\,dx/(b-a) = \frac{1}{3}$. ***If you pick a number randomly between* 0 *and* 1, *its expected value is* $\frac{1}{2}$ *and its expected square is* $\frac{1}{3}$.**

To me that sentence is a puzzle. First, we don't expect the number to be exactly $\frac{1}{2}$—so we need to define "expected value." Second, if the expected value is $\frac{1}{2}$, why is the expected square equal to $\frac{1}{3}$ instead of $\frac{1}{4}$? The ideas come from probability theory, and calculus is leading us to ***continuous probability***. We introduce it briefly here, and come back to it in Chapter 8.

PREDICTABLE AVERAGES FROM RANDOM EVENTS

Suppose you throw a pair of dice. The outcome is not predictable. Otherwise why throw them? *But the average over more and more throws is totally predictable.* We don't know what will happen, but we know its probability.

For dice, we are adding two numbers between 1 and 6. The outcome is between 2 and 12. The probability of 2 is the chance of two ones: $(1/6)(1/6) = 1/36$. Beside each outcome we can write its probability:

$$2\left(\frac{1}{36}\right) 3\left(\frac{2}{36}\right) 4\left(\frac{3}{36}\right) 5\left(\frac{4}{36}\right) 6\left(\frac{5}{36}\right) 7\left(\frac{6}{36}\right) 8\left(\frac{5}{36}\right) 9\left(\frac{4}{36}\right) 10\left(\frac{3}{36}\right) 11\left(\frac{2}{36}\right) 12\left(\frac{1}{36}\right)$$

To repeat, one roll is unpredictable. Only the probabilities are known, and they add to 1. (Those fractions add to 36/36; all possibilities are covered.) The total from a million rolls is even more unpredictable—it can be anywhere between 2,000,000 and 12,000,000. Nevertheless the *average* of those million outcomes is almost completely predictable. This *expected value* is found by adding the products in that line above:

***Expected value*: *multiply* (*outcome*) *times* (*probability of outcome*) *and add*:**

$$\frac{2}{36} + \frac{6}{36} + \frac{12}{36} + \frac{20}{36} + \frac{30}{36} + \frac{42}{36} + \frac{40}{36} + \frac{36}{36} + \frac{30}{36} + \frac{22}{36} + \frac{12}{36} = 7.$$

If you throw the dice 1000 times, and the average is not between 6.9 and 7.1, you get an A. Use the random number generator on a computer and round off to integers.

Now comes ***continuous probability***. Suppose all numbers between 2 and 12 are equally probable. This means all numbers—not just integers. What is the probability of hitting the particular number $x = \pi$? *It is zero*! By any reasonable measure, π has *no chance* to occur. In the continuous case, every x has probability zero. But an *interval of* x's has a nonzero probability:

the probability of an outcome between 2 and 3 is 1/10
the probability of an outcome between x and $x + \Delta x$ is $\Delta x/10$

To find the average, add up each outcome times the probability of that outcome. First divide 2 to 12 into intervals of length $\Delta x = 1$ and probability $p = 1/10$. If we round off x, the average is $6\frac{1}{2}$:

$$2\left(\frac{1}{10}\right) + 3\left(\frac{1}{10}\right) + \cdots + 11\left(\frac{1}{10}\right) = 6.5.$$

Here all outcomes are integers (as with dice). It is more accurate to use 20 intervals of length 1/2 and probability 1/20. The average is $6\frac{3}{4}$, and you see what is coming. These are rectangular areas (Riemann sums). As $\Delta x \to 0$ they approach an integral. The probability of an outcome between x and $x + dx$ is $p(x)\,dx$, and this problem has $p(x) = 1/10$. ***The average outcome in the continuous case is not a sum but an integral:***

$$\textbf{\textit{expected value}}\ E(x) = \int_2^{12} xp(x)\,dx = \int_2^{12} x\frac{dx}{10} = \frac{x^2}{20}\bigg]_2^{12} = 7.$$

That is a big jump. From the point of view of integration, it is a limit of sums. From the point of view of probability, the chance of each outcome is zero but the ***probability density*** at x is $p(x) = 1/10$. The integral of $p(x)$ is 1, because some outcome must happen. The integral of $xp(x)$ is $x_{\text{ave}} = 7$, the expected value. Each choice of x is random, but the average is predictable.

This completes a first step in probability theory. The second step comes after more calculus. Decaying probabilities use e^{-x} and e^{-x^2}—then the chance of a large x is very small. Here we end with the expected values of x^n and $1/\sqrt{x}$ and $1/x$, for a random choice between 0 and 1 (so $p(x) = 1$):

$$E(x^n) = \int_0^1 x^n\,dx = \frac{1}{n+1} \qquad E\left(\frac{1}{\sqrt{x}}\right) = \int_0^1 \frac{dx}{\sqrt{x}} = 2 \qquad E\left(\frac{1}{x}\right) = \int_0^1 \frac{dx}{x} = \infty(!)$$

A CONFUSION ABOUT "EXPECTED" CLASS SIZE

A college can advertise an average class size of 29, while most students are in large classes most of the time. I will show quickly how that happens.

Suppose there are 95 classes of 20 students and 5 classes of 200 students. The total enrollment in 100 classes is 1900 + 1000 = 2900. A random professor has expected class size 29. But a random student sees it differently. The probability is 1900/2900 of being in a small class and 1000/2900 of being in a large class. Adding class size times probability gives the expected class size *for the student*:

$$(20)\left(\frac{1900}{2900}\right) + (200)\left(\frac{1000}{2900}\right) = 82 \text{ students in the class.}$$

Similarly, the average waiting time at a restaurant seems like 40 minutes (to the customer). To the hostess, who averages over the whole day, it is 10 minutes. If you came at a random time it *would* be 10, but if you are a random customer it is 40.

Traffic problems could be eliminated by raising the average number of people per car to 2.5, or even 2. But that is virtually impossible. Part of the problem is the difference between (a) the percentage of cars with one person and (b) the percentage of people alone in a car. Percentage (b) is smaller. In practice, most people would be in crowded cars. See Problems 37–38.

5.6 EXERCISES

Read-through questions

The integrals $\int_0^b v(x)\,dx$ and $\int_b^5 v(x)\,dx$ add to __a__. The integral $\int_3^1 v(x)\,dx$ equals __b__. The reason is __c__. If $v(x) \leqslant x$ then $\int_0^1 v(x)\,dx \leqslant$ __d__. The average value of $v(x)$ on the interval $1 \leqslant x \leqslant 9$ is defined by __e__. It is equal to $v(c)$ at a point $x = c$ which is __f__. The rectangle across this interval with height $v(c)$ has the same area as __g__. The average value of $v(x) = x + 1$ on the interval $1 \leqslant x \leqslant 9$ is __h__.

If x is chosen from 1, 3, 5, 7 with equal probabilities $\frac{1}{4}$, its expected value (average) is __i__. The expected value of x^2 is __j__. If x is chosen from 1, 2, ..., 8 with probabilities $\frac{1}{8}$, its expected value is __k__. If x is chosen from $1 \leqslant x \leqslant 9$, the chance of hitting an integer is __l__. The chance of falling between x and $x + dx$ is $p(x)\,dx =$ __m__. The expected value $E(x)$ is the integral __n__. It equals __o__.

In 1–6 find the average value of $v(x)$ between a and b, and find all points c where $v_{ave} = v(c)$.

1 $v = x^4$, $a = -1$, $b = 1$

2 $v = x^5$, $a = -1$, $b = 1$

3 $v = \cos^2 x$, $a = 0$, $b = \pi$

4 $v = \sqrt{x}$, $a = 0$, $b = 4$

5 $v = 1/x^2$, $a = 1$, $b = 2$

6 $v = (\sin x)^9$, $a = -\pi$, $b = \pi$.

7 At $x = 8$, $F(x) = \int_3^x v(t)\,dt + \int_x^5 v(t)\,dt$ is ______.

8 $\int_1^3 x\,dx + \int_3^5 x\,dx - \int_5^1 x\,dx =$ ______.

Are 9–16 true or false? Give a reason or an example.

9 The minimum of $\int_4^x v(t)\,dt$ is at $x = 4$.

10 The value of $\int_x^{x+3} v(t)\,dt$ does not depend on x.

11 The average value from $x = 0$ to $x = 3$ equals

$$\tfrac{1}{3}(v_{ave} \text{ on } 0 \leqslant x \leqslant 1) + \tfrac{2}{3}(v_{ave} \text{ on } 1 \leqslant x \leqslant 3).$$

12 The ratio $(f(b) - f(a))/(b - a)$ is the average value of $f(x)$ on $a \leqslant x \leqslant b$.

13 On the symmetric interval $-1 \leqslant x \leqslant 1$, $v(x) - v_{ave}$ is an odd function.

14 If $l(x) \leqslant v(x) \leqslant u(x)$ then $dl/dx \leqslant dv/dx \leqslant du/dx$.

15 The average of $v(x)$ from 0 to 2 plus the average from 2 to 4 equals the average from 0 to 4.

16 (a) Antiderivatives of even functions are odd functions.
(b) Squares of odd functions are odd functions.

