CHAPTER 6

Exponentials and Logarithms

This chapter is devoted to exponentials like 2^x and 10^x and above all e^x. The goal is to understand them, differentiate them, integrate them, solve equations with them, and invert them (to reach the logarithm). The overwhelming importance of e^x makes this a crucial chapter in pure and applied mathematics.

In the traditional order of calculus books, e^x waits until other applications of the integral are complete. I would like to explain why it is placed earlier here. I believe that the equation $dy/dx = y$ has to be emphasized above techniques of integration. The laws of nature are expressed by ***differential equations***, and at the center is e^x. Its applications are to life sciences and physical sciences and economics and engineering (and more—wherever change is influenced by the present state). The model produces a differential equation and I want to show what calculus can do.

The key is always $b^{m+n} = (b^m)(b^n)$. Section 6.1 applies that rule in three ways:

1. to understand the ***logarithm*** as the ***exponent***;
2. to draw ***graphs*** on ordinary and semilog and log-log paper;
3. to find ***derivatives***. The slope of b^x will use $b^{x+\Delta x} = (b^x)(b^{\Delta x})$.

6.1 An Overview

There is a good chance you have met logarithms. They turn multiplication into addition, which is a lot simpler. They are the basis for slide rules (not so important) and for graphs on log paper (very important). Logarithms are mirror images of exponentials—and those I know you have met.

Start with exponentials. The numbers 10 and 10^2 and 10^3 are basic to the decimal system. For completeness I also include 10^0, which is "ten to the zeroth power" or 1. ***The logarithms of those numbers are the exponents.*** The logarithms of 1 and 10 and 100 and 1000 are 0 and 1 and 2 and 3. These are logarithms "*to base* 10," because the powers are powers of 10.

Question When the base changes from 10 to b, what is the logarithm of 1?
Answer Since $b^0 = 1$, $\log_b 1$ is always *zero*. To base b, ***the logarithm of*** b^n ***is*** n.

Negative powers are also needed. The number 10^x is positive, but its exponent x can be negative. The first examples are 1/10 and 1/100, which are the same as 10^{-1} and 10^{-2}. *The logarithms are the exponents* -1 *and* -2:

$$1000 = 10^3 \quad \text{and} \quad \log 1000 = 3$$
$$1/1000 = 10^{-3} \quad \text{and} \quad \log 1/1000 = -3.$$

Multiplying 1000 times 1/1000 gives $1 = 10^0$. Adding logarithms gives $3 + (-3) = 0$. Always 10^m times 10^n equals 10^{m+n}. In particular 10^3 times 10^2 produces five tens:

$$(10)(10)(10) \text{ times } (10)(10) \text{ equals } (10)(10)(10)(10)(10) = 10^5.$$

The law for b^m times b^n extends to all exponents, as in $10^{4.6}$ times 10^{π}. Furthermore the law applies to all bases (we restrict the base to $b > 0$ and $b \neq 1$). In every case ***multiplication of numbers is addition of exponents.***

6A b^m times b^n equals b^{m+n}, so logarithms (exponents) add
b^m divided by b^n equals b^{m-n}, so logarithms (exponents) subtract

$$\log_b(yz) = \log_b y + \log_b z \quad \text{and} \quad \log_b(y/z) = \log_b y - \log_b z. \qquad (1)$$

Historical note In the days of slide rules, 1.2 and 1.3 were multiplied by sliding one edge across to 1.2 and reading the answer under 1.3. A slide rule made in Germany would give the third digit in 1.56. Its photograph shows the numbers on a log scale. The distance from 1 to 2 equals the distance from 2 to 4 and from 4 to 8. By sliding the edges, you add distances and multiply numbers.

Division goes the other way. *Notice how* $1000/10 = 100$ *matches* $3 - 1 = 2$. To divide 1.56 by 1.3, look back along line D for the answer 1.2.

The second figure, though smaller, is the important one. ***When*** x ***increases by*** 1, 2^x ***is multiplied by*** 2. *Adding to* x *multiplies* y. This rule easily gives $y = 1, 2, 4, 8$, but look ahead to calculus—which doesn't stay with whole numbers.

Calculus will add Δx. Then y is multiplied by $2^{\Delta x}$. This number is near 1. If $\Delta x = \frac{1}{10}$ then $2^{\Delta x} \approx 1.07$—the tenth root of 2. ***To find the slope, we have to consider*** $(2^{\Delta x} - 1)/\Delta x$. The limit is near $(1.07 - 1)/\frac{1}{10} = .7$, but the exact number will take time.

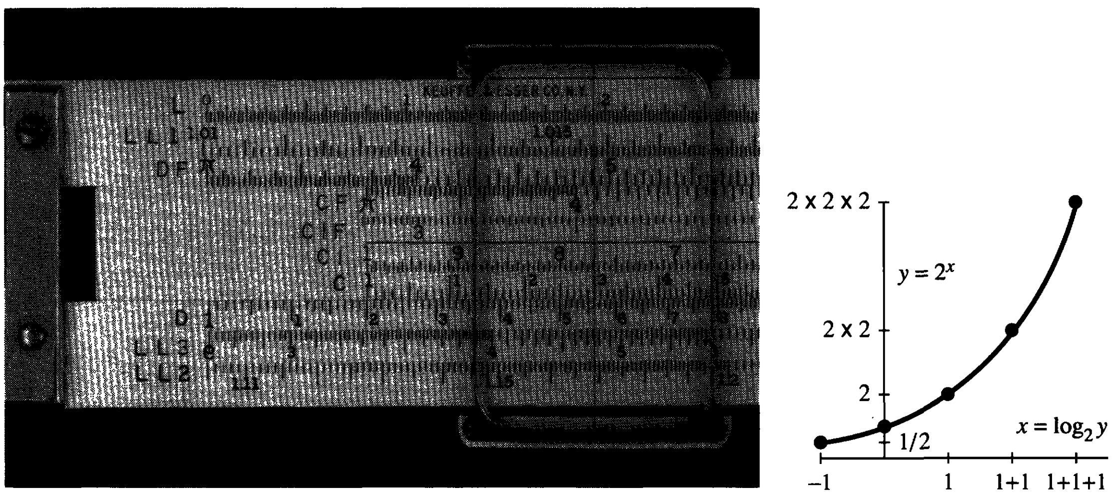


Fig. 6.1 An ancient relic (the slide rule). When exponents x add, powers 2^x multiply.

Base Change Bases other than 10 and exponents other than 1, 2, 3, ... are needed for applications. The population of the world x years from now is predicted to grow by a factor close to 1.02^x. Certainly x does not need to be a whole number of years. And certainly the base 1.02 should not be 10 (or we are in real trouble). This prediction will be refined as we study the differential equations for growth. It can be rewritten to base 10 if that is preferred (*but look at the exponent*):

$$1.02^x \quad \textbf{\textit{is the same as}} \quad 10^{(\log 1.02)x}.$$

When the base changes from 1.02 to 10, the exponent is multiplied—as we now see.

For practice, start with base b and change to base a. The logarithm to base a will be written "log." Everything comes from the rule that logarithm = exponent:

$$\textbf{\textit{base change for numbers}}: \quad b = a^{\log b}.$$

Now raise both sides to the power x. You see the change in the exponent:

$$\textbf{\textit{base change for exponentials}}: \quad b^x = a^{(\log b)x}.$$

Finally set $y = b^x$. Its logarithm to base b is x. Its logarithm to base a is the exponent on the right hand side: $\log_a y = (\log_a b)x$. Now replace x by $\log_b y$:

$$\textbf{\textit{base change for logarithms}}: \quad \log_a y = (\log_a b)(\log_b y).$$

We absolutely need this ability to change the base. An example with $a = 2$ is

$$b = 8 = 2^3 \qquad 8^2 = (2^3)^2 = 2^6 \qquad \log_2 64 = 3 \cdot 2 = (\log_2 8)(\log_8 64).$$

The rule behind base changes is $(a^m)^x = a^{mx}$. When the mth power is raised to the xth power, *the exponents multiply*. The square of the cube is the sixth power:

$$(a)(a)(a) \text{ times } (a)(a)(a) \text{ equals } (a)(a)(a)(a)(a)(a): \quad (a^3)^2 = a^6.$$

Another base will soon be more important than 10—here are the rules for base changes:

6B To change *numbers*, *powers*, and *logarithms* from base b to base a, use

$$b = a^{\log_a b} \qquad b^x = a^{(\log_a b)x} \qquad \log_a y = (\log_a b)(\log_b y) \tag{2}$$

The first is the definition. The second is the xth power of the first. The third is the logarithm of the second (remember y is b^x). An important case is $y = a$:

$$\log_a a = (\log_a b)(\log_b a) = 1 \text{ so } \log_a b = 1/\log_b a. \tag{3}$$

EXAMPLE $8 = 2^3$ means $8^{1/3} = 2$. Then $(\log_2 8)(\log_8 2) = (3)(1/3) = 1$.

This completes the algebra of logarithms. The addition rules **6A** came from $(b^m)(b^n) = b^{m+n}$. The multiplication rule **6B** came from $(a^m)^x = a^{mx}$. *We still need to define b^x and a^x for all real numbers x.* When x is a fraction, the definition is easy. The square root of a^8 is a^4 ($m = 8$ times $x = 1/2$). When x is not a fraction, as in 2^π, the graph suggests one way to fill in the hole.

We could define 2^π as the limit of $2^3, 2^{31/10}, 2^{314/100}, \ldots$. As the fractions r approach π, the powers 2^r approach 2^π. This makes $y = 2^x$ into a continuous function, with the desired properties $(2^m)(2^n) = 2^{m+n}$ and $(2^m)^x = 2^{mx}$—whether m and n and x are integers or not. But the ε's and δ's of continuity are not attractive, and we eventually choose (in Section 6.4) a smoother approach based on integrals.

GRAPHS OF b^x AND $\log_b y$

It is time to draw graphs. In principle one graph should do the job for both functions, because $y = b^x$ means the same as $x = \log_b y$. ***These are inverse functions.*** What one function does, its inverse undoes. The logarithm of $g(x) = b^x$ is x:

$$g^{-1}(g(x)) = \log_b(b^x) = x. \tag{4}$$

In the opposite direction, the exponential of the logarithm of y is y:

$$g(g^{-1}(y)) = b^{(\log_b y)} = y. \tag{5}$$

This holds for every base b, and it is valuable to see $b = 2$ and $b = 4$ on the same graph. Figure 6.2a shows $y = 2^x$ and $y = 4^x$. Their mirror images in the 45° line give the logarithms to base 2 and base 4, which are in the right graph.

When x is negative, $y = b^x$ is still positive. If the first graph is extended to the left, it stays above the x axis. ***Sketch it in with your pencil.*** Also extend the second graph down, to be the mirror image. Don't cross the vertical axis.

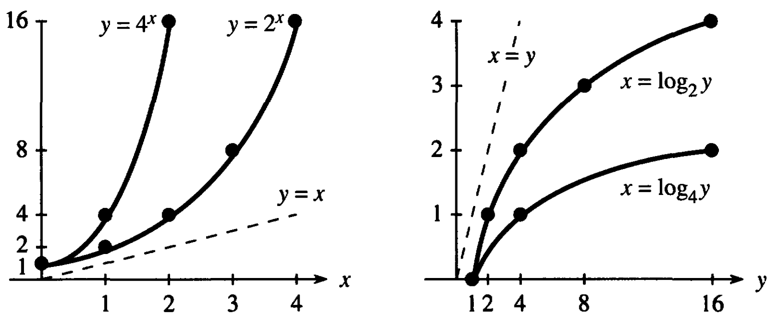


Fig. 6.2 Exponentials and mirror images (logarithms). Different scales for x and y.

There are interesting relations within the left figure. All exponentials start at 1, because b^0 is always 1. At the height $y = 16$, one graph is above $x = 2$ (because $4^2 = 16$). The other graph is above $x = 4$ (because $2^4 = 16$). ***Why does 4^x in one graph equal 2^{2x} in the other?*** This is the base change for powers, since $4 = 2^2$.