17 What number $\bar{v}$ gives $\int_a^b (v(x) - \bar{v})\,dx = 0$?

18 If $f(2) = 6$ and $f(6) = 2$ then the average of df/dx from $x = 2$ to $x = 6$ is ______.

19 (a) The averages of $\cos x$ and $|\cos x|$ from 0 to π are ______.
(b) The average of the numbers $v_1, \ldots, v_n$ is ______ than the average of $|v_1|, \ldots, |v_n|$.

20 (a) Which property of integrals proves $\int_0^1 v(x)\,dx \leqslant \int_0^1 |v(x)|\,dx$?
(b) Which property proves $-\int_0^1 v(x)\,dx \leqslant \int_0^1 |v(x)|\,dx$?

Together these are ***Property* 8**: $|\int_0^1 v(x)\,dx| \leqslant \int_0^1 |v(x)|\,dx$.

21 What function has v_{ave} (from 0 to x) equal to $\frac{1}{3}v(x)$ at all x? What functions have $v_{ave} = v(x)$ at all x?

22 (a) If $v(x)$ is increasing, explain from Property **6** why $\int_0^x v(t)\,dt \leqslant xv(x)$ for $x > 0$.
(b) Take derivatives of both sides for a second proof.

23 The average of $v(x) = 1/(1 + x^2)$ on the interval $[0, b]$ approaches ______ as $b \to \infty$. The average of $V(x) = x^2/(1 + x^2)$ approaches ______.

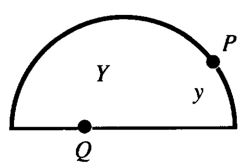
24 If the positive numbers v_n approach zero as $n \to \infty$ prove that their average $(v_1 + \cdots + v_n)/n$ also approaches zero.

25 Find the average distance from $x = a$ to points in the interval $0 \leqslant x \leqslant 2$. Is the formula different if $a < 2$?

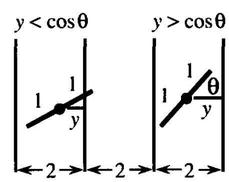
26 (Computer experiment) Choose random numbers x between 0 and 1 until the average value of x^2 is between .333 and .334. How many values of x^2 are above and below? If possible repeat ten times.

27 A point P is chosen randomly along a semicircle (see figure: equal probability for equal arcs). What is the average distance y from the x axis? The radius is 1.

28 A point Q is chosen randomly between -1 and 1.
(a) What is the average distance Y up to the semicircle?
(b) Why is this different from Problem 27?



Buffon needle



29 (A classic way to compute π) A 2″ needle is tossed onto a floor with boards 2″ wide. Find the probability of falling across a crack. (This happens when $\cos\theta > y$ = distance from midpoint of needle to nearest crack. In the rectangle $0 \leqslant \theta \leqslant \pi/2$, $0 \leqslant y \leqslant 1$, shade the part where $\cos\theta > y$ and find the fraction of area that is shaded.)

30 If Buffon's needle has length $2x$ instead of 2, find the probability $P(x)$ of falling across the same cracks.

31 If you roll *three* dice at once, what are the probabilities of each outcome between 3 and 18? What is the expected value?

32 If you choose a random point in the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, what is the chance that its coordinates have $y^2 \leqslant x$?

33 The voltage $V(t) = 220\cos 2\pi t/60$ has frequency 60 hertz and amplitude 220 volts. Find V_{ave} from 0 to t.

34 (a) Show that $v_{\text{even}}(x) = \frac{1}{2}(v(x) + v(-x))$ is always even.
(b) Show that $v_{\text{odd}}(x) = \frac{1}{2}(v(x) - v(-x))$ is always odd.

35 By Problem 34 or otherwise, write $(x+1)^3$ and $1/(x+1)$ as an even function plus an odd function.

36 Prove from the definition of df/dx that it is an odd function if $f(x)$ is even.

37 Suppose four classes have 6, 8, 10, and 40 students, averaging ________. The chance of being in the first class is ________. The expected class size (for the student) is

$$E(x) = 6\left(\frac{6}{64}\right) + 8\left(\frac{8}{64}\right) + 10\left(\frac{10}{64}\right) + 40\left(\frac{40}{64}\right) = \_\_\_\_\_\_\_\_.$$

38 With groups of sizes $x_1, \ldots, x_n$ adding to G, the average size is ________. The chance of an individual belonging to group 1 is ________. The expected size of his or her group is $E(x) = x_1(x_1/G) + \cdots + x_n(x_n/G)$. *Prove $\Sigma_1^n\, x_i^2/G \geqslant G/n$.

39 **True or false**, 15 seconds each:
(a) If $f(x) \leqslant g(x)$ then $df/dx \leqslant dg/dx$.
(b) If $df/dx \leqslant dg/dx$ then $f(x) \leqslant g(x)$.
(c) $xv(x)$ is odd if $v(x)$ is even.
(d) If $v_{\text{ave}} \leqslant w_{\text{ave}}$ on all intervals then $v(x) \leqslant w(x)$ at all points.

40 If $v(x) = \begin{cases} 2x \text{ for } x < 3 \\ -2x \text{ for } x > 3 \end{cases}$ then $f(x) = \begin{cases} x^2 \text{ for } x < 3 \\ -x^2 \text{ for } x > 3 \end{cases}$.
Thus $\int_0^4 v(x)\,dx = f(4) - f(0) = -16$. Correct the mistake.

41 If $v(x) = |x - 2|$ find $f(x)$. Compute $\int_0^5 v(x)\,dx$.

42 Why are there equal areas above and below v_{ave}?

5.7 The Fundamental Theorem and Its Applications

When the endpoints are fixed at a and b, we have a *definite* integral. When the upper limit is a variable point x, we have an *indefinite* integral. More generally: When the endpoints depend in any way on x, ***the integral is a function of*** x. Therefore we can find its derivative. This requires the Fundamental Theorem of Calculus.

The essence of the Theorem is: ***Derivative of integral of*** v ***equals*** v. We also compute the derivative when the integral goes from $a(x)$ to $b(x)$—both limits variable.

Part **2** of the Fundamental Theorem reverses the order: ***Integral of derivative of*** f ***equals*** $f + C$. That will follow quickly from Part **1**, with help from the Mean Value Theorem. It is Part **2** that we use most, since integrals are harder than derivatives.

After the proofs we go to new applications, beyond the standard problem of area under a curve. Integrals can add up rings and triangles and shells—not just rectangles. The answer can be a volume or a probability—not just an area.

THE FUNDAMENTAL THEOREM, PART 1

Start with a continuous function v. Integrate it from a fixed point a to a variable point x. For each x, this integral $f(x)$ is a number. We do not require or expect a formula for $f(x)$—it is the area out to the point x. It is a function of x! The Fundamental Theorem says that this area function has a derivative (another limiting process). ***The derivative*** df/dx ***equals the original*** $v(x)$.

5C (***Fundamental Theorem, Part*** **1**) Suppose $v(x)$ is a continuous function:

If $f(x)=\int_a^x v(t)\,dt$ *then* $df/dx=v(x)$.

The dummy variable is written as t, so we can concentrate on the limits. The value of the integral depends on the limits a and x, not on t.

To find df/dx, start with $\Delta f=f(x+\Delta x)-f(x)=$ *difference of areas*:

$$\Delta f=\int_a^{x+\Delta x} v(t)\,dt-\int_a^x v(t)\,dt=\int_x^{x+\Delta x} v(t)\,dt. \tag{1}$$

Officially, this is Property 1. The area out to $x+\Delta x$ minus the area out to x equals the small part from x to $x+\Delta x$. Now divide by Δx:

$$\frac{\Delta f}{\Delta x}=\frac{1}{\Delta x}\int_x^{x+\Delta x} v(t)\,dt=\textbf{\textit{average value}}=v(c). \tag{2}$$

This is Property 7, the Mean Value Theorem for integrals. The average value on this short interval equals $v(c)$. This point c is somewhere between x and $x+\Delta x$ (exact position not known), and we let Δx approach zero. That squeezes c toward x, so $v(c)$ approaches $v(x)$—remember that v is continuous. The limit of equation (2) is the Fundamental Theorem:

$$\frac{\Delta f}{\Delta x}\rightarrow\frac{df}{dx} \quad\text{and}\quad v(c)\rightarrow v(x) \quad\text{so}\quad \frac{df}{dx}=v(x). \tag{3}$$

If Δx is negative the reasoning still holds. Why assume that $v(x)$ is continuous? Because if v is a step function, then $f(x)$ has a corner where df/dx is not $v(x)$.

We could skip the Mean Value Theorem and simply bound v above and below:

$$\begin{aligned}
&\text{for } t \text{ between } x \text{ and } x+\Delta x: && v_{\min}\leqslant v(t)\leqslant v_{\max}\\
&\text{integrate over that interval:} && v_{\min}\Delta x\leqslant \Delta f\leqslant v_{\max}\Delta x\\
&\text{divide by } \Delta x: && v_{\min}\leqslant \Delta f/\Delta x\leqslant v_{\max}
\end{aligned} \tag{4}$$

As $\Delta x\rightarrow 0$, $v_{\min}$ and $v_{\max}$ approach $v(x)$. In the limit df/dx again equals $v(x)$.

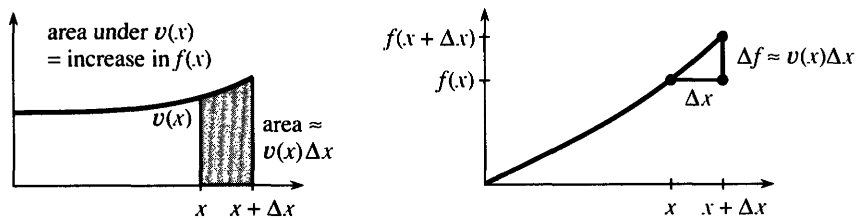


Fig. 5.14 Fundamental Theorem, Part 1: (thin area Δf)/(base length Δx) → height $v(x)$.

Graphical meaning The f-graph gives the area under the v-graph. The thin strip in Figure 5.14 has area Δf. ***That area is approximately*** $v(x)$ ***times*** Δx. Dividing by its base, $\Delta f/\Delta x$ is close to the height $v(x)$. When $\Delta x\rightarrow 0$ and the strip becomes infinitely thin, the expression "close to" converges to "equals." Then df/dx is the height at $v(x)$.

DERIVATIVES WITH VARIABLE ENDPOINTS

When the upper limit is x, the derivative is $v(x)$. Suppose the *lower limit* is x. The integral goes from x to b, instead of a to x. When x moves, the lower limit moves.

The change in area is on the left side of Figure 5.15. *As x goes forward, **area is removed.*** So there is a minus sign in the derivative of area:

$$\textit{The derivative of}\quad g(x)=\int_x^b v(t)\,dt \quad \textit{is}\quad \frac{dg}{dx}=-v(x). \tag{5}$$

The quickest proof is to reverse b and x, which reverses the sign (Property **3**):

$$g(x)=-\int_b^x v(t)\,dt \quad \text{so by Part } \mathbf{1} \quad \frac{dg}{dx}=-v(x).$$

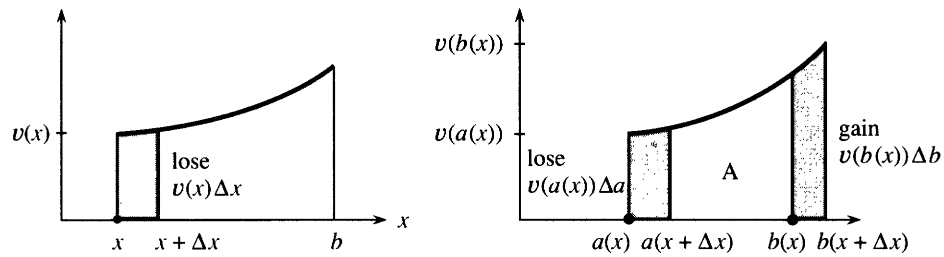


Fig. 5.15 Area from x to b has $dg/dx=-v(x)$. Area $v(b)db$ is added, area $v(a)da$ is lost

The general case is messier but not much harder (it is quite useful). Suppose *both limits* are changing. The upper limit $b(x)$ is not necessarily x, but it depends on x. The lower limit $a(x)$ can also depend on x (Figure 5.15b). The area A between those limits changes as x changes, and we want dA/dx:

$$\textbf{\textit{If}}\quad A=\int_{a(x)}^{b(x)} v(t)\,dt \quad \textbf{\textit{then}}\quad \frac{dA}{dx}=v(b(x))\,\frac{db}{dx}-v(a(x))\,\frac{da}{dx}. \tag{6}$$

The figure shows two thin strips, one added to the area and one subtracted.

First check the two cases we know. When $a=0$ and $b=x$, we have $da/dx=0$ and $db/dx=1$. The derivative according to (6) is $v(x)$ times 1—the Fundamental Theorem. The other case has $a=x$ and $b=$ constant. Then the lower limit in (6) produces $-v(x)$. When the integral goes from $a=2x$ to $b=x^3$, its derivative is new:

EXAMPLE 1 $\quad A=\int_{2x}^{x^3}\cos t\,dt=\sin x^3-\sin 2x$

$dA/dx=(\cos x^3)(3x^2)-(\cos 2x)(2).$

That fits with (6), because db/dx is $3x^2$ and da/dx is 2 (with minus sign). It also looks like the chain rule—which it is! To prove (6) we use the letters v and f:

$$A=\int_{a(x)}^{b(x)} v(t)\,dt=f(b(x))-f(a(x)) \qquad \text{(by Part } \mathbf{2} \text{ below)}$$

$$\frac{dA}{dx}=f'(b(x))\,\frac{db}{dx}-f'(a(x))\,\frac{da}{dx} \qquad \text{(by the chain rule)}$$

Since $f'=v$, equation (6) is proved. In the next example the area turns out to be constant, although it seems to depend on x. Note that $v(t)=1/t$ so $v(3x)=1/3x$.

EXAMPLE 2 $\quad A=\int_{2x}^{3x}\frac{1}{t}\,dt$ has $\dfrac{dA}{dx}=\left(\dfrac{1}{3x}\right)(3)-\left(\dfrac{1}{2x}\right)(2)=0.$

Question $A = \int_{-x}^{x} v(t)\,dt$ has $\frac{dA}{dx} = v(x) + v(-x)$. *Why does $v(-x)$ have a plus sign?*

THE FUNDAMENTAL THEOREM, PART 2

We have used a hundred times the Theorem that is now to be proved. It is the key to integration. "*The integral of df/dx is $f(x)+C$.*" The application starts with $v(x)$. We search for an $f(x)$ with this derivative. If $df/dx = v(x)$, the Theorem says that

$$\int v(x)\,dx = \int \frac{df}{dx}\,dx = f(x) + C.$$

We can't rely on knowing formulas for v and f—only the definitions of $\int$ and d/dx.

The proof rests on one extremely special case: df/dx is the ***zero function***. We easily find $f(x) = constant$. The problem is to prove that there are no other possibilities: f *must* be constant. When the slope is zero, the graph *must* be flat. Everybody knows this is true, but intuition is not the same as proof.

Assume that $df/dx = 0$ ***in an interval***. If $f(x)$ is not constant, there are points where $f(a) \neq f(b)$. By the Mean Value Theorem, there is a point c where

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{(this is not zero because } f(a) \neq f(b)\text{)}.$$

But $f'(c) \neq 0$ directly contradicts $df/dx = 0$. Therefore $f(x)$ must be constant.

Note the crucial role of the Mean Value Theorem. A *local* hypothesis ($df/dx = 0$ at each point) yields a *global* conclusion ($f =$ constant in the whole interval). The derivative narrows the field of view, the integral widens it. The Mean Value Theorem connects instantaneous to average, local to global, points to intervals. This special case (the zero function) applies when $A(x)$ and $f(x)$ have the same derivative:

$$\textbf{\textit{If }} dA/dx = df/dx \textbf{\textit{ on an interval, then }} A(x) = f(x) + C. \tag{7}$$

Reason: The derivative of $A(x) - f(x)$ is zero. So $A(x) - f(x)$ must be constant.

Now comes the big theorem. It assumes that $v(x)$ is continuous, and integrates using $f(x)$:

> **5D** (***Fundamental Theorem, Part 2***) If $v(x) = \frac{df}{dx}$ then $\int_a^b v(x)\,dx = f(b) - f(a)$.

Proof The antiderivative is $f(x)$. But Part **1** gave another antiderivative for the same $v(x)$. It was the integral—constructed from rectangles and now called $A(x)$:

$$A(x) = \int_a^x v(t)\,dt \quad \textbf{\textit{also has}} \quad \frac{dA}{dx} = v(x).$$

Since $A' = v$ and $f' = v$, the special case in equation (7) states that $A(x) = f(x) + C$. That is the essential point: ***The integral from rectangles equals*** $f(x) + C$.

At the lower limit the area integral is $A = 0$. So $f(a) + C = 0$. At the upper limit $f(b) + C = A(b)$. Subtract to find $A(b)$, the definite integral:

$$A(b) = \int_a^b v(x)\,dx = f(b) - f(a).$$

Calculus is beautiful—its Fundamental Theorem is also its most useful theorem.

Another proof of Part **2** starts with $f' = v$ and looks at subintervals:

$$f(x_1) - f(a) = v(x_1^*)(x_1 - a) \qquad \text{(by the Mean Value Theorem)}$$
$$f(x_2) - f(x_1) = v(x_2^*)(x_2 - x_1) \qquad \text{(by the Mean Value Theorem)}$$
$$\cdots = \cdots$$
$$f(b) - f(x_{n-1}) = v(x_n^*)(b - x_{n-1}) \qquad \text{(by the Mean Value Theorem).}$$

The left sides add to $f(b) - f(a)$. The sum on the right, as $\Delta x \to 0$, is $\int_a^b v(x)\,dx$.