The figure on the right shows the mirror image—the logarithm. All logarithms start from zero at $y = 1$. The graphs go down to $-\infty$ at $y = 0$. (Roughly speaking $2^{-\infty}$ is zero.) Again x in one graph corresponds to $2x$ in the other (base change for logarithms). Both logarithms climb slowly, since the exponentials climb so fast.

The number $\log_2 10$ is between 3 and 4, because 10 is between 2^3 and 2^4. The slope of 2^x is proportional to 2^x—which never happened for x^n. But there are two practical difficulties with those graphs:

1. 2^x and 4^x increase too fast. The curves turn virtually straight up.
2. The most important fact about Ab^x is the value of b—and the base doesn't stand out in the graph.

There is also another point. In many problems we don't know the function $y = f(x)$. We are looking for it! All we have are measured values of y (with errors mixed in). When the values are plotted on a graph, we want to discover $f(x)$.

Fortunately there is a solution. ***Scale the y axis differently.*** On ordinary graphs, each unit upward adds a fixed amount to y. *On a log scale each unit **multiplies** y by* *a fixed amount*. The step from $y = 1$ to $y = 2$ is the same length as the step from 3 to 6 or 10 to 20.

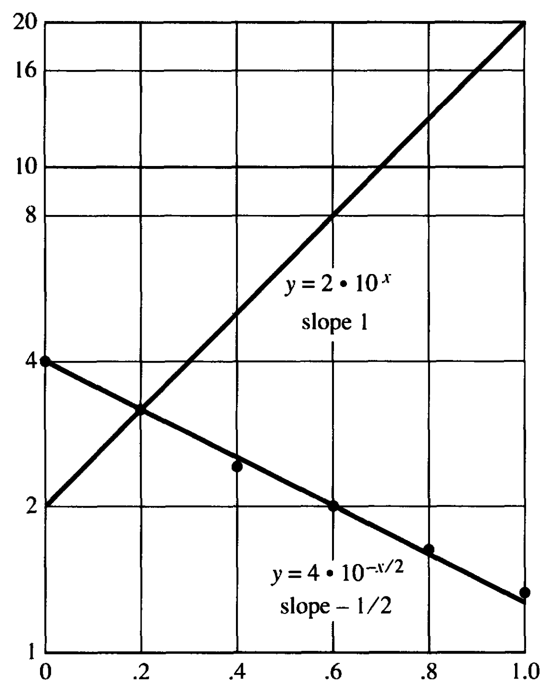
On a log scale, $y = 11$ is not halfway between 10 and 12. And $y = 0$ is not there at all. Each step down divides by a fixed amount—*we never reach zero*. This is completely satisfactory for Ab^x, which also never reaches zero.

Figure 6.3 is on ***semilog paper*** (also known as *log-linear*), with an ordinary x axis. ***The graph of*** $y = Ab^x$ ***is a straight line***. To see why, take logarithms of that equation:

$$\log y = \log A + x \log b. \tag{6}$$

The relation between x ***and*** $\log y$ ***is linear***. It is really $\log y$ that is plotted, so the graph is straight. The markings on the y axis allow you to enter y without looking up its logarithm—you get an ordinary graph of $\log y$ against x.

Figure 6.3 shows two examples. One graph is an exact plot of $y = 2 \cdot 10^x$. It goes upward with slope 1, because a unit across has the same length as multiplication by 10 going up. 10^x ***has slope*** 1 ***and*** $10^{(\log b)x}$ (which is b^x) ***will have slope*** $\log b$. The crucial number $\log b$ can be measured directly as the slope.



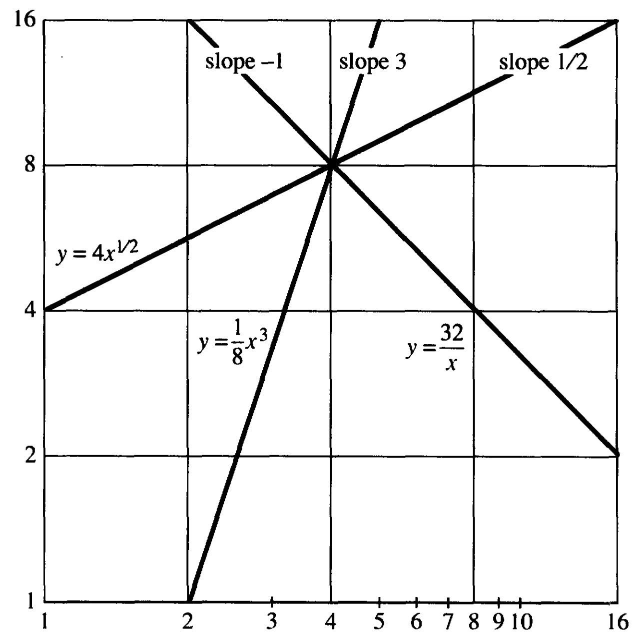


Fig. 6.3 $2 \cdot 10^x$ and $4 \cdot 10^{-x/2}$ on semilog paper. **Fig. 6.4** Graphs of Ax^k on log-log paper.

The second graph in Figure 6.3 is more typical of actual practice, in which we start with measurements and look for $f(x)$. Here are the data points:

$$x = 0.0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0$$

$$y = 4.0 \quad 3.2 \quad 2.4 \quad 2.0 \quad 1.6 \quad 1.3$$

We don't know in advance whether these values fit the model $y = Ab^x$. The graph is strong evidence that they do. The points lie close to a line with negative slope—indicating $\log b < 0$ and $b < 1$. The slope down is half of the earlier slope up, so the model is consistent with

$$y = A \cdot 10^{-x/2} \quad \text{or} \quad \log y = \log A - \tfrac{1}{2}x. \tag{7}$$

When x reaches 2, y drops by a factor of 10. At $x = 0$ we see $A \approx 4$.

Another model—a ***power*** $y = Ax^k$ instead of an exponential—also stands out with logarithmic scaling. This time we use ***log–log paper***, with both axes scaled. The logarithm of $y = Ax^k$ gives a linear relation between $\log y$ and $\log x$:

$$\log y = \log A + k \log x. \tag{8}$$

The exponent** k **becomes the slope on log–log paper. The base b makes no difference. We just measure the slope, and a straight line is a lot more attractive than a power curve.

The graphs in Figure 6.4 have slopes 3 and $\frac{1}{2}$ and -1. They represent Ax^3 and $A\sqrt{x}$ and A/x. To find the A's, look at one point on the line. At $x = 4$ the height is 8, so adjust the A's to make this happen: The functions are $x^3/8$ and $4\sqrt{x}$ and $32/x$. On semilog paper those graphs would not be straight!

You can buy log paper or create it with computer graphics.

THE DERIVATIVES OF $y = b^x$ AND $x = \log_b y$

This is a calculus book. *We have to ask about slopes.* The algebra of exponents is done, the rules are set, and on log paper the graphs are straight. Now come limits.

The central question is the derivative. ***What is** dy/dx **when** $y = b^x$? **What is** dx/dy **when** x **is the logarithm*** $\log_b y$? Those questions are closely related, because b^x and $\log_b y$ are inverse functions. If one slope can be found, the other is known from $dx/dy = 1/(dy/dx)$. The problem is to find one of them, and the exponential comes first.

You will now see that those questions have quick (and beautiful) answers, *except for a mysterious constant*. There is a multiplying factor c which needs more time. I think it is worth separating out the part that can be done immediately, leaving c in dy/dx and $1/c$ in dx/dy. Then Section 6.2 discovers c by studying the special number called e (but $c \neq e$).

> **6C** The derivative of b^x is a multiple cb^x. The number c depends on the base b.

The product and power and chain rules do not yield this derivative. We are pushed all the way back to the original definition, the limit of $\Delta y/\Delta x$:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h} = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h}. \tag{9}$$

Key idea: Split b^{x+h} into b^x times b^h. Then the crucial quantity b^x factors out. More than that, b^x comes *outside the limit* because it does not depend on h. The remaining limit, inside the brackets, is the number c that we don't yet know:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{b^x b^h - b^x}{h} = b^x \left[\lim_{h \to 0} \frac{b^h - 1}{h} \right] = cb^x. \tag{10}$$

This equation is central to the whole chapter: dy/dx ***equals*** cb^x ***which equals*** cy. *The rate of change of y is proportional to y.* The slope increases in the same way that b^x increases (except for the factor c). A typical example is money in a bank, where interest is proportional to the principal. The rich get richer, and the poor get slightly richer. We will come back to compound interest, and identify b and c.

The inverse function is $x = \log_b y$. Now the unknown factor is $1/c$:

6D The slope of $\log_b y$ is $1/cy$ with the same c (depending on b).

Proof If $dy/dx = cb^x$ then $dx/dy = 1/cb^x = 1/cy$. (11)

That proof was like a Russian toast, powerful but too quick! We go more carefully:

$$f(b^x) = x \qquad \text{(logarithm of exponential)}$$

$$f'(b^x)(cb^x) = 1 \qquad \text{(x derivative by chain rule)}$$

$$f'(b^x) = 1/cb^x \qquad \text{(divide by cb^x)}$$

$$f'(y) = 1/cy \qquad \text{(identify b^x as y)}$$

The logarithm gives another way to find c. From its slope we can discover $1/c$. *This is the way that finally works* (next section).

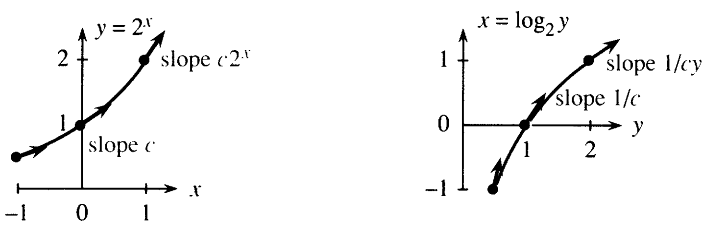


Fig. 6.5 The slope of 2^x is about $.7 \cdot 2^x$. The slope of $\log_2 y$ is about $1/.7y$.

Final remark It is extremely satisfying to meet an $f(y)$ whose derivative is $1/cy$. At last the "-1 power" has an antiderivative. Remember that $\int x^n dx = x^{n+1}/(n+1)$ is a failure when $n = -1$. The derivative of x^0 (a constant) does not produce x^{-1}. ***We had no integral for x^{-1}, and the logarithm fills that gap***. If y is replaced by x or t (all dummy variables) then

$$\frac{d}{dx}\log_b x = \frac{1}{cx} \quad \text{and} \quad \frac{d}{dt}\log_b t = \frac{1}{ct}. \tag{12}$$

The base b can be chosen so that $c = 1$. Then the derivative is $1/x$. This final touch comes from the magic choice $b = e$—the highlight of Section 6.2.

6.1 EXERCISES

Read-through questions

In $10^4 = 10{,}000$, the exponent 4 is the __a__ of 10,000. The base is $b =$ __b__. The logarithm of 10^m times 10^n is __c__. The logarithm of $10^m/10^n$ is __d__. The logarithm of $10{,}000^x$ is __e__. If $y = b^x$ then $x =$ __f__. Here x is any number, and y is always __g__.

A base change gives $b = a^{\text{h}}$ and $b^x = a^{\text{i}}$. Then 8^5 is 2^{15}. In other words $\log_2 y$ is __j__ times $\log_8 y$. When $y = 2$ it follows that $\log_2 8$ times $\log_8 2$ equals __k__.

On ordinary paper the graph of $y =$ __l__ is a straight line. Its slope is __m__. On semilog paper the graph of $y =$ __n__ is a straight line. Its slope is __o__. On log–log paper the graph of $y =$ __p__ is a straight line. Its slope is __q__.