APPLICATIONS OF INTEGRATION

Up to now the integral has been the area under a curve. There are many other applications, quite different from areas. ***Whenever addition becomes*** "*continuous,*" ***we have integrals instead of sums.*** Chapter 8 has space to develop more applications, but four examples can be given immediately—which will make the point.

We stay with geometric problems, rather than launching into physics or engineering or biology or economics. All those will come. The goal here is to take a first step away from rectangles.

EXAMPLE 3 (for circles) ***The area*** A ***and circumference*** C ***are related by*** $dA/dr = C$.

The question is why. The area is πr^2. Its derivative $2\pi r$ is the circumference. By the Fundamental Theorem, the integral of C is A. What is missing is the geometrical reason. Certainly πr^2 is the integral of $2\pi r$, but what is the *real* explanation for $A = \int C(r)\,dr$?

My point is that *the pieces are not rectangles.* We could squeeze rectangles under a circular curve, but their heights would have nothing to do with C. Our intuition has to take a completely different direction, and add up the *thin rings* in Figure 5.16.

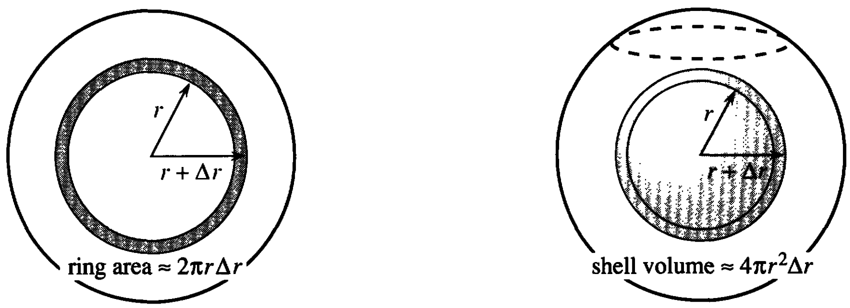


Fig. 5.16 Area of circle = integral over rings. Volume of sphere = integral over shells.

Suppose the ring thickness is Δr. Then the ring area is close to C times Δr. This is precisely the kind of approximation we need, because its error is of higher order $(\Delta r)^2$. *The integral adds ring areas* just as it added rectangular areas:

$$A = \int_0^r C\,dr = \int_0^r 2\pi r\,dr = \pi r^2.$$

That is our first step toward freedom, away from rectangles to rings.

The ring area ΔA can be checked exactly—it is the difference of circles:

$$\Delta A = \pi(r+\Delta r)^2 - \pi r^2 = 2\pi r\, \Delta r + \pi(\Delta r)^2.$$

This is $C\Delta r$ plus a correction. Dividing both sides by $\Delta r \to 0$ leaves $dA/dr = C$.

Finally there is a geometrical reason. The ring unwinds into a thin strip. Its width is Δr and its length is close to C. The inside and outside circles have different perimeters, so this is not a true rectangle—but the area is near $C\Delta r$.

EXAMPLE 4 For a sphere, surface area and volume satisfy $A = dV/dr$.

What worked for circles will work for spheres. The thin rings become *thin shells.* A shell goes from radius r to radius $r + \Delta r$, so its thickness is Δr. We want the volume of the shell, but we don't need it exactly. The surface area is $4\pi r^2$, so the volume is about $4\pi r^2 \Delta r$. That is close enough!

Again we are correct except for $(\Delta r)^2$. Infinitesimally speaking $dV = A\, dr$:

$$V = \int_0^r A\, dr = \int_0^r 4\pi r^2\, dr = \frac{4}{3}\pi r^3.$$

This is the volume of a sphere. The derivative of V is A, and the shells explain why. Main point: ***Integration is not restricted to rectangles.***

EXAMPLE 5 The distance around a square is $4s$. Why does the area have $dA/ds = 2s$?

The side is s and the area is s^2. Its derivative $2s$ goes only ***half way around the square.*** I tried to understand that by drawing a figure. Normally this works, but in the figure dA/ds looks like $4s$. Something is wrong. The bell is ringing so I leave this as an exercise.

EXAMPLE 6 Find the area under $v(x) = \cos^{-1} x$ from $x = 0$ to $x = 1$.

That is a conventional problem, but we have no antiderivative for $\cos^{-1} x$. We could look harder, and find one. However there is another solution—unconventional but correct. ***The region can be filled with horizontal rectangles*** (not vertical rectangles). Figure 5.17b shows a typical strip of length $x = \cos v$ (the curve has $v = \cos^{-1} x$). As the thickness Δv approaches zero, the total area becomes $\int x\, dv$. We are integrating upward, so *the limits are on v not on x*:

$$\text{area} = \int_0^{\pi/2} \cos v\, dv = \sin v\Big]_0^{\pi/2} = 1.$$

The exercises ask you to set up other integrals—not always with rectangles. Archimedes used triangles instead of rings to find the area of a circle.

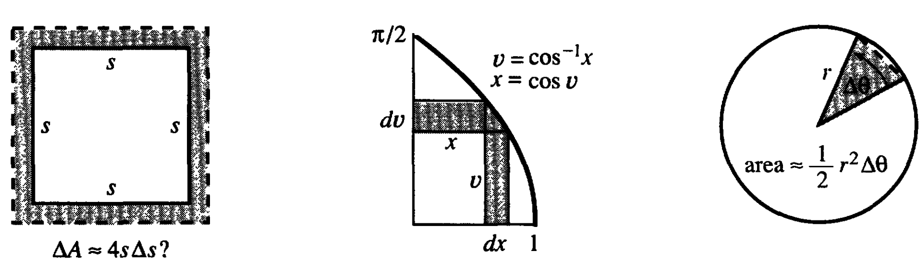


Fig. 5.17 Trouble with a square. Success with horizontal strips and triangles.

5.7 EXERCISES

Read-through questions

The area $f(x) = \int_a^x v(t)\,dt$ is a function of __a__. By Part 1 of the Fundamental Theorem, its derivative is __b__. In the proof, a small change Δx produces the area of a thin __c__. This area Δf is approximately __d__ times __e__. So the derivative of $\int_a^x t^2\,dt$ is __f__.

The integral $\int_x^b t^2\,dt$ has derivative __g__. The minus sign is because __h__. When both limits $a(x)$ and $b(x)$ depend on x, the formula for df/dx becomes __i__ minus __j__. In the example $\int_2^{3x} t\,dt$, the derivative is __k__.

By Part 2 of the Fundamental Theorem, the integral of df/dx is __l__. In the special case when $df/dx = 0$, this says that __m__. From this special case we conclude: If $dA/dx = dB/dx$ then $A(x) =$ __n__. If an antiderivative of $1/x$ is $\ln x$ (whatever that is), then automatically $\int_a^b dx/x =$ __o__.

The square $0 \leqslant x \leqslant s$, $0 \leqslant y \leqslant s$ has area $A =$ __p__. If s is increased by Δs, the extra area has the shape of __q__. That area ΔA is approximately __r__. So $dA/ds =$ __s__.

Find the derivatives of the following functions $F(x)$.

1 $\int_1^x \cos^2 t\,dt$

2 $\int_x^1 \cos 3t\,dt$

3 $\int_0^2 t^n\,dt$

4 $\int_0^2 x^n\,dt$

5 $\int_1^{x^2} u^3\,du$

6 $\int_{-x}^{x/2} v(u)\,du$

7 $\int_x^{x+1} v(t)\,dt$ (a "*running average*" of v)

8 $\dfrac{1}{x}\displaystyle\int_0^x v(t)\,dt$ (the average of v; use product rule)

9 $\dfrac{1}{x}\displaystyle\int_0^x \sin^2 t\,dt$

10 $\dfrac{1}{2}\displaystyle\int_x^{x+2} t^3\,dt$

11 $\int_0^x [\int_0^t v(u)\,du]\,dt$

12 $\int_0^x (df/dt)^2\,dt$

13 $\int_0^x v(t)\,dt + \int_x^1 v(t)\,dt$

14 $\int_0^x v(-t)\,dt$

15 $\int_{-x}^x \sin t^2\,dt$

16 $\int_{-x}^x \sin t\,dt$

17 $\int_0^x u(t)v(t)\,dt$

18 $\int_{a(x)}^{b(x)} 5\,dt$

19 $\displaystyle\int_0^{\sin x} \sin^{-1} t\,dt$

20 $\displaystyle\int_0^{f(x)} \frac{df}{dt}\,dt$

21 **True or false**

(a) If $df/dx = dg/dx$ then $f(x) = g(x)$.

(b) If $d^2f/dx^2 = d^2g/dx^2$ then $f(x) = g(x) + C$.