The slope of $y = b^x$ is $dy/dx =$ __r__, where c depends on b. The number c is the limit as $h \to 0$ of __s__. Since $x = \log_b y$ is the inverse, $(dx/dy)(dy/dx) =$ __t__. Knowing $dy/dx = cb^x$ yields $dx/dy =$ __u__. Substituting b^x for y, the slope of $\log_b y$ is __v__. With a change of letters, the slope of $\log_b x$ is __w__.

Problems 1–10 use the rules for logarithms.

1 Find these logarithms (or exponents):
(a) $\log_2 32$ (b) $\log_2(1/32)$ (c) $\log_{32}(1/32)$
(d) $\log_{32} 2$ (e) $\log_{10}(10\sqrt{10})$ (f) $\log_2(\log_2 16)$

2 Without a calculator find the values of
(a) $3^{\log_3 5}$ (b) $3^{2\log_3 5}$
(c) $\log_{10}5 + \log_{10}2$ (d) $(\log_3 b)(\log_b 9)$
(e) $10^5 10^{-4} 10^3$ (f) $\log_2 56 - \log_2 7$

3 Sketch $y = 2^{-x}$ and $y = \frac{1}{2}(4^x)$ from -1 to 1 on the same graph. Put their mirror images $x = -\log_2 y$ and $x = \log_4 2y$ on a second graph.

4 Following Figure 6.2 sketch the graphs of $y = (\frac{1}{2})^x$ and $x = \log_{1/2} y$. What are $\log_{1/2} 2$ and $\log_{1/2} 4$?

5 Compute without a computer:
(a) $\log_2 3 + \log_2 \frac{2}{3}$ (b) $\log_2(\frac{1}{2})^{10}$
(c) $\log_{10} 100^{40}$ (d) $(\log_{10} e)(\log_e 10)$
(e) $2^{2^3}/(2^2)^3$ (f) $\log_e(1/e)$

6 Solve the following equations for x:
(a) $\log_{10}(10^x) = 7$ (b) $\log 4x - \log 4 = \log 3$
(c) $\log_x 10 = 2$ (d) $\log_2(1/x) = 2$
(e) $\log x + \log x = \log 8$ (f) $\log_x(x^x) = 5$

7 The logarithm of $y = x^n$ is $\log_b y =$ ________.

*8 Prove that $(\log_b a)(\log_d c) = (\log_d a)(\log_b c)$.

9 2^{10} is close to 10^3 (1024 versus 1000). If they were equal then $\log_2 10$ would be ________. Also $\log_{10} 2$ would be ________ instead of 0.301.

10 The number 2^{1000} has approximately how many (decimal) digits?

Questions 11–19 are about the graphs of $y = b^x$ and $x = \log_b y$.

11 By hand draw the axes for semilog paper and the graphs of $y = 1.1^x$ and $y = 10(1.1)^x$.

12 Display a set of axes on which the graph of $y = \log_{10} x$ is a straight line. What other equations give straight lines on those axes?

13 When noise is measured in *decibels*, amplifying by a factor A increases the decibel level by $10 \log A$. If a whisper is 20db and a shout is 70db then $10 \log A = 50$ and $A =$ ________.

14 Draw semilog graphs of $y = 10^{1-x}$ and $y = \frac{1}{2}(\sqrt{10})^x$.

15 The Richter scale measures earthquakes by $\log_{10}(I/I_0) = R$. What is R for the standard earthquake of intensity I_0? If the 1989 San Francisco earthquake measured $R = 7$, how did its intensity I compare to I_0? The 1906 San Francisco quake had $R = 8.3$. The record quake was four times as intense with $R =$ ________.

16 The frequency of A above middle C is 440/second. The frequency of the next higher A is ________. Since $2^{7/12} \approx 1.5$, the note with frequency 660/sec is ________.

17 Draw your own semilog paper and plot the data

$$y = 7,\ 11,\ 16,\ 28,\ 44 \quad \text{at} \quad x = 0,\ 1/2,\ 1,\ 3/2,\ 2.$$

Estimate A and b in $y = Ab^x$.

18 Sketch log–log graphs of $y = x^2$ and $y = \sqrt{x}$.

19 On log–log paper, printed or homemade, plot $y = 4$, 11, 21, 32, 45 at $x = 1$, 2, 3, 4, 5. Estimate A and k in $y = Ax^k$.

Questions 20–29 are about the derivative $dy/dx = cb^x$.

20 $g(x) = b^x$ has slope $g' = cg$. Apply the chain rule to $g(f(y)) = y$ to prove that $df/dy = 1/cy$.

21 If the slope of $\log x$ is $1/cx$, find the slopes of $\log(2x)$ and $\log(x^2)$ and $\log(2^x)$.

22 What is the equation (including c) for the tangent line to $y = 10^x$ at $x = 0$? Find also the equation at $x = 1$.

23 What is the equation for the tangent line to $x = \log_{10} y$ at $y = 1$? Find also the equation at $y = 10$.

24 With $b = 10$, the slope of 10^x is $c10^x$. Use a calculator for small h to estimate $c = \lim\,(10^h - 1)/h$.

25 The unknown constant in the slope of $y = (.1)^x$ is $L = \lim\,(.1^h - 1)/h$. (a) Estimate L by choosing a small h. (b) Change h to $-h$ to show that $L = -c$ from Problem 24.

26 Find a base b for which $(b^h - 1)/h \approx 1$. Use $h = 1/4$ by hand or $h = 1/10$ and $1/100$ by calculator.

27 Find the second derivative of $y = b^x$ and also of $x = \log_b y$.

28 Show that $C = \lim\,(100^h - 1)/h$ is twice as large as $c = \lim\,(10^h - 1)/h$. (Replace the last h's by $2h$.)

29 In 28, the limit for $b = 100$ is twice as large as for $b = 10$. So c probably involves the ________ of b.

6.2 The Exponential e^x

The last section discussed b^x and $\log_b y$. The base b was arbitrary—it could be 2 or 6 or 9.3 or any positive number except 1. But in practice, only a few bases are used. I have never met a logarithm to base 6 or 9.3. Realistically there are two leading candidates for b, and 10 is one of them. This section is about the other one, which is an extremely remarkable number. This number is not seen in arithmetic or algebra or geometry, where it looks totally clumsy and out of place. In calculus it comes into its own.

The number is e. That symbol was chosen by Euler (initially in a fit of selfishness, but he was a wonderful mathematician). It is the base of the ***natural logarithm***. It also controls the exponential e^x, which is much more important than $\ln x$. Euler also chose π to stand for perimeter—anyway, our first goal is to find e.

Remember that the derivatives of b^x and $\log_b y$ include a constant c that depends on b. Equations (10) and (11) in the previous section were

$$\frac{d}{dx} b^x = cb^x \quad \text{and} \quad \frac{d}{dy} \log_b y = \frac{1}{cy}. \tag{1}$$

At $x = 0$, the graph of b^x starts from $b^0 = 1$. The slope is c. At $y = 1$, the graph of $\log_b y$ starts from $\log_b 1 = 0$. The logarithm has slope $1/c$. ***With the right choice of the base b those slopes will equal*** 1 (because c will equal 1).

For $y = 2^x$ the slope c is near .7. We already tried $\Delta x = .1$ and found $\Delta y \approx .07$. The base has to be larger than 2, for a starting slope of $c = 1$.

We begin with a direct computation of the slope of $\log_b y$ at $y = 1$:

$$\frac{1}{c} = \text{slope at } 1 = \lim_{h \to 0} \frac{1}{h}\left[\log_b(1 + h) - \log_b 1\right] = \lim_{h \to 0} \log_b\left[(1 + h)^{1/h}\right]. \tag{2}$$

Always $\log_b 1 = 0$. The fraction in the middle is $\log_b(1 + h)$ times the number $1/h$. This number can go up into the exponent, and it did.

The quantity $(1 + h)^{1/h}$ is unusual, to put it mildly. As $h \to 0$, the number $1 + h$ is approaching 1. At the same time, $1/h$ is approaching infinity. ***In the limit we have*** 1^∞. But that expression is meaningless (like 0/0). Everything depends on the balance between "nearly 1" and "nearly ∞." This balance produces the extraordinary number e:

DEFINITION ***The number e is equal to*** $\lim_{h \to 0} (1 + h)^{1/h}$. ***Equivalently*** $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$.

Before computing e, look again at the slope $1/c$. At the end of equation (2) is the logarithm of e:

$$1/c = \log_b e. \tag{3}$$

When the base is $b = e$, the slope is $\log_e e = 1$. ***That base e has $c = 1$ as desired:***

$$\textit{The derivative of } e^x \textit{ is } 1 \cdot e^x \textit{ and the derivative of } \log_e y \textit{ is } \frac{1}{1 \cdot y}. \tag{4}$$

This is why the base e is all-important in calculus. It makes $c = 1$.

To compute the actual number e from $(1 + h)^{1/h}$, choose $h = 1, 1/10, 1/100, \ldots$. Then the exponents $1/h$ are $n = 1, 10, 100, \ldots$. (All limits and derivatives will become official in Section 6.4.) The table shows $(1 + h)^{1/h}$ approaching e as $h \to 0$ and $n \to \infty$:

n	$h = \frac{1}{n}$	$1 + h = 1 + \frac{1}{n}$	$(1+h)^{1/h} = \left(1 + \frac{1}{n}\right)^n$
1	1.0	2.0	2.0
2	0.5	1.5	2.25
10	0.1	1.1	2.593742
100	0.01	1.01	2.704814
1000	0.001	1.001	2.716924
10000	0.0001	1.0001	2.718146

The last column is converging to e (not quickly). There is an infinite series that converges much faster. We know 125,000 digits of e (and a billion digits of π). There are no definite patterns, although you might think so from the first sixteen digits:

$$e = 2.7\ 1828\ 1828\ 45\ 90\ 45 \cdots \quad (\text{and } 1/e \approx .37).$$

The powers of e produce $y = e^x$. At $x = 2.3$ and 5, we are close to $y = 10$ and 150.

The logarithm is the inverse function. The logarithms of 150 and 10, to the base e, are close to $x = 5$ and $x = 2.3$. There is a special name for this logarithm—the ***natural logarithm***. There is also a special notation "ln" to show that the base is e:

$\ln y$ ***means the same as*** $\log_e y$. ***The natural logarithm is the exponent in*** $e^x = y$.

The notation $\ln y$ (or $\ln x$—it is the function that matters, not the variable) is standard in calculus courses. After calculus, the base is generally assumed to be e. In most of science and engineering, the natural logarithm is the automatic choice. The symbol "$\exp(x)$" means e^x, and the truth is that the symbol "$\log x$" generally means $\ln x$. Base e is understood even without the letters ln. But in any case of doubt—on a calculator key for example—the symbol "$\ln x$" emphasizes that the base is e.

THE DERIVATIVES OF e^x AND $\ln x$

Come back to derivatives and slopes. The derivative of b^x is cb^x, and the derivative of $\log_b y$ is $1/cy$. ***If*** $b = e$ ***then*** $c = 1$. For all bases, equation (3) is $1/c = \log_b e$. This gives c—the slope of b^x at $x = 0$:

6E ***The number*** c ***is*** $1/\log_b e = \log_e b$. ***Thus*** c ***equals*** $\ln b$. (5)

$c = \ln b$ ***is the mysterious constant that was not available earlier.*** The slope of 2^x is $\ln 2$ times 2^x. The slope of e^x is $\ln e$ times e^x (but $\ln e = 1$). We have the derivatives on which this chapter depends:

6F The derivatives of e^x and $\ln y$ are e^x and $1/y$. For other bases

$$\frac{d}{dx} b^x = (\ln b) b^x \quad \text{and} \quad \frac{d}{dy} \log_b y = \frac{1}{(\ln b) y}. \tag{6}$$

To make clear that those derivatives come from the functions (and not at all from the dummy variables), we rewrite them using t and x:

$$\frac{d}{dt} e^t = e^t \quad \text{and} \quad \frac{d}{dx} \ln x = \frac{1}{x}. \tag{7}$$

Remark on slopes at $x = 0$: It would be satisfying to see directly that the slope of 2^x is below 1, and the slope of 4^x is above 1. Quick proof: e is between 2 and 4. But the idea is to see the slopes graphically. This is a small puzzle, which is fun to solve but can be skipped.

2^x rises from 1 at $x = 0$ to 2 at $x = 1$. On that interval its average slope is 1. Its slope at the beginning is *smaller* than average, so it must be less than 1—as desired. On the other hand 4^x rises from $\frac{1}{2}$ at $x = -\frac{1}{2}$ to 1 at $x = 0$. Again the average slope is $\frac{1}{2}/\frac{1}{2} = 1$. Since $x = 0$ comes at the *end* of this new interval, the slope of 4^x at that point exceeds 1. Somewhere between 2^x and 4^x is e^x, which starts out with slope 1.

This is the graphical approach to e. There is also the infinite series, and a fifth definition through integrals which is written here for the record:

1. e is the number such that e^x has slope 1 at $x = 0$

2. e is the base for which $\ln y = \log_e y$ has slope 1 at $y = 1$

3. e is the limit of $\left(1 + \frac{1}{n}\right)^n$ as $n \to \infty$

4. $e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots$

5. the area $\int_1^e x^{-1}\,dx$ equals 1.

The connections between **1**, **2**, and **3** have been made. The slopes are 1 when e is the limit of $(1 + 1/n)^n$. Multiplying this out wlll lead to **4**, the infinite series in Section 6.6. The official definition of $\ln x$ comes from $\int dx/x$, and then **5** says that $\ln e = 1$. This approach to e (Section 6.4) seems less intuitive than the others.

Figure 6.6b shows the graph of e^{-x}. It is the mirror image of e^x across the vertical axis. Their product is $e^x e^{-x} = 1$. Where e^x grows exponentially, e^{-x} decays exponentially—or it grows as x approaches $-\infty$. ***Their growth and decay are faster than any power of*** x. Exponential growth is more rapid than polynomial growth, so that e^x/x^n goes to infinity (Problem 59). It is the fact that e^x has slope e^x which keeps the function climbing so fast.

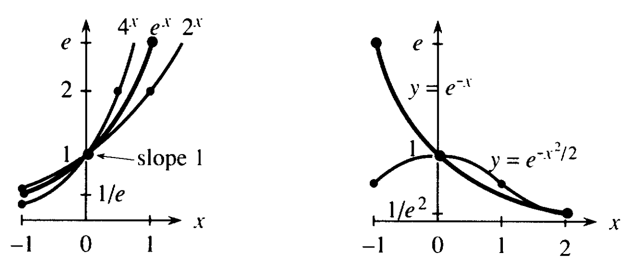


Fig. 6.6 e^x grows between 2^x and 4^x. Decay of e^{-x}, faster decay of $e^{-x^2/2}$.

The other curve is $y = e^{-x^2/2}$. This is the famous "*bell-shaped curve*" of probability theory. After dividing by $\sqrt{2\pi}$, it gives the ***normal distribution***, which applies to so many averages and so many experiments. The Gallup Poll will be an example in Section 8.4. The curve is symmetric around its mean value $x = 0$, since changing x to $-x$ has no effect on x^2.

About two thirds of the area under this curve is between $x = -1$ and $x = 1$. If you pick points at random below the graph, 2/3 of all samples are expected in that interval. The points $x = -2$ and $x = 2$ are "two standard deviations" from the center, enclosing 95% of the area. There is only a 5% chance of landing beyond. The decay is even faster than an ordinary exponential, because $\frac{1}{2}x^2$ has replaced x.

THE DERIVATIVES OF e^{cx} AND $e^{u(x)}$

The slope of e^x is e^x. This opens up a whole world of functions that calculus can deal with. The chain rule gives the slope of e^{3x} and $e^{\sin x}$ and every $e^{u(x)}$:

> **6G** The derivative of $e^{u(x)}$ is $e^{u(x)}$ times du/dx. (8)
>
> Special case $u = cx$: The derivative of e^{cx} is ce^{cx}. (9)

EXAMPLE 1 The derivative of e^{3x} is $3e^{3x}$ (here $c = 3$). The derivative of $e^{\sin x}$ is $e^{\sin x}\cos x$ (here $u = \sin x$). The derivative of $f(u(x))$ is df/du times du/dx. Here $f = e^u$ so $df/du = e^u$. ***The chain rule demands that second factor*** du/dx.

EXAMPLE 2 $e^{(\ln 2)x}$ is the same as 2^x. Its derivative is $\ln 2$ times 2^x. The chain rule rediscovers our constant $c = \ln 2$. In the slope of b^x it rediscovers the factor $c = \ln b$.

Generally e^{cx} is preferred to the original b^x. The derivative just brings down the constant c. ***It is better to agree on e as the base***, and put all complications (like $c = \ln b$) up in the exponent. The second derivative of e^{cx} is c^2e^{cx}.

EXAMPLE 3 The derivative of $e^{-x^2/2}$ is $-xe^{-x^2/2}$ (here $u = -x^2/2$ so $du/dx = -x$).

EXAMPLE 4 The second derivative of $f = e^{-x^2/2}$, by the chain rule and product rule, is

$$f'' = (-1)\cdot e^{-x^2/2} + (-x)^2 e^{-x^2/2} = (x^2 - 1)e^{-x^2/2}. \tag{10}$$

Notice how ***the exponential survives***. With every derivative it is multiplied by more factors, but it is still there to dominate growth or decay. The *points of inflection*, where the bell-shaped curve has $f'' = 0$ in equation (10), are $x = 1$ and $x = -1$.

EXAMPLE 5 ($u = n \ln x$). Since $e^{n \ln x}$ is x^n in disguise, its slope must be nx^{n-1}:

$$\text{slope} = e^{n \ln x}\frac{d}{dx}(n \ln x) = x^n\left(\frac{n}{x}\right) = nx^{n-1}. \tag{11}$$

This slope is correct for all n, ***integer or not***. Chapter 2 produced $3x^2$ and $4x^3$ from the binomial theorem. Now nx^{n-1} comes from ln and exp and the chain rule.

EXAMPLE 6 An extreme case is $x^x = (e^{\ln x})^x$. Here $u = x \ln x$ and we need du/dx:

$$\frac{d}{dx}(x^x) = e^{x \ln x}\left(\ln x + x \cdot \frac{1}{x}\right) = x^x(\ln x + 1).$$

INTEGRALS OF e^{cx} AND $e^u\, du/dx$

The integral of e^x is e^x. *The integral of* e^{cx} *is not* e^{cx}. The derivative multiplies by c so the integral divides by c. ***The integral of*** e^{cx} ***is*** e^{cx}/c (plus a constant).

EXAMPLES $\displaystyle\int e^{2x}dx = \frac{1}{2}e^{2x} + C \qquad \int b^x dx = \frac{b^x}{\ln b} + C$

$$\int e^{3(x+1)}dx = \frac{1}{3}e^{3(x+1)} + C \qquad \int e^{-x^2/2}\,dx \to \text{failure}$$

The first one has $c = 2$. The second has $c = \ln b$—remember again that $b^x = e^{(\ln b)x}$. The integral divides by $\ln b$. In the third one, $e^{3(x+1)}$ is e^{3x} times the number e^3 and that number is carried along. Or more likely we see $e^{3(x+1)}$ as e^u. The missing $du/dx = 3$ is fixed by dividing by 3. The last example fails because du/dx is not there. ***We cannot integrate without*** du/dx:

6H ***The indefinite integral*** $\int e^u \frac{du}{dx}dx$ ***equals*** $e^{u(x)} + C$.

Here are three examples with du/dx and one without it:

$$\int e^{\sin x}\cos x\,dx = e^{\sin x} + C \qquad \int xe^{x^2/2}\,dx = e^{x^2/2} + C$$

$$\int \frac{e^{\sqrt{x}}dx}{\sqrt{x}} = 2e^{\sqrt{x}} + C \qquad \int \frac{e^x dx}{(1+e^x)^2} = \frac{-1}{1+e^x} + C$$

The first is a pure $e^u du$. So is the second. The third has $u = \sqrt{x}$ and $du/dx = 1/2\sqrt{x}$, so only the factor 2 had to be fixed. The fourth example does not belong with the others. It is the integral of du/u^2, not the integral of $e^u du$. I don't know any way to tell you which substitution is best—except that *the complicated part is* $1 + e^x$ *and it is natural to substitute u*. If it works, good.

Without an extra e^x for du/dx, the integral $\int dx/(1+e^x)^2$ looks bad. But $u = 1 + e^x$ is still worth trying. It has $du = e^x dx = (u-1)dx$:

$$\int \frac{dx}{(1+e^x)^2} = \int \frac{du}{(u-1)u^2} = \int du\left(\frac{1}{u-1} - \frac{1}{u} - \frac{1}{u^2}\right). \tag{12}$$

That last step is "*partial fractions.*" The integral splits into simpler pieces (explained in Section 7.4) and we integrate each piece. Here are three other integrals:

$$\int e^{1/x}dx \qquad \int e^x(4+e^x)dx \qquad \int e^{-x}(4+e^x)dx$$

The first can change to $-\int e^u du/u^2$, which is not much better. (It is just as impossible.) The second is actually $\int u\,du$, but I prefer a split: $\int 4e^x$ and $\int e^{2x}$ are safer to do separately. The third is $\int (4e^{-x} + 1)dx$, which also separates. The exercises offer practice in reaching $e^u du/dx$ — ready to be integrated.

Warning about definite integrals When the lower limit is $x = 0$, there is a natural tendency to expect $f(0) = 0$—in which case the lower limit contributes nothing. For a power $f = x^3$ that is true. For an exponential $f = e^{3x}$ it is definitely *not true*, because $f(0) = 1$:

$$\int_0^1 e^{3x}dx = \frac{1}{3}e^{3x}\Big]_0^1 = \frac{1}{3}(e^3 - 1) \qquad \int_0^1 xe^{x^2}dx = \frac{1}{2}e^{x^2}\Big]_0^1 = \frac{1}{2}(e-1).$$

6.2 EXERCISES

Read-through questions

The number e is approximately __a__. It is the limit of $(1+h)$ to the power __b__. This gives 1.01^{100} when $h=$ __c__. An equivalent form is $e=\lim($ __d__ $)^n$.

When the base is $b=e$, the constant c in Section 6.1 is __e__. Therefore the derivative of $y=e^x$ is $dy/dx=$ __f__. The derivative of $x=\log_e y$ is $dx/dy=$ __g__. The slopes at $x=0$ and $y=1$ are both __h__. The notation for $\log_e y$ is __i__, which is the __j__ logarithm of y.

The constant c in the slope of b^x is $c=$ __k__. The function b^x can be rewritten as __l__. Its derivative is __m__. The derivative of $e^{u(x)}$ is __n__. The derivative of $e^{\sin x}$ is __o__. The derivative of e^{cx} brings down a factor __p__.

The integral of e^x is __q__. The integral of e^{cx} is __r__. The integral of $e^{u(x)}du/dx$ is __s__. In general the integral of $e^{u(x)}$ by itself is __t__ to find.

Find the derivatives of the functions in 1–18.

1 $7e^{7x}$	**2** $-7e^{-7x}$
3 $(e^x)^8$	**4** $(e^{-x})^{-8}$
5 3^x	**6** $e^{x\ln 3}$
7 $(2/3)^x$	**8** 4^{4x}
9 $1/(1+e^x)$	**10** $e^{1/(1+x)}$
11 $e^{\ln x}+x^{\ln e}$	**12** $xe^{1/x}$
13 xe^x-e^x	**14** $x^2e^x-2xe^x+2e^x$
15 $\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$	**16** $e^{\ln(x^2)}+\ln(e^{x^2})$
17 $e^{\sin x}+\sin e^x$	**18** $x^{-1/x}$ (which is e—)

19 The difference between e and $(1+1/n)^n$ is approximately Ce/n. Subtract the calculated values for $n=10, 100, 1000$ from 2.7183 to discover the number C.