(c) If $3 > x$ then the derivative of $\int_3^x v(t)\,dt$ is $-v(x)$.

(d) The derivative of $\int_1^3 v(x)\,dx$ is zero.

22 For $F(x) = \int_x^{2x} \sin t\,dt$, locate $F(\pi + \Delta x) - F(\pi)$ on a sine graph. Where is $F(\Delta x) - F(0)$?

23 Find the function $v(x)$ whose average value between 0 and x is $\cos x$. Start from $\int_0^x v(t)\,dt = x \cos x$.

24 Suppose $df/dx = 2x$. We know that $d(x^2)/dx = 2x$. How do we prove that $f(x) = x^2 + C$?

25 If $\int_{-x}^0 v(t)\,dt = \int_0^x v(t)\,dt$ (equal areas left and right of zero), then $v(x)$ is an _______ function. Take derivatives to prove it.

26 Example 2 said that $\int_{2x}^{3x} dt/t$ does not really depend on x (or t!). Substitute xu for t and watch the limits on u.

27 **True or false**, with reason:

(a) All continuous functions have derivatives.

(b) All continuous functions have antiderivatives.

(c) All antiderivatives have derivatives.

(d) $A(x) = \int_{2x}^{3x} dt/t^2$ has $dA/dx = 0$.

Find $\int_1^x v(t)\,dt$ from the facts in 28–29.

28 $\dfrac{d(x^n)}{dx} = v(x)$

29 $\displaystyle\int_0^x v(t)\,dt = \frac{x}{x+2}.$

30 What is wrong with Figure 5.17? It seems to show that $dA = 4s\,ds$, which would mean $A = \int 4s\,ds = 2s^2$.

31 The cube $0 \leqslant x, y, z \leqslant s$ has volume $V =$ _______. The three square faces with $x = s$ or $y = s$ or $z = s$ have total area $A =$ _______. If s is increased by Δs, the extra volume has the shape of _______. That volume ΔV is approximately _______. So $dV/ds =$ _______.

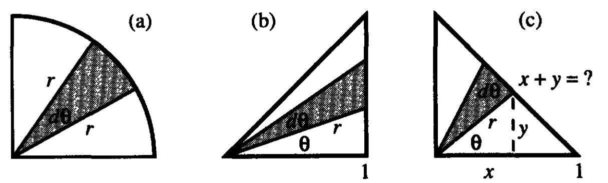
32 The four-dimensional cube $0 \leqslant x, y, z, t \leqslant s$ has hypervolume $H =$ _______. The face with $x = s$ is really a _______. Its volume is $V =$ _______. The total volume of the four faces with $x = s$, $y = s$, $z = s$, or $t = s$ is _______. When s is increased by Δs, the extra hypervolume is $\Delta H \approx$ _______. So $dH/ds =$ _______.

33 The hypervolume of a four-dimensional sphere is $H = \frac{1}{2}\pi^2 r^4$. Therefore the area (volume?) of its three-dimensional surface $x^2 + y^2 + z^2 + t^2 = r^2$ is _______.

34 The area above the parabola $y = x^2$ from $x = 0$ to $x = 1$ is $\frac{2}{3}$. Draw a figure with horizontal strips and integrate.

35 The wedge in Figure (a) has area $\frac{1}{2}r^2\,d\theta$. One reason: It is a fraction $d\theta/2\pi$ of the total area πr^2. Another reason: It is close to a triangle with small base $r\,d\theta$ and height _______. Integrating $\frac{1}{2}r^2\,d\theta$ from $\theta = 0$ to $\theta =$ _______ gives the area _______ of a quarter-circle.

36 $A = \int_0^r \sqrt{r^2 - x^2}\,dx$ is also the area of a quarter-circle. Show why, with a graph and thin rectangles. Calculate this integral by substituting $x = r \sin\theta$ and $dx = r\cos\theta\,d\theta$.



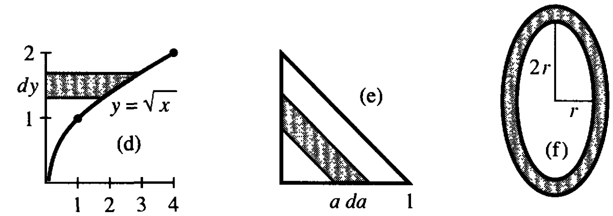
37 The distance r in Figure (b) is related to θ by $r =$ ________. Therefore the area of the thin triangle is $\frac{1}{2}r^2\,d\theta =$ ________. Integration to $\theta =$ ________ gives the total area $\frac{1}{2}$.

38 The x and y coordinates in Figure (c) add to $r\cos\theta + r\sin\theta =$ ________. Without integrating explain why

$$\int_0^{\pi/2} \frac{d\theta}{(\cos\theta + \sin\theta)^2} = 1.$$

39 The horizontal strip at height y in Figure (d) has width dy and length $x =$ ________. So the area up to $y = 2$ is ________. What length are the vertical strips that give the same area?

40 Use thin rings to find the area between the circles $r = 2$ and $r = 3$. Draw a picture to show why thin rectangles would be extra difficult.



41 The length of the strip in Figure (e) is approximately ________. The width is ________. Therefore the triangle has area $\int_0^1$ ________ da (do you get $\frac{1}{2}$?).

42 The area of the ellipse in Figure (f) is $2\pi r^2$. Its derivative is $4\pi r$. But this is not the correct perimeter. Where does the usual reasoning go wrong?

43 The derivative of the integral of $v(x)$ is $v(x)$. What is the corresponding statement for sums and differences of the numbers v_j? Prove that statement.

44 The integral of the derivative of $f(x)$ is $f(x) + C$. What is the corresponding statement for sums of differences of f_j? Prove that statement.

45 Does $d^2f/dx^2 = a(x)$ lead to $\int_0^1 (\int_0^x a(t)\,dt)\,dx = f(1) - f(0)$?

46 The mountain $y = -x^2 + t$ has an area $A(t)$ above the x axis. As t increases so does the area. Draw an xy graph of the mountain at $t = 1$. What line gives dA/dt? Show with words or derivatives that $d^2A/dt^2 > 0$.

5.8 Numerical Integration

This section concentrates on definite integrals. The inputs are $y(x)$ and two endpoints a and b. The output is the integral I. Our goal is to find that number $\int_a^b y(x)\,dx = I$, accurately and in a short time. Normally this goal is achievable—as soon as we have a good method for computing integrals.

Our two approaches so far have weaknesses. The search for an antiderivative succeeds in important cases, and Chapter 7 extends that range—but generally $f(x)$ is not available. The other approach (by rectangles) is in the right direction but too crude. The height is set by $y(x)$ at the right and left end of each small interval. The ***right and left rectangle rules*** add the areas (Δx times y):

$$R_n = (\Delta x)(y_1 + y_2 + \cdots + y_n) \quad \text{and} \quad L_n = (\Delta x)(y_0 + y_1 + \cdots + y_{n-1}).$$

The value of $y(x)$ at the end of interval j is y_j. The extreme left value $y_0 = y(a)$ enters L_n. With n equal intervals of length $\Delta x = (b - a)/n$, the extreme right value is $y_n = y(b)$. It enters R_n. Otherwise the sums are the same—simple to compute, easy to visualize, but very inaccurate.

This section goes from slow methods (*rectangles*) to better methods (*trapezoidal and midpoint*) to good methods (*Simpson and Gauss*). Each improvement cuts down the error. You could discover the formulas without the book, by integrating x and

x^2 and x^4. The rule R_n would come out on one side of the answer, and L_n would be on the other side. You would figure out what to do next, to come closer to the exact integral. The book can emphasize one key point:

> ***The quality of a formula depends on how many integrals*** $\int 1\,dx, \int x\,dx, \int x^2\,dx, \ldots,$ ***it computes exactly. If*** $\int x^p\,dx$ ***is the first to be wrong, the order of accuracy*** is p.

By testing the integrals of $1, x, x^2, \ldots,$ we decide how accurate the formulas are.

Figure 5.18 shows the rectangle rules R_n and L_n. ***They are already wrong when*** $y = x$. These methods are ***first-order***: $p = 1$. The errors involve the first power of Δx—where we would much prefer a higher power. A larger p in $(\Delta x)^p$ means a smaller error.

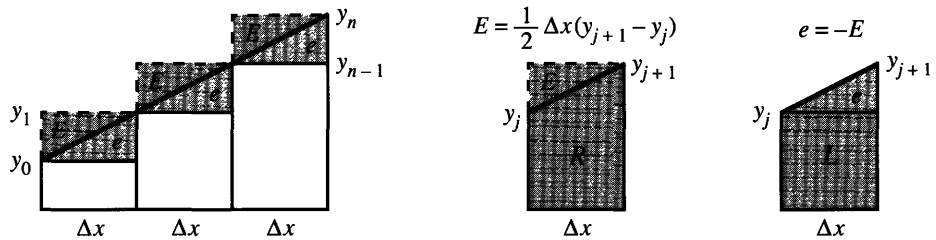


Fig. 5.18 Errors E and e in R_n and L_n are the areas of triangles.