20 By algebra or a calculator find the limits of $(1+1/n)^{2n}$ and $(1+1/n)^{\sqrt{n}}$.

21 The limit of $(11/10)^{10}$, $(101/100)^{100}$, ... is e. So the limit of $(10/11)^{10}$, $(100/101)^{100}$, ... is ______. So the limit of $(10/11)^{11}$, $(100/101)^{101}$, ... is ______. The last sequence is $(1-1/n)^n$.

22 Compare the number of correct decimals of e for $(1.001)^{1000}$ and $(1.0001)^{10000}$ and if possible $(1.00001)^{100000}$. Which power n would give all the decimals in 2.71828?

23 The function $y=e^x$ solves $dy/dx=y$. Approximate this equation by $\Delta Y/\Delta x=Y$, which is $Y(x+h)-Y(x)=hY(x)$. With $h=\frac{1}{10}$ find $Y(h)$ after one step starting from $Y(0)=1$. What is $Y(1)$ after ten steps?

24 The function that solves $dy/dx=-y$ starting from $y=1$ at $x=0$ is ______. Approximate by $Y(x+h)-Y(x)=-hY(x)$. If $h=\frac{1}{4}$ what is $Y(h)$ after one step and what is $Y(1)$ after four steps?

25 Invent three functions f, g, h such that for $x>10$

$$(1+1/x)^x<f(x)<e^x<g(x)<e^{2x}<h(x)<x^x.$$

26 Graph e^x and $\sqrt{e^x}$ at $x=-2, -1, 0, 1, 2$. Another form of $\sqrt{e^x}$ is ______.

Find antiderivatives for the functions in 27–36.

27 $e^{3x}+e^{7x}$	**28** $(e^{3x})(e^{7x})$
29 $1^x+2^x+3^x$	**30** 2^{-x}
31 $(2e)^x+2e^x$	**32** $(1/e^x)+(1/x^e)$
33 $xe^{x^2}+xe^{-x^2}$	**34** $(\sin x)e^{\cos x}+(\cos x)e^{\sin x}$
35 $\sqrt{e^x}+(e^x)^2$	**36** xe^x (trial and error)

37 Compare e^{-x} with e^{-x^2}. Which one decreases faster near $x=0$? Where do the graphs meet again? When is the ratio of e^{-x^2} to e^{-x} less than 1/100?

38 Compare e^x with x^x: Where do the graphs meet? What are their slopes at that point? Divide x^x by e^x and show that the ratio approaches infinity.

39 Find the tangent line to $y=e^x$ at $x=a$. From which point on the graph does the tangent line pass through the origin?

40 By comparing slopes, prove that if $x>0$ then

(a) $e^x>1+x$ (b) $e^{-x}>1-x$.

41 Find the minimum value of $y=x^x$ for $x>0$. Show from d^2y/dx^2 that the curve is concave upward.

42 Find the slope of $y=x^{1/x}$ and the point where $dy/dx=0$. Check d^2y/dx^2 to show that the maximum of $x^{1/x}$ is ______.

43 If $dy/dx=y$ find the derivative of $e^{-x}y$ by the product rule. Deduce that $y(x)=Ce^x$ for some constant C.

44 Prove that $x^e=e^x$ has only one positive solution.

Evaluate the integrals in 45–54. With infinite limits, 49–50 are "improper."

45 $\displaystyle\int_0^1 e^{2x}\,dx$	**46** $\displaystyle\int_0^\pi \sin x\, e^{\cos x}\,dx$
47 $\displaystyle\int_{-1}^1 2^x\,dx$	**48** $\displaystyle\int_{-1}^1 2^{-x}\,dx$
49 $\displaystyle\int_0^\infty e^{-x}\,dx$	**50** $\displaystyle\int_0^\infty xe^{-x^2}\,dx$
51 $\displaystyle\int_0^1 e^{1+x}\,dx$	**52** $\displaystyle\int_0^1 e^{1+x^2}\,x\,dx$

53 $\int_0^\pi 2^{\sin x}\cos x\,dx$ **54** $\int_0^1 (1-e^x)^{10}\,e^x\,dx$

55 Integrate the integrals that can be integrated:

$$\int \frac{e^u}{du/dx}\,dx \qquad \int \frac{du/dx}{e^u}\,dx$$

$$\int e^u\left(\frac{du}{dx}\right)^2 dx \qquad \int (e^u)^2\,\frac{du}{dx}\,dx.$$

56 Find a function that solves $y'(x)=5y(x)$ with $y(0)=2$.

57 Find a function that solves $y'(x)=1/y(x)$ with $y(0)=2$.

58 With electronic help graph the function $(1+1/x)^x$. What are its asymptotes? Why?

59 This exercise shows that $F(x)=x^n/e^x \to 0$ as $x\to\infty$.
(a) Find dF/dx. Notice that $F(x)$ decreases for $x>n>0$. The maximum of x^n/e^x, at $x=n$, is n^n/e^n.
(b) $F(2x)=(2x)^n/e^{2x}=2^n x^n/e^x\cdot e^x \leqslant 2^n n^n/e^n\cdot e^x$.
Deduce that $F(2x)\to 0$ as $x\to\infty$. Thus $F(x)\to 0$.

60 With $n=6$, graph $F(x)=x^6/e^x$ on a calculator or computer. Estimate its maximum. Estimate x when you reach $F(x)=1$. Estimate x when you reach $F(x)=\frac{1}{2}$.

61 Stirling's formula says that $n!\approx\sqrt{2\pi n}\,n^n/e^n$. Use it to estimate $6^6/e^6$ to the nearest whole number. Is it correct? How many decimal digits in 10!?

62 $x^6/e^x\to 0$ is also proved by l'Hôpital's rule (at $x=\infty$):

$$\lim x^6/e^x = \lim 6x^5/e^x = \underline{\text{fill this in}} = 0.$$

6.3 Growth and Decay in Science and Economics

The derivative of $y=e^{cx}$ has taken time and effort. ***The result was*** $y'=ce^{cx}$, ***which means that*** $y'=cy$. That computation brought others with it, virtually for free—the derivatives of b^x and x^x and $e^{u(x)}$. But I want to stay with $y'=cy$—which is the most important differential equation in applied mathematics.

Compare $y'=x$ ***with*** $y'=y$. The first only asks for an antiderivative of x. We quickly find $y=\frac{1}{2}x^2+C$. The second has dy/dx equal to y itself—which we rewrite as $dy/y=dx$. The integral is $\ln y=x+C$. Then y itself is e^xe^C. Notice that the first solution is $\frac{1}{2}x^2$ *plus* a constant, and the second solution is e^x *times* a constant.

There is a way to graph slope x versus slope y. Figure 6.7 shows "*tangent arrows*," which give the slope at each x and y. For parabolas, the arrows grow steeper as x

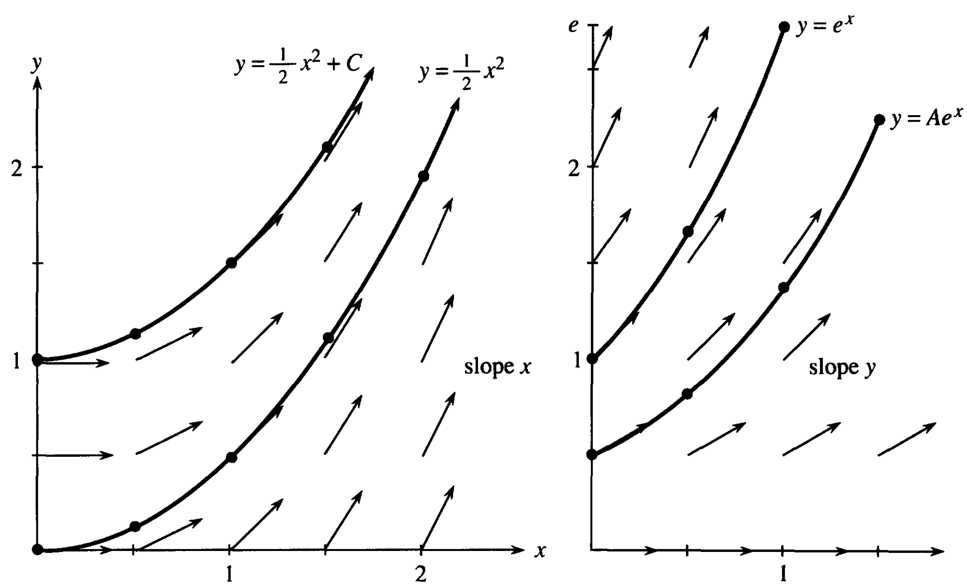


Fig. 6.7 The slopes are $y'=x$ and $y'=y$. The solution curves fit those slopes.

grows—because $y' = \text{slope} = x$. For exponentials, the arrows grow steeper as y grows—the equation is $y' = \text{slope} = y$. Now the arrows are connected by $y = Ae^x$. ***A differential equation gives a field of arrows*** (slopes). *Its solution is a curve that stays tangent to the arrows* — then the curve has the right slope.

A field of arrows can show many solutions at once (this comes in a differential equations course). Usually a single y_0 is not sacred. To understand the equation we start from many y_0—on the left the parabolas stay parallel, on the right the heights stay proportional. For $y' = -y$ all solution curves go to zero.

From $y' = y$ it is a short step to $y' = cy$. To make c appear in the derivative, *put c into the exponent*. The derivative of $y = e^{cx}$ is ce^{cx}, which is c times y. We have reached the key equation, which comes with an *initial condition*—a starting value y_0:

$$dy/dt = cy \text{ with } y = y_0 \text{ at } t = 0. \tag{1}$$

A small change: x has switched to t. In most applications *time* is the natural variable, rather than space. The factor c becomes the "growth rate" or "decay rate"—and e^{cx} converts to e^{ct}.

The last step is to match the initial condition. The problem requires $y = y_0$ at $t = 0$. Our e^{ct} starts from $e^{c0} = 1$. *The constant of integration is needed now—the solutions are* $y = Ae^{ct}$. By choosing $A = y_0$, we match the initial condition and solve equation (1). ***The formula to remember is*** $y_0 e^{ct}$.

6I The exponential law $y = y_0 e^{ct}$ solves $y' = cy$ starting from y_0.

The rate of growth or decay is c. May I call your attention to a basic fact? *The formula* $y_0 e^{ct}$ *contains three quantities* y_0, c, t. If two of them are given, plus one additional piece of information, the third is determined. Many applications have one of these three forms: ***find*** t, ***find*** c, ***find*** y_0.

1. Find the doubling time T if $c = 1/10$. At that time $y_0 e^{cT}$ equals $2y_0$:

$$e^{cT} = 2 \text{ yields } cT = \ln 2 \text{ so that } T = \frac{\ln 2}{c} \approx \frac{.7}{.1}. \tag{2}$$

The question asks for an exponent T. The answer involves logarithms. If a cell grows at a continuous rate of $c = 10\%$ per day, it takes about $.7/.1 = 7$ days to double in size. (Note that .7 is close to ln 2.) If a savings account earns 10% continuous interest, it doubles in 7 years.

In this problem we knew c. In the next problem we know T.

2. Find the decay constant c for carbon-14 if $y = \frac{1}{2}y_0$ in $T = 5568$ years.

$$e^{cT} = \tfrac{1}{2} \text{ yields } cT = \ln \tfrac{1}{2} \text{ so that } c \approx (\ln \tfrac{1}{2})/5568. \tag{3}$$

After the half-life $T = 5568$, the factor e^{cT} equals $\frac{1}{2}$. Now c is negative ($\ln \frac{1}{2} = -\ln 2$).

Question **1** was about growth. Question **2** was about decay. Both answers found e^{cT} as the ratio $y(T)/y(0)$. Then cT is its logarithm. Note how c sticks to T. T has the units of time, c has the units of "1/time."

Main point: The doubling time is $(\ln 2)/c$, because $cT = \ln 2$. The time to multiply by e is $1/c$. The time to multiply by 10 is $(\ln 10)/c$. The time to divide by e is $-1/c$, when a negative c brings decay.

3. Find the initial value y_0 if $c = 2$ and $y(1) = 5$:

$$y(t) = y_0 e^{ct} \text{ yields } y_0 = y(t)e^{-ct} = 5e^{-2}.$$

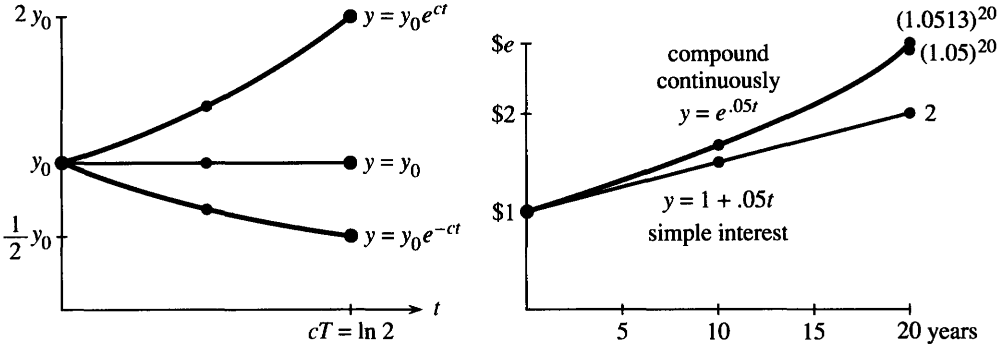


Fig. 6.8 Growth ($c > 0$) and decay ($c < 0$). Doubling time $T = (\ln 2)/c$. Future value at 5%.

All we do is run the process backward. Start from 5 and go back to y_0. With time reversed, e^{ct} becomes e^{-ct}. The product of e^2 and e^{-2} is 1—growth forward and decay backward.

Equally important is $T+t$. ***Go forward to time T and go on to*** $T+t$:

$$y(T+t) \text{ is } y_0e^{c(T+t)} \text{ which is } (y_0e^{cT})e^{ct}. \tag{4}$$

Every step t, at the start or later, multiplies by the same e^{ct}. This uses the fundamental property of exponentials, that $e^{T+t} = e^T e^t$.

EXAMPLE 1 ***Population growth from birth rate b and death rate d*** (both constant):

$$dy/dt = by - dy = cy \quad \text{(the net rate is } c = b - d).$$

The population in this model is $y_0e^{ct} = y_0e^{bt}e^{-dt}$. It grows when $b > d$ (which makes $c > 0$). One estimate of the growth rate is $c = 0.02$/year:

$$\textit{The earth's population doubles in about } T = \frac{\ln 2}{c} \approx \frac{.7}{.02} = 35 \textit{ years.}$$

First comment: We predict the future based on c. We count the past population to find c. Changes in c are a serious problem for this model.

Second comment: y_0e^{ct} is not a whole number. You may prefer to think of bacteria instead of people. (***This section begins a major application of mathematics to economics and the life sciences.***) Malthus based his theory of human population on this equation $y' = cy$—and with large numbers a fraction of a person doesn't matter so much. To use calculus we go from discrete to continuous. The theory must fail when t is very large, since populations cannot grow exponentially forever. Section 6.5 introduces the *logistic equation* $y' = cy - by^2$, with a competition term $-by^2$ to slow the growth.

Third comment: The dimensions of b, c, d are "1/time." The dictionary gives birth rate = number of births per person in a unit of time. It is a *relative* rate—people divided by people and time. The product ct is dimensionless and e^{ct} makes sense (also dimensionless). Some texts replace c by λ (lambda). Then $1/\lambda$ is the growth time or decay time or drug elimination time or diffusion time.

EXAMPLE 2 *Radioactive dating* A gram of charcoal from the cave paintings in France gives 0.97 disintegrations per minute. A gram of living wood gives 6.68 disintegrations per minute. Find the age of those Lascaux paintings.

The charcoal stopped adding radiocarbon when it was burned (at $t = 0$). The amount has decayed to y_0e^{ct}. In living wood this amount is still y_0, because cosmic rays maintain the balance. *Their ratio is* $e^{ct} = 0.97/6.68$. Knowing the decay rate c from Question **2** above, we know the present time t:

$$ct = \ln\left(\frac{0.97}{6.68}\right) \quad \text{yields} \quad t = \frac{5568}{-.7}\ln\left(\frac{0.97}{6.68}\right) = 14{,}400 \text{ years.}$$

Here is a related problem—*the age of uranium*. Right now there is 140 times as much U-238 as U-235. Nearly equal amounts were created, with half-lives of $(4.5)10^9$ and $(0.7)10^9$ years. ***Question***: How long since uranium was created? *Answer*: Find t by substituting $c = (\ln \frac{1}{2})/(4.5)10^9$ and $C = (\ln \frac{1}{2})/(0.7)10^9$:

$$e^{ct}/e^{Ct} = 140 \Rightarrow ct - Ct = \ln 140 \Rightarrow t = \frac{\ln 140}{c - C} = 6(10^9) \text{ years.}$$

EXAMPLE 3 ***Calculus in Economics*: *price inflation and the value of money***

We begin with two inflation rates — a *continuous rate* and an *annual rate*. For the price change Δy over a year, use the annual rate:

$$\Delta y = (\textit{annual rate}) \text{ times } (y) \text{ times } (\Delta t). \tag{5}$$

Calculus applies the continuous rate to each instant *dt*. ***The price change is*** *dy*:

$$dy = (\textit{continuous rate}) \text{ times } (y) \text{ times } (dt). \tag{6}$$

Dividing by dt, this is a differential equation for the price:

$$dy/dt = (\textbf{\textit{continuous rate}}) \text{ times } (y) = .05y.$$

The solution is $y_0 e^{.05t}$. Set $t = 1$. Then $e^{.05} \approx 1.0513$ and the annual rate is 5.13%.

When you ask a bank what interest they pay, they give both rates: 8% and 8.33%. The higher one they call the "effective rate." It comes from compounding (and depends how often they do it). If the compounding is continuous, every dt brings an increase of dy—and $e^{.08}$ is near 1.0833.

Section 6.6 returns to compound interest. The interval drops from a month to a day to a second. That leads to $(1 + 1/n)^n$, and in the limit to e. Here we compute the effect of 5% continuous interest:

Future value A dollar now has the same value as $e^{.05T}$ dollars in T years.

Present value A dollar in T years has the same value as $e^{-.05T}$ dollars now.

Doubling time Prices double ($e^{.05T} = 2$) in $T = \ln 2/.05 \approx 14$ years.

With no compounding, the doubling time is 20 years. Simple interest adds on 20 times 5% = 100%. With continuous compounding the time is reduced by the factor $\ln 2 \approx .7$, regardless of the interest rate.

EXAMPLE 4 In 1626 the Indians sold Manhattan for \$24. Our calculations indicate that they knew what they were doing. Assuming 8% compound interest, the original \$24 is multiplied by $e^{.08t}$. After $t = 365$ years the multiplier is $e^{29.2}$ and the \$24 has grown to 115 trillion dollars. With that much money they could buy back the land and pay off the national debt.

This seems farfetched. Possibly there is a big flaw in the model. It is absolutely true that Ben Franklin left money to Boston and Philadelphia, to be invested for 200 years. In 1990 it yielded millions (not trillions, that takes longer). Our next step is a new model.

Question How can you estimate $e^{29.2}$ with a \$24 calculator (log but not ln)?
Answer Multiply 29.2 by $\log_{10} e = .434$ to get 12.7. This is the exponent to base 10. After that base change, we have $10^{12.7}$ or more than a trillion.

GROWTH OR DECAY WITH A SOURCE TERM

The equation $y' = y$ will be given a new term. Up to now, all growth or decay has started from y_0. No deposit or withdrawal was made later. The investment grew by itself—a pure exponential. ***The new term s allows you to add or subtract from the account.*** It is a "*source*"—or a "sink" if s is negative. The source $s = 5$ adds $5dt$, proportional to dt but not to y:

$$\textit{Constant source:} \quad dy/dt = y + 5 \text{ starting from } y = y_0.$$

Notice y on both sides! My first guess $y = e^{t+5}$ failed completely. Its derivative is e^{t+5} again, which is not $y + 5$. The class suggested $y = e^t + 5t$. But its derivative $e^t + 5$ is still not $y + 5$. We tried other ways to produce 5 in dy/dt. This idea is doomed to failure. *Finally we thought of* $y = Ae^t - 5$. That has $y' = Ae^t = y + 5$ as required.

Important: A ***is not*** y_0. Set $t = 0$ to find $y_0 = A - 5$. The source contributes $5e^t - 5$:

$$\textbf{\textit{The solution is}}\ (y_0 + 5)e^t - 5. \ \textbf{\textit{That is the same as}}\ y_0e^t + 5(e^t - 1).$$

$s = 5$ multiplies the growth term $e^t - 1$ that starts at zero. y_0e^t grows as before.

EXAMPLE 5 $dy/dt = -y + 5$ has $y = (y_0 - 5)e^{-t} + 5$. This is $y_0e^{-t} + 5(1 - e^{-t})$.

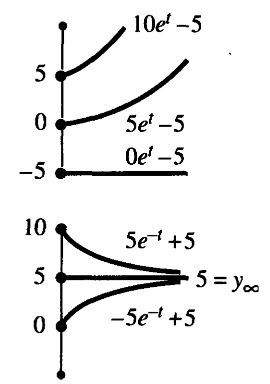


Fig. 6.9

That final term from the source is still positive. The other term y_0e^{-t} decays to zero. ***The limit as*** $t \to \infty$ ***is*** $y_\infty = 5$. A negative c leads to a steady state y_∞.

Based on these examples with $c = 1$ and $c = -1$, we can find y for any c and s.

EQUATION WITH SOURCE $\dfrac{dy}{dt} = cy + s$ starts from $y = y_0$ at $t = 0$. (7)

The source could be a deposit of $s = \$1000$/year, after an initial investment of $y_0 = \$8000$. Or we can withdraw funds at $s = -\$200$/year. The units are "dollars per year" to match dy/dt. The equation feeds in \$1000 or removes \$200 *continuously*—not all at once.

Note again that $y = e^{(c+s)t}$ is not a solution. Its derivative is $(c + s)y$. The combination $y = e^{ct} + s$ is also not a solution (but closer). ***The analysis of*** $y' = cy + s$ ***will be our main achievement for differential equations*** (in this section). The equation is not restricted to finance—far from it—but that produces excellent examples.

I propose to find y in four ways. You may feel that one way is enough.† The first way is the fastest—only three lines—but please give the others a chance. There is no point in preparing for real problems if we don't solve them.

Solution by Method 1 (fast way) Substitute the combination $y = Ae^{ct} + B$. ***The solution has this form—exponential plus constant.*** From two facts we find A and B:

$$\text{the equation } y' = cy + s \text{ gives } cAe^{ct} = c(Ae^{ct} + B) + s$$

$$\text{the initial value at } t = 0 \text{ gives } A + B = y_0.$$

†My class says one way is *more* than enough. They just want the answer. Sometimes I cave in and write down the formula: y is y_0e^{ct} plus $s(e^{ct} - 1)/c$ from the source term.

The first line has cAe^{ct} on both sides. Subtraction leaves $cB + s = 0$, or $B = -s/c$. Then the second line becomes $A = y_0 - B = y_0 + (s/c)$:

KEY FORMULA $$y = \left(y_0 + \frac{s}{c}\right)e^{ct} - \frac{s}{c} \quad \text{or} \quad y = y_0 e^{ct} + \frac{s}{c}(e^{ct} - 1). \tag{8}$$

With $s = 0$ this is the old solution $y_0 e^{ct}$ (no source). The example with $c = 1$ and $s = 5$ produced $(y_0 + 5)e^t - 5$. Separating the source term gives $y_0 e^t + 5(e^t - 1)$.