When the graph of $y(x)$ is a straight line, the integral I is known. The error triangles E and e have base Δx. Their heights are the differences $y_{j+1} - y_j$. The areas are $\frac{1}{2}$(base)(height), and the only difference is a minus sign. (L is too low, so the error $L - I$ is negative.) The total error in R_n is the sum of the E's:

$$R_n - I = \tfrac{1}{2}\Delta x(y_1 - y_0) + \cdots + \tfrac{1}{2}\Delta x(y_n - y_{n-1}) = \tfrac{1}{2}\Delta x(y_n - y_0). \tag{1}$$

All y's between y_0 and y_n cancel. Similarly for the sum of the e's:

$$L_n - I = -\tfrac{1}{2}\Delta x(y_n - y_0) = -\tfrac{1}{2}\Delta x[y(b) - y(a)]. \tag{2}$$

The greater the slope of $y(x)$, the greater the error—since rectangles have zero slope.

Formulas (1) and (2) are nice—*but those errors are large*. To integrate $y = x$ from $a = 0$ to $b = 1$, the error is $\frac{1}{2}\Delta x(1 - 0)$. It takes 500,000 rectangles to reduce this error to 1/1,000,000. This accuracy is reasonable, but that many rectangles is unacceptable.

The beauty of the error formulas is that they are "*asymptotically correct*" for all functions. When the graph is curved, the errors don't fit exactly into triangles. But the ratio of predicted error to actual error approaches 1. As $\Delta x \to 0$, the graph is almost straight in each interval—this is linear approximation.

The error prediction $\frac{1}{2}\Delta x[y(b) - y(a)]$ is so simple that we test it on $y(x) = \sqrt{x}$:

$I = \int_0^1 \sqrt{x}\,dx = \frac{2}{3}$	$n =$	1	10	100	1000
	error $R_n - I =$	.33	.044	.0048	.00049
	error $L_n - I =$	−.67	−.056	−.0052	−.00051

The error decreases along each row. So does $\Delta x = .1, .01, .001, .0001$. Multiplying n by 10 divides Δx by 10. The error is also divided by 10 (almost). ***The error is nearly proportional to*** Δx—typical of first-order methods.

The predicted error is $\frac{1}{2}\Delta x$, since here $y(1) = 1$ and $y(0) = 0$. The computed errors in the table come closer and closer to $\frac{1}{2}\Delta x = .5, .05, .005, .0005$. The prediction is the "leading term" in the actual error.

The table also shows a curious fact. Subtracting the last row from the row above gives exact numbers 1, .1, .01, and .001. This is $(R_n - I) - (L_n - I)$, which is $R_n - L_n$. It comes from an extra rectangle at the right, included in R_n but not L_n. Its height is 1 and its area is 1, .1, .01, .001.

The errors in R_n and L_n almost cancel. The average $T_n = \frac{1}{2}(R_n + L_n)$ has less error—it is the "trapezoidal rule." First we give the rectangle prediction two final tests:

		$n=1$	$n=10$	$n=100$	$n=1000$
$\int (x^2 - x)\,dx$:	errors	$1.7 \cdot 10^{-1}$	$1.7 \cdot 10^{-3}$	$1.7 \cdot 10^{-5}$	$1.7 \cdot 10^{-7}$
$\int dx/(10 + \cos 2\pi x)$:	errors	$-1 \cdot 10^{-3}$	$2 \cdot 10^{-14}$	"0"	"0"

Those errors are falling *faster* than Δx. For $y = x^2 - x$ the prediction explains why: $y(0)$ equals $y(1)$. The leading term, with $y(b)$ minus $y(a)$, is *zero*. The exact errors are $\frac{1}{6}(\Delta x)^2$, dropping from 10^{-1} to 10^{-3} to 10^{-5} to 10^{-7}. In these examples L_n is identical to R_n (and also to T_n), because the end rectangles are the same. We will see these $\frac{1}{6}(\Delta x)^2$ errors in the trapezoidal rule.

The last row in the table is more unusual. It shows practically no error. Why do the rectangle rules suddenly achieve such an outstanding success?

The reason is that $y(x) = 1/(10 + \cos 2\pi x)$ is ***periodic***. The leading term in the error is zero, because $y(0) = y(1)$. Also the next term will be zero, because $y'(0) = y'(1)$. Every power of Δx is multiplied by zero, when we integrate over a complete period. So the errors go to zero exponentially fast.

Personal note I tried to integrate $1/(10 + \cos 2\pi x)$ by hand and failed. Then I was embarrassed to discover the answer in my book on applied mathematics. The method was a special trick using complex numbers, which applies over an exact period. Finally I found the antiderivative (quite complicated) in a handbook of integrals, and verified the area $1/\sqrt{99}$.

THE TRAPEZOIDAL AND MIDPOINT RULES

We move to integration formulas that are exact when $y = x$. They have ***second-order accuracy***. The Δx error term disappears. The formulas give the correct area under straight lines. The predicted error is a multiple of $(\Delta x)^2$. That multiple is found by testing $y = x^2$—for which the answers are not exact.

The first formula combines R_n and L_n. To cancel as much error as possible, take the average $\frac{1}{2}(R_n + L_n)$. This yields the ***trapezoidal rule***, which approximates $\int y(x)\,dx$ by T_n:

$$T_n = \tfrac{1}{2}R_n + \tfrac{1}{2}L_n = \Delta x(\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \tfrac{1}{2}y_n). \qquad (3)$$

Another way to find T_n is from the area of the "trapezoid" below $y = x$ in Figure 5.19a.

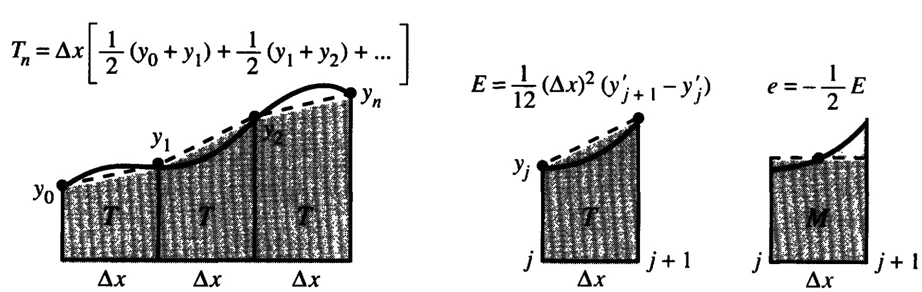


Fig. 5.19 Second-order accuracy: The error prediction is based on $v = x^2$.

The base is Δx and the sides have heights y_{j-1} and y_j. Adding those areas gives $\frac{1}{2}(L_n + R_n)$ in formula (3)—the coefficients of y_j combine into $\frac{1}{2}+\frac{1}{2}=1$. Only the first and last intervals are missing a neighbor, so the rule has $\frac{1}{2}y_0$ and $\frac{1}{2}y_n$. Because trapezoids (unlike rectangles) fit under a sloping line, T_n is exact when $y = x$.

What is the difference from rectangles? The trapezoidal rule gives weight $\frac{1}{2}\Delta x$ to y_0 and y_n. The rectangle rule R_n gives full weight Δx to y_n (and no weight to y_0). $R_n - T_n$ is exactly the leading error $\frac{1}{2}y_n - \frac{1}{2}y_0$. The change to T_n knocks out that error.

Another important formula is exact for $y(x) = x$. A rectangle has the same area as a trapezoid, if the height of the rectangle is halfway between y_{j-1} and y_j. On a straight line graph that is achieved at the *midpoint* of the interval. By evaluating $y(x)$ at the halfway points $\frac{1}{2}\Delta x, \frac{3}{2}\Delta x, \frac{5}{2}\Delta x, \ldots$, we get much better rectangles. This leads to the ***midpoint rule*** M_n:

$$M_n = \Delta x(y_{1/2} + y_{3/2} + \cdots + y_{n-1/2}). \quad (4)$$

For $\int_0^4 x\,dx$, trapezoids give $\frac{1}{2}(0)+1+2+3+\frac{1}{2}(4)=8$. The midpoint rule gives $\frac{1}{2}+\frac{3}{2}+\frac{5}{2}+\frac{7}{2}=8$, again correct. The rules become different when $y = x^2$, because $y_{1/2}$ is no longer the average of y_0 and y_1. Try both second-order rules on x^2:

$I = \int_0^1 x^2\,dx$	$n =$	1	10	100
	error $T_n - I =$	1/6	1/600	1/60000
	error $M_n - I =$	$-1/12$	$-1/1200$	$-1/120000$

The errors fall by 100 when n is multiplied by 10. The midpoint rule is twice as good ($-1/12$ vs. $1/6$). Since all smooth functions are close to parabolas (quadratic approximation in each interval), the leading errors come from Figure 5.19. The trapezoidal error is exactly $\frac{1}{6}(\Delta x)^2$ when $y(x)$ is x^2 (the 12 in the formula divides the 2 in y'):

$$T_n - I \approx \frac{1}{12}(\Delta x)^2\left[(y_1' - y_0') + \cdots + (y_n' - y_{n-1}')\right] = \frac{1}{12}(\Delta x)^2\left[y_n' - y_0'\right] \quad (5)$$

$$M_n - I \approx -\frac{1}{24}(\Delta x)^2\left[y_n' - y_0'\right] = -\frac{1}{24}(\Delta x)^2\left[y'(b) - y'(a)\right] \quad (6)$$

For exact error formulas, change $y'(b) - y'(a)$ to $(b-a)y''(c)$. The location of c is unknown (as in the Mean Value Theorem). In practice these formulas are not much used—they involve the pth derivative at an unknown location c. The main point about the error is the factor $(\Delta x)^p$.

One crucial fact is easy to overlook in our tests. *Each value of* $y(x)$ *can be extremely hard to compute.* Every time a formula asks for y_j, a computer calls a subroutine. The goal of numerical integration is to get below the error tolerance, while calling for *a minimum number of values of y*. Second-order rules need about a thousand values for a typical tolerance of 10^{-6}. The next methods are better.

FOURTH-ORDER RULE: SIMPSON

The trapezoidal error is nearly twice the midpoint error (1/6 vs. $-1/12$). So a good combination will have twice as much of M_n as T_n. That is ***Simpson's rule***:

$$S_n = \frac{1}{3}T_n + \frac{2}{3}M_n = \frac{1}{6}\Delta x\left[y_0 + 4y_{1/2} + 2y_1 + 4y_{3/2} + 2y_2 + \cdots + 4y_{n-1/2} + y_n\right]. \quad (7)$$

Multiply the midpoint values by $2/3 = 4/6$. The endpoint values are multiplied by

2/6, except at the far ends a and b (with heights y_0 and y_n). This 1–4–2–4–2–4–1 pattern has become famous.

Simpson's rule goes deeper than a combination of T and M. It comes from a *parabolic* approximation to $y(x)$ in each interval. When a parabola goes through y_0, $y_{1/2}$, y_1, the area under it is $\frac{1}{6}\Delta x(y_0 + 4y_{1/2} + y_1)$. This is S over the first interval. ***All our rules are constructed this way*: *Integrate correctly as many powers*** $1, x, x^2, \ldots$ ***as possible***. Parabolas are better than straight lines, which are better than flat pieces. S beats M, which beats R. Check Simpson's rule on powers of x, with $\Delta x = 1/n$:

	$n=1$	$n=10$	$n=100$
error if $y=x^2$	0	0	0
error if $y=x^3$	0	0	0
error if $y=x^4$	$8.33 \cdot 10^{-3}$	$8.33 \cdot 10^{-7}$	$8.33 \cdot 10^{-11}$

Exact answers for x^2 are no surprise. S_n was selected to get parabolas right. But the zero errors for x^3 were not expected. The accuracy has jumped to ***fourth order***, with errors proportional to $(\Delta x)^4$. That explains the popularity of Simpson's rule.

To understand why x^3 is integrated exactly, look at the interval $[-1, 1]$. The odd function x^3 has zero integral, and Simpson agrees by symmetry:

$$\int_{-1}^{1} x^3\,dx = \frac{1}{4}x^4\Big]_{-1}^{1} = 0 \quad \text{and} \quad \frac{2}{6}\Big[(-1)^3 + 4(0)^3 + 1^3\Big] = 0. \tag{8}$$

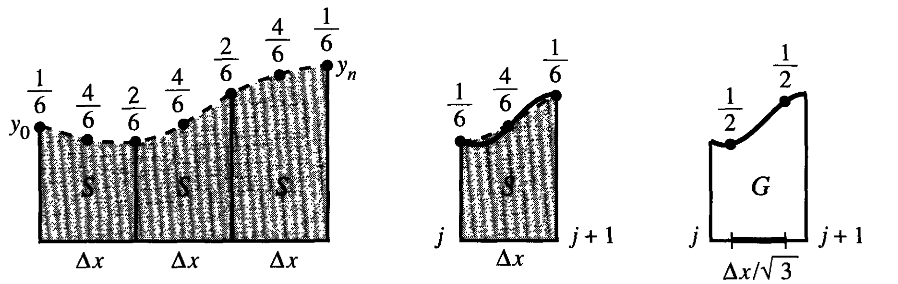


Fig. 5.20 Simpson versus Gauss: $E = c(\Delta x)^4(y'''_{j+1} - y'''_j)$ with $c_S = 1/2880$ and $c_G = -1/4320$.

THE GAUSS RULE (OPTIONAL)

We need a competitor for Simpson, and Gauss can compete with anybody. He calculated integrals in astronomy, and discovered that *two points are enough for a fourth-order method*. From -1 to 1 (a single interval) his rule is

$$\int_{-1}^{1} y(x)\,dx \approx y(-1/\sqrt{3}) + y(1/\sqrt{3}). \tag{9}$$

Those "***Gauss points***" $x = -1/\sqrt{3}$ and $x = 1/\sqrt{3}$ can be found directly. By placing them symmetrically, all odd powers $x, x^3, \ldots$ are correctly integrated. The key is in $y = x^2$, whose integral is 2/3. The Gauss points $-x_G$ and $+x_G$ get this integral right:

$$\frac{2}{3} = (-x_G)^2 + (x_G)^2, \text{ so } x_G^2 = \frac{1}{3} \quad \text{and} \quad x_G = \pm\frac{1}{\sqrt{3}}.$$

Figure 5.20c shifts to the interval from 0 to Δx. The Gauss points are $(1 \pm 1/\sqrt{3})\Delta x/2$. They are not as convenient as Simpson's (which hand calculators prefer). Gauss is good for thousands of integrations over one interval. Simpson is good when intervals go back to back—then Simpson also uses two y's per interval. For $y = x^4$, you see both errors drop by 10^{-4} in comparing $n = 1$ to $n = 10$:

$I = \int_0^1 x^4\,dx$	Simpson error	$8.33 \cdot 10^{-3}$	$8.33 \cdot 10^{-7}$
	Gauss error	$-5.56 \cdot 10^{-3}$	$-5.56 \cdot 10^{-7}$

DEFINITE INTEGRALS ON A CALCULATOR

It is fascinating to know how numerical integration is actually done. The points are not equally spaced! For an integral from 0 to 1, Hewlett-Packard machines might internally replace x by $3u^2 - 2u^3$ (the limits on u are also 0 and 1). The machine remembers to change dx. For example,

$$\int_0^1 \frac{dx}{\sqrt{x}} \quad \text{becomes} \quad \int_0^1 \frac{6(u-u^2)\,du}{\sqrt{3u^2-2u^3}} = \int_0^1 \frac{6(1-u)\,du}{\sqrt{3-2u}}.$$

Algebraically that looks worse—but the infinite value of $1/\sqrt{x}$ at $x = 0$ disappears at $u = 0$. The differential $6(u - u^2)\,du$ was chosen to vanish at $u = 0$ and $u = 1$. We don't need $y(x)$ at the endpoints—where infinity is most common. In the u variable the integration points are equally spaced—therefore in x they are not.

When a difficult point is *inside* $[a, b]$, break the interval in two pieces. And chop off integrals that go out to infinity. The integral of e^{-x^2} should be stopped by $x = 10$, since the tail is so thin. (It is bad to go too far.) Rapid oscillations are among the toughest—the answer depends on cancellation of highs and lows, and the calculator requires many integration points.

The change from x to u affects periodic functions. I thought equal spacing was good, since $1/(10 + \cos 2\pi x)$ was integrated above to enormous accuracy. But there is a danger called ***aliasing***. If $\sin 8\pi x$ is sampled with $\Delta x = 1/8$, it is always zero. A high frequency 8 is confused with a low frequency 0 (its "alias" which agrees at the sample points). With unequal spacing the problem disappears. *Notice how any integration method can be deceived*:

> Ask for the integral of $y = 0$ and specify the accuracy. The calculator samples y at $x_1, \ldots, x_k$. (With a PAUSE key, the x's may be displayed.) Then integrate $Y(x) = (x - x_1)^2 \cdots (x - x_k)^2$. That also returns the answer zero (now wrong), because the calculator follows the same steps.

On the HP-28S you enter the function, the endpoints, and the accuracy. The variable x can be named or not (see the margin). The outputs 4.67077 and 4.7E-5 are the requested integral $\int_1^2 e^x\,dx$ and the estimated error bound. Your input accuracy .00001 guarantees

```
3: 'EXP(X)'
2: {X 1 2}
1: .00001
```

$$\text{relative error in } y = \left|\frac{\text{true } y - \text{computed } y}{\text{computed } y}\right| \leqslant .00001.$$

```
3: ((EXP))
2: {1 2}
1: .00001
```

The machine estimates accuracy based on its experience in sampling $y(x)$. If you guarantee e^x within .00000000001, it thinks you want high accuracy and takes longer.