Solution by Method 2 (slow way) The input y_0 produces the output $y_0 e^{ct}$. After t years any deposit is multiplied by e^{ct}. *That also applies to deposits made after the account is opened.* If the deposit enters at time T, the growing time is only $t - T$. Therefore the multiplying factor is only $e^{c(t-T)}$. This growth factor applies to the small deposit (amount $s\,dT$) made between time T and $T + dT$.

Now add up all outputs at time t. The output from y_0 is $y_0 e^{ct}$. The small deposit $s\,dT$ near time T grows to $e^{c(t-T)} s\,dT$. The total is an integral:

$$y(t) = y_0 e^{ct} + \int_{T=0}^{t} e^{c(t-T)} s\,dT. \tag{9}$$

This principle of Duhamel would still apply when the source s varies with time. Here s is constant, and the integral divides by c:

$$s\int_{T=0}^{t} e^{c(t-T)}dT = \frac{se^{c(t-T)}}{-c}\Big]_0^t = -\frac{s}{c} + \frac{s}{c}e^{ct}. \tag{10}$$

That agrees with the source term from Method 1, at the end of equation (8). There we looked for "exponential plus constant," here we added up outputs.

Method 1 was easier. It succeeded because we knew the form $Ae^{ct} + B$—with "undetermined coefficients." Method 2 is more complete. The form for y is part of the output, not the input. The source s is a continuous supply of new deposits, all growing separately. Section 6.5 starts from scratch, by directly integrating $y' = cy + s$.

Remark Method 2 is often described in terms of an *integrating factor*. First write the equation as $y' - cy = s$. Then multiply by a magic factor that makes integration possible:

$(y' - cy)e^{-ct} = se^{-ct}$	multiply by the factor e^{-ct}
$ye^{-ct}\Big]_0^t = -\frac{s}{c}e^{-ct}\Big]_0^t$	integrate both sides
$ye^{-ct} - y_0 = -\frac{s}{c}(e^{-ct} - 1)$	substitute 0 and t
$y = e^{ct}y_0 + \frac{s}{c}(e^{ct} - 1)$	isolate y to reach formula (8)

The integrating factor produced a perfect derivative in line 1. I prefer Duhamel's idea, that all inputs y_0 and s grow the same way. Either method gives formula (8) for y.

THE MATHEMATICS OF FINANCE (AT A CONTINUOUS RATE)

The question from finance is this: *What inputs give what outputs?* The inputs can come at the start by y_0, or continuously by s. The output can be paid at the end or continuously. There are six basic questions, two of which are already answered.

The future value is $y_0 e^{ct}$ from a deposit of y_0. To produce y in the future, deposit the present value ye^{-ct}. *Questions* **3–6** *involve the source term s.* We fix the continuous rate at 5% per year ($c = .05$), and start the account from $y_0 = 0$. The answers come fast from equation (8).

Question 3 With deposits of $s = \$1000$/year, how large is y after 20 years?

$$y = \frac{s}{c}(e^{ct} - 1) = \frac{1000}{.05}(e^{(.05)(20)} - 1) = 20{,}000(e - 1) \approx \$34{,}400.$$

One big deposit yields $20{,}000e \approx \$54{,}000$. The same 20,000 via s yields \$34,400.

Notice a small by-product (for mathematicians). When the interest rate is $c = 0$, our formula $s(e^{ct} - 1)/c$ turns into 0/0. We are absolutely sure that depositing \$1000/year with no interest produces \$20,000 after 20 years. But this is not obvious from 0/0. By l'Hôpital's rule we take c-derivatives in the fraction:

$$\lim_{c \to 0} \frac{s(e^{ct} - 1)}{c} = \lim_{c \to 0} \frac{ste^{ct}}{1} = st. \text{ This is } (1000)(20) = 20{,}000. \qquad (11)$$

Question 4 What continuous deposit of s per year yields \$20,000 after 20 years?

$$20{,}000 = \frac{s}{.05}(e^{(.05)(20)} - 1) \text{ requires } s = \frac{1000}{e - 1} \approx 582.$$

Deposits of \$582 over 20 years total \$11,640. A single deposit of $y_0 = 20{,}000/e =$ \$7,360 produces the same \$20,000 at the end. Better to be rich at $t = 0$.

Questions **1** and **2** had $s = 0$ (no source). Questions **3** and **4** had $y_0 = 0$ (no initial deposit). Now we come to $y = 0$. In **5**, everything is paid out by an ***annuity***. In **6**, everything is paid up on a ***loan***.

Question 5 What deposit y_0 provides \$1000/year for 20 years? End with $y = 0$.

$$y = y_0 e^{ct} + \frac{s}{c}(e^{ct} - 1) = 0 \text{ requires } y_0 = \frac{-s}{c}(1 - e^{-ct}).$$

Substituting $s = -1000$, $c = .05$, $t = 20$ gives $y_0 \approx 12{,}640$. If you win \$20,000 in a lottery, and it is paid over 20 years, the lottery only has to put in \$12,640. Even less if the interest rate is above 5%.

Question 6 What payments s will clear a loan of $y_0 = \$20{,}000$ in 20 years?

Unfortunately, s exceeds \$1000 per year. The bank gives up more than the \$20,000 to buy your car (and pay tuition). *It also gives up the interest on that money*. You pay that back too, but you don't have to stay even at every moment. Instead you repay at a *constant rate* for 20 years. Your payments mostly cover interest at the start and principal at the end. After $t = 20$ years you are even and your debt is $y = 0$.

This is like Question **5** (also $y = 0$), but now we know y_0 and we want s:

$$y = y_0 e^{ct} + \frac{s}{c}(e^{ct} - 1) = 0 \text{ requires } s = -cy_0 e^{ct}/(e^{ct} - 1).$$

The loan is $y_0 = \$20{,}000$, the rate is $c = .05$/year, the time is $t = 20$ years. Substituting in the formula for s, your payments are \$1582 per year.

Puzzle How is $s = \$1582$ for loan payments related to $s = \$582$ for deposits?

$$0 \to \$582 \text{ per year} \to \$20{,}000 \quad \text{and} \quad \$20{,}000 \to -\$1582 \text{ per year} \to 0.$$

That difference of exactly 1000 cannot be an accident. 1582 and 582 came from

$$1000\frac{e}{e-1} \text{ and } 1000\frac{1}{e-1} \text{ with difference } 1000\frac{e-1}{e-1} = 1000.$$

Why? Here is the real reason. Instead of repaying 1582 we can pay only 1000 (to keep even with the interest on 20,000). The other 582 goes into a separate account. After 20 years the continuous 582 has built up to 20,000 (including interest as in Question **4**). From that account we pay back the loan.

Section 6.6 deals with daily compounding—which differs from continuous compounding by only a few cents. Yearly compounding differs by a few dollars.

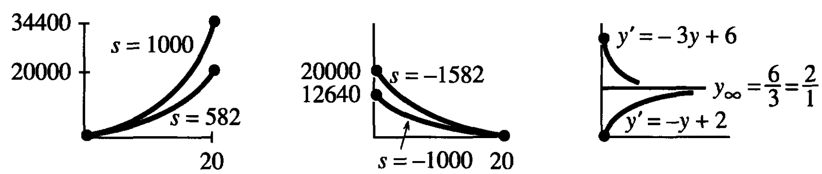


Fig. 6.10 Questions **3–4** deposit s. Questions **5–6** repay loan or annuity. Steady state $-s/c$.

TRANSIENTS VS. STEADY STATE

Suppose there is decay instead of growth. The constant c is negative and y_0e^{ct} dies out. That is the "*transient*" term, which disappears as $t \to \infty$. What is left is the "*steady state*." We denote that limit by y_∞.

Without a source, y_∞ is zero (total decay). When s is present, $y_\infty = -s/c$:

> **6J** The solution $y = \left(y_0 + \frac{s}{c}\right)e^{ct} - \frac{s}{c}$ approaches $y_\infty = -\frac{s}{c}$ when $e^{ct} \to 0$.

At this steady state, the source s exactly balances the decay cy. In other words $cy + s = 0$. From the left side of the differential equation, this means $dy/dt = 0$. *There is no change*. That is why y_∞ is steady.

Notice that y_∞ depends on the source and on c—*but not on* y_0.

EXAMPLE 6 Suppose Bermuda has a birth rate $b = .02$ and death rate $d = .03$. The net decay rate is $c = -.01$. There is also immigration from outside, of $s = 1200$/year. The initial population might be $y_0 = 5$ thousand or $y_0 = 5$ million, but that number has no effect on y_∞. ***The steady state is independent of*** y_0.

In this case $y_\infty = -s/c = 1200/.01 = 120{,}000$. The population grows to 120,000 if y_0 is smaller. It decays to 120,000 if y_0 is larger.

EXAMPLE 7 ***Newton's Law of Cooling***: $\qquad dy/dt = c(y - y_\infty). \qquad (12)$

This is back to physics. The temperature of a body is y. The temperature around it is y_∞. Then y starts at y_0 and approaches y_∞, following Newton's rule: *The rate is proportional to* $y - y_\infty$. The bigger the difference, the faster heat flows.

The equation has $-cy_\infty$ where before we had s. That fits with $y_\infty = -s/c$. For the solution, replace s by $-cy_\infty$ in formula (8). Or use this new method:

Solution by Method 3 ***The new idea is to look at the difference*** $y - y_\infty$. Its derivative is dy/dt, since y_∞ is constant. But dy/dt is $c(y - y_\infty)$—this is our equation. The difference starts from $y_0 - y_\infty$, and grows or decays as a pure exponential:

$$\frac{d}{dt}(y - y_\infty) = c(y - y_\infty) \quad \textit{has the solution} \quad (y - y_\infty) = (y_0 - y_\infty)e^{ct}. \qquad (13)$$

This solves the law of cooling. We repeat Method 3 using the letters s and c:

$$\frac{d}{dt}\left(y + \frac{s}{c}\right) = c\left(y + \frac{s}{c}\right) \quad \textit{has the solution} \quad \left(y + \frac{s}{c}\right) = \left(y_0 + \frac{s}{c}\right)e^{ct}. \qquad (14)$$

Moving s/c to the right side recovers formula (8). There is a *constant term* and an *exponential term*. In a differential equations course, those are the "***particular solution***" and the "***homogeneous solution.***" In a calculus course, it's time to stop.

EXAMPLE 8 In a 70° room, Newton's corpse is found with a temperature of 90°. A day later the body registers 80°. When did he stop integrating (at 98.6°)?

Solution Here $y_\infty = 70$ and $y_0 = 90$. Newton's equation (13) is $y = 20e^{ct} + 70$. Then $y = 80$ at $t = 1$ gives $20e^c = 10$. The rate of cooling is $c = \ln \frac{1}{2}$. Death occurred when $20e^{ct} + 70 = 98.6$ or $e^{ct} = 1.43$. The time was $t = \ln 1.43/\ln \frac{1}{2}$ = half a day earlier.

6.3 EXERCISES

Read-through exercises

If $y' = cy$ then $y(t)$ = __a__. If $dy/dt = 7y$ and $y_0 = 4$ then $y(t)$ = __b__. This solution reaches 8 at t = __c__. If the doubling time is T then c = __d__. If $y' = 3y$ and $y(1) = 9$ then y_0 was __e__. When c is negative, the solution approaches __f__ as $t \to \infty$.

The constant solution to $dy/dt = y + 6$ is y = __g__. The general solution is $y = Ae^t - 6$. If $y_0 = 4$ then A = __h__. The solution of $dy/dt = cy + s$ starting from y_0 is $y = Ae^{ct} + B$ = __i__. The output from the source s is __j__. An input at time T grows by the factor __k__ at time t.