In consulting for HP, William Kahan chose formulas using 1, 3, 7, 15, ... sample points. Each new formula uses the samples in the previous formula. The calculator stops when answers are close. The last paragraphs are based on Kahan's work.

TI-81 Program to Test the Integration Methods L, R, T, M, S

```
PrgmI:NUM INT
:Disp "A="
:Input A
:Disp "B="
:Input B
:Lbl N
:Disp "N="
:Input N
:(B-A)/N→D
:D/2→H
:A→X
:Y1→L
:1→J
:Ø→R
:Ø→M
:Lbl 1
:X+H→X
:M+Y1→M
:A+JD→X
:R+Y1→R
:IS>(J,N)
:Goto 1
:(L+R-Y1)D→L
:RD→R
:MD→M
:(L+R)/2→T
:(2M+T)/3→S
:Disp "L, R, M,
   T, S"
:Disp L
:Disp R
:Disp M
:Disp T
:Disp S
:Pause
:Goto N
```

Place the integrand $y(x)$ in the Y1 position on the Y= function edit screen. Execute this program, indicating the interval $[A, B]$ and the number of subintervals N. Rules L and R and M use N evaluations of $y(x)$. The trapezoidal rule uses $N + 1$ and Simpson's rule uses $2N + 1$. The program pauses to display the results. Press ENTER to continue by choosing a different N. The program never terminates (only pauses). You break out by pressing ON. Don't forget that IS, Goto, ... are on menus.

5.8 EXERCISES

Read-through questions

To integrate $y(x)$, divide $[a, b]$ into n pieces of length $\Delta x =$ ___a___. R_n and L_n place a ___b___ over each piece, using the height at the right or ___c___ endpoint: $R_n = \Delta x(y_1 + \cdots + y_n)$ and $L_n =$ ___d___. These are ___e___ order methods, because they are incorrect for $y =$ ___f___. The total error on $[0,1]$ is approximately ___g___. For $y = \cos \pi x$ this leading term is ___h___. For $y = \cos 2\pi x$ the error is very small because $[0, 1]$ is a complete ___i___.

A much better method is $T_n = \frac{1}{2}R_n +$ ___j___ $= \Delta x[\frac{1}{2}y_0 +$ ___k___ $y_1 + \cdots +$ ___l___ $y_n]$. This ___m___ rule is ___n___-order because the error for $y = x$ is ___o___. The error for $y = x^2$ from a to b is ___p___. The ___q___ rule is twice as accurate, using $M_n = \Delta x[$ ___r___ $]$.

Simpson's method is $S_n = \frac{2}{3}M_n +$ ___s___. It is ___t___-order, because the powers ___u___ are integrated correctly. The coefficients of $y_0, y_{1/2}, y_1$ are ___v___ times Δx. Over three intervals the weights are $\Delta x/6$ times 1–4–___w___. Gauss uses ___x___ points in each interval, separated by $\Delta x/\sqrt{3}$. For a method of order p the error is nearly proportional to ___y___.

1 What is the difference $L_n - T_n$? Compare with the leading error term in (2).

2 If you cut Δx in half, by what factor is the trapezoidal error reduced (approximately)? By what factor is the error in Simpson's rule reduced?

3 Compute R_n and L_n for $\int_0^1 x^3\,dx$ and $n = 1, 2, 10$. Either verify (with computer) or use (without computer) the formula $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$.

4 One way to compute T_n is by averaging $\frac{1}{2}(L_n + R_n)$. Another way is to add $\frac{1}{2}y_0 + y_1 + \cdots + \frac{1}{2}y_n$. Which is more efficient? Compare the number of operations.

5 Test three different rules on $I = \int_0^1 x^4\,dx$ for $n = 2, 4, 8$.

6 Compute π to six places as $4\int_0^1 dx/(1+x^2)$, using any rule.

7 Change Simpson's rule to $\Delta x(\frac{1}{4}y_0 + \frac{1}{2}y_{1/2} + \frac{1}{4}y_1)$ in each interval and find the order of accuracy p.

8 Demonstrate superdecay of the error when $1/(3 + \sin x)$ is integrated from 0 to 2π.

9 Check that $(\Delta x)^2(y'_{j+1} - y'_j)/12$ is the correct error for $y = 1$ and $y = x$ and $y = x^2$ from the first trapezoid ($j = 0$). Then it is correct for every parabola over every interval.

10 Repeat Problem 9 for the midpoint error $-(\Delta x)^2(y'_{j+1} - y'_j)/24$. Draw a figure to show why the rectangle M has the same area as any trapezoid through the midpoint (including the trapezoid tangent to $y(x)$).

11 In principle $\int_{-\infty}^{\infty} \sin^2 x\,dx/x^2 = \pi$. With a symbolic algebra code or an HP-28S, how many decimal places do you get? Cut off the integral to $\int_{-A}^{A}$, and test large and small A.

12 These four integrals all equal π:

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} \qquad \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx \qquad \frac{8}{3}\int_0^{\pi} \sin^4 x\,dx \qquad \int_0^{\infty} \frac{x^{-1/2}\,dx}{1+x}$$

(a) Apply the midpoint rule to two of them until $\pi \approx 3.1416$.

(b) Optional: Pick the other two and find $\pi \approx 3$.

13 To compute $\ln 2 = \int_1^2 dx/x = .69315$ with error less than .001, how many intervals should T_n need? Its leading error is $(\Delta x)^2[y'(b) - y'(a)]/12$. Test the actual error with $y = 1/x$.

14 Compare T_n with M_n for $\int_0^1 \sqrt{x}\, dx$ and $n = 1, 10, 100$. The error prediction breaks down because $y'(0) = \infty$.

15 Take $f(x) = \int_0^x y(x)\, dx$ in error formula **3R** to prove that $\int_0^{\Delta x} y(x)\, dx - y(0)\,\Delta x$ is exactly $\frac{1}{2}(\Delta x)^2 y'(c)$ for some point c.

16 For the periodic function $y(x) = 1/(2 + \cos 6\pi x)$ from -1 to 1, compare T and S and G for $n = 2$.

17 For $I = \int_0^1 \sqrt{1 - x^2}\, dx$, the leading error in the trapezoidal rule is ________. Try $n = 2, 4, 8$ to defy the prediction.

18 Change to $x = \sin\theta$, $\sqrt{1 - x^2} = \cos\theta$, $dx = \cos\theta\, d\theta$, and repeat T_4 on $\int_0^{\pi/2} \cos^2\theta\, d\theta$. What is the predicted error after the change to θ?

19 Write down the three equations $Ay(0) + By(\frac{1}{2}) + Cy(1) = I$ for the three integrals $I = \int_0^1 1\, dx$, $\int_0^1 x\, dx$, $\int_0^1 x^2\, dx$. Solve for A, B, C and name the rule.

20 Can you invent a rule using $Ay_0 + By_{1/4} + Cy_{1/2} + Dy_{3/4} + Ey_1$ to reach higher accuracy than Simpson's?

21 Show that T_n is the only combination of L_n and R_n that has second-order accuracy.

22 Calculate $\int e^{-x^2}\, dx$ with ten intervals from 0 to 5 and 0 to 20 and 0 to 400. The integral from 0 to ∞ is $\frac{1}{2}\sqrt{\pi}$. What is the best point to chop off the infinite integral?

23 The graph of $y(x) = 1/(x^2 + 10^{-10})$ has a sharp spike and a long tail. Estimate $\int_0^1 y\, dx$ from T_{10} and T_{100} (don't expect much). Then substitute $x = 10^{-5} \tan\theta$, $dx = 10^{-5} \sec^2\theta\, d\theta$ and integrate 10^5 from 0 to $\pi/4$.

24 Compute $\int_0^4 |x - \pi|\, dx$ from T_4 and compare with the divide and conquer method of separating $\int_0^\pi |x - \pi|\, dx$ from $\int_\pi^4 |x - \pi|\, dx$.

25 Find a, b, c so that $y = ax^2 + bx + c$ equals 1, 3, 7 at $x = 0, \frac{1}{2}, 1$ (three equations). Check that $\frac{1}{6}\cdot 1 + \frac{4}{6}\cdot 3 + \frac{1}{6}\cdot 7$ equals $\int_0^1 y\, dx$.

26 Find c in $S - I = c(\Delta x)^4[y'''(1) - y'''(0)]$ by taking $y = x^4$ and $\Delta x = 1$.

27 Find c in $G - I = c(\Delta x)^4[y'''(1) - y'''(-1)]$ by taking $y = x^4$, $\Delta x = 2$, and $G = (-1/\sqrt{3})^4 + (1/\sqrt{3})^4$.

28 What condition on $y(x)$ makes $L_n = R_n = T_n$ for the integral $\int_a^b y(x)\, dx$?

29 Suppose $y(x)$ is *concave up*. Show from a picture that the trapezoidal answer is too high and the midpoint answer is too low. How does $y'' > 0$ make equation (5) positive and (6) negative?