At $c = 10\%$, the interest in time dt is dy = __l__. This equation yields $y(t)$ = __m__. With a source term instead of y_0, a continuous deposit of $s = 4000$/year yields y = __n__ after 10 years. The deposit required to produce 10,000 in 10 years is s = __o__ (exactly or approximately). An income of 4000/year forever (!) comes from y_0 = __p__. The deposit to give 4000/year for 20 years is y_0 = __q__. The payment rate s to clear a loan of 10,000 in 10 years is __r__.

The solution to $y' = -3y + s$ approaches y_∞ = __s__.

Solve 1–4 starting from $y_0 = 1$ and from $y_0 = -1$. Draw both solutions on the same graph.

1 $\frac{dy}{dt} = 2t$ 2 $\frac{dy}{dt} = -t$ 3 $\frac{dy}{dt} = 2y$ 4 $\frac{dy}{dt} = -y$

Solve 5–8 starting from $y_0 = 10$. At what time does y increase to 100 or drop to 1?

5 $\frac{dy}{dt} = 4y$ 6 $\frac{dy}{dt} = 4t$ 7 $\frac{dy}{dt} = e^{4t}$ 8 $\frac{dy}{dt} = e^{-4t}$

9 Draw a field of "tangent arrows" for $y' = -y$, with the solution curves $y = e^{-x}$ and $y = -e^{-x}$.

10 Draw a direction field of arrows for $y' = y - 1$, with solution curves $y = e^x + 1$ and $y = 1$.

Problems 11–27 involve y_0e^{ct}. They ask for c or t or y_0.

11 If a culture of bacteria doubles in two hours, how many hours to multiply by 10? First find c.

12 If bacteria increase by factor of ten in ten hours, how many hours to increase by 100? What is c?

13 How old is a skull that contains $\frac{1}{5}$ as much radiocarbon as a modern skull?

14 If a relic contains 90% as much radiocarbon as new material, could it come from the time of Christ?

15 The population of Cairo grew from 5 million to 10 million in 20 years. From $y' = cy$ find c. When was $y = 8$ million?

16 The populations of New York and Los Angeles are growing at 1% and 1.4% a year. Starting from 8 million (NY) and 6 million (LA), when will they be equal?

17 Suppose the value of \$1 in Japanese yen decreases at 2% per year. Starting from \$1 = Y240, when will 1 dollar equal 1 yen?

18 The effect of advertising decays exponentially. If 40% remember a new product after three days, find c. How long will 20% remember it?

19 If $y = 1000$ at $t = 3$ and $y = 3000$ at $t = 4$ (exponential growth), what was y_0 at $t = 0$?

20 If $y = 100$ at $t = 4$ and $y = 10$ at $t = 8$ (exponential decay) when will $y = 1$? What was y_0?

21 Atmospheric pressure decreases with height according to $dp/dh = cp$. The pressures at $h = 0$ (sea level) and $h = 20$ km are 1013 and 50 millibars. Find c. Explain why $p = \sqrt{1013 \cdot 50}$ halfway up at $h = 10$.

22 For exponential decay show that $y(t)$ is the square root of $y(0)$ times $y(2t)$. How could you find $y(3t)$ from $y(t)$ and $y(2t)$?

23 Most drugs in the bloodstream decay by $y' = cy$ (***first-order kinetics***). (a) The half-life of morphine is 3 hours. Find its decay constant c (with units). (b) The half-life of nicotine is 2 hours. After a six-hour flight what fraction remains?

24 How often should a drug be taken if its dose is 3 mg, it is cleared at $c = .01$/hour, and 1 mg is required in the bloodstream at all times? (The doctor decides this level based on body size.)

25 The antiseizure drug dilantin has constant clearance rate $y' = -a$ until $y = y_1$. Then $y' = -ay/y_1$. Solve for $y(t)$ in two pieces from y_0. When does y reach y_1?

26 The actual elimination of nicotine is multiexponential: $y = Ae^{ct} + Be^{Ct}$. The first-order equation $(d/dt - c)y = 0$ changes to the second-order equation $(d/dt - c)(d/dt - C)y = 0$. Write out this equation starting with y'', and show that it is satisfied by the given y.

27 ***True or false.*** **If false, say what's true.**

(a) The time for $y = e^{ct}$ to double is $(\ln 2)/(\ln c)$.

(b) If $y' = cy$ and $z' = cz$ then $(y + z)' = 2c(y + z)$.

(c) If $y' = cy$ and $z' = cz$ then $(y/z)' = 0$.

(d) If $y' = cy$ and $z' = Cz$ then $(yz)' = (c + C)yz$.

28 A rocket has velocity v. Burnt fuel of mass Δm leaves at velocity $v - 7$. Total momentum is constant:

$$mv = (m - \Delta m)(v + \Delta v) + \Delta m(v - 7).$$

What differential equation connects m to v? Solve for $v(m)$ not $v(t)$, starting from $v_0 = 20$ and $m_0 = 4$.

Problems 29–36 are about solutions of $y' = cy + s$.

29 Solve $y' = 3y + 1$ with $y_0 = 0$ by assuming $y = Ae^{3t} + B$ and determining A and B.

30 Solve $y' = 8 - y$ starting from y_0 and $y = Ae^{-t} + B$.

Solve 31–34 with $y_0 = 0$ and graph the solution.

31 $\dfrac{dy}{dt} = y + 1$ **32** $\dfrac{dy}{dt} = y - 1$

33 $\dfrac{dy}{dt} = -y + 1$ **34** $\dfrac{dy}{dt} = -y - 1$

35 (a) What value $y =$ constant solves $dy/dt = -2y + 12$?

(b) Find the solution with an arbitrary constant A.

(c) What solutions start from $y_0 = 0$ and $y_0 = 10$?

(d) What is the steady state y_∞?

36 Choose $\pm$ signs in $dy/dt = \pm 3y \pm 6$ to achieve the following results starting from $y_0 = 1$. Draw graphs.

(a) y increases to ∞ (b) y increases to 2

(c) y decreases to -2 (d) y decreases to $-\infty$

37 What value $y =$ constant solves $dy/dt = 4 - y$? Show that $y(t) = Ae^{-t} + 4$ is also a solution. Find $y(1)$ and y_∞ if $y_0 = 3$.

38 Solve $y' = y + e^t$ from $y_0 = 0$ by Method 2, where the deposit e^T at time T is multiplied by e^{t-T}. The total output at time t is $y(t) = \int_0^t e^T e^{t-T}\, dT =$ ________. Substitute back to check $y' = y + e^t$.

39 Rewrite $y' = y + e^t$ as $y' - y = e^t$. Multiplying by e^{-t}, the left side is the derivative of ________. Integrate both sides from $y_0 = 0$ to find $y(t)$.

40 Solve $y' = -y + 1$ from $y_0 = 0$ by rewriting as $y' + y = 1$, multiplying by e^t, and integrating both sides.

41 Solve $y' = y + t$ from $y_0 = 0$ by assuming $y = Ae^t + Bt + C$.

Problems 42–57 are about the mathematics of finance.

42 Dollar bills decrease in value at $c = -.04$ per year because of inflation. If you hold \$1000, what is the decrease in dt years? At what rate s should you print money to keep even?

43 If a bank offers annual interest of $7\frac{1}{2}\%$ or continuous interest of $7\frac{1}{4}\%$, which is better?

44 What continuous interest rate is equivalent to an annual rate of 9%? Extra credit: Telephone a bank for both rates and check their calculation.

45 At 100% interest ($c = 1$) how much is a continuous deposit of s per year worth after one year? What initial deposit y_0 would have produced the same output?

46 To have \$50,000 for college tuition in 20 years, what gift y_0 should a grandparent make now? Assume $c = 10\%$. What continuous deposit should a parent make during 20 years? If the parent saves $s = \$1000$ per year, when does he or she reach \$50,000 and retire?

47 Income per person grows 3%, the population grows 2%, the total income grows ________. Answer if these are (a) annual rates (b) continuous rates.

48 When $dy/dt = cy + 4$, how much is the deposit of $4dT$ at time T worth at the later time t? What is the value at $t = 2$ of deposits $4dT$ from $T = 0$ to $T = 1$?

49 Depositing $s = \$1000$ per year leads to $34,400 after 20 years (Question 3). To reach the same result, when should you deposit $20,000 all at once?

50 For how long can you withdraw $s = \$500$/year after depositing $y_0 = \$5000$ at 8%, before you run dry?

51 What continuous payment s clears a $1000 loan in 60 days, if a loan shark charges 1% per day continuously?

52 You are the loan shark. What is $1 worth after a year of continuous compounding at 1% per day?

53 You can afford payments of $s = \$100$ per month for 48 months. If the dealer charges $c = 6\%$, how much can you borrow?

54 Your income is I_0e^{2ct} per year. Your expenses are E_0e^{ct} per year. (a) At what future time are they equal? (b) If you borrow the difference until then, how much money have you borrowed?

55 If a student loan in your freshman year is repaid plus 20% four years later, what was the effective interest rate?

56 Is a variable rate mortgage with $c = .09 + .001t$ for 20 years better or worse than a fixed rate of 10%?

57 At 10% instead of 8%, the $24 paid for Manhattan is worth ________ after 365 years.

Problems 58–65 approach a steady state y_∞ as $t \to \infty$.

58 If $dy/dt = -y + 7$ what is y_∞? What is the derivative of $y - y_\infty$? Then $y - y_\infty$ equals $y_0 - y_\infty$ times ________.

59 Graph $y(t)$ when $y' = 3y - 12$ and y_0 is

(a) below 4 (b) equal to 4 (c) above 4

60 The solutions to $dy/dt = c(y - 12)$ converge to $y_\infty =$ ________ provided c is ________.

61 Suppose the time unit in $dy/dt = cy$ changes from minutes to hours. How does the equation change? How does $dy/dt = -y + 5$ change? How does y_∞ change?

62 ***True or false***, when y_1 and y_2 both satisfy $y' = cy + s$.

(a) The sum $y = y_1 + y_2$ also satisfies this equation.

(b) The average $y = \frac{1}{2}(y_1 + y_2)$ satisfies the same equation.

(c) The derivative $y = y_1'$ satisfies the same equation.

63 If Newton's coffee cools from 80° to 60° in 12 minutes (room temperature 20°), find c. When was the coffee at 100°?

64 If $y_0 = 100$ and $y(1) = 90$ and $y(2) = 84$, what is y_∞?

65 If $y_0 = 100$ and $y(1) = 90$ and $y(2) = 81$, what is y_∞?

66 To cool down coffee, should you add milk now or later? The coffee is at 70°C, the milk is at 10°, the room is at 20°.

(a) Adding 1 part milk to 5 parts coffee makes it 60°. With $y_\infty = 20°$, the white coffee cools to $y(t) =$ ________.

(b) The black coffee cools to $y_c(t) =$ ________. The milk warms to $y_m(t) =$ ________. Mixing at time t gives $(5y_c + y_m)/6 =$ ________.

6.4 Logarithms

We have given first place to e^x and a lower place to $\ln x$. In applications that is absolutely correct. But logarithms have one important theoretical advantage (plus many applications of their own). The advantage is that the derivative of $\ln x$ is $1/x$, whereas the derivative of e^x is e^x. We can't define e^x as its own integral, without circular reasoning. But we can and do define $\ln x$ (the ***natural logarithm***) as the integral of the "-1 power" which is $1/x$:

$$\ln x = \int_1^x \frac{1}{x}\,dx \quad \text{or} \quad \ln y = \int_1^y \frac{1}{u}\,du. \tag{1}$$

Note the dummy variables, first x then u. Note also the live variables, first x then y. Especially note the lower limit of integration, which is 1 and not 0. ***The logarithm is the area measured from*** 1. Therefore $\ln 1 = 0$ at that starting point—as required.

Earlier chapters integrated all powers except this "-1 power." The logarithm is that missing integral. The curve in Figure 6.11 has height $y = 1/x$—it is a hyperbola. At $x = 0$ the height goes to infinity and the area becomes infinite: $\log 0 = -\infty$. The minus sign is because the integral goes backward from 1 to 0. The integral does not extend past zero to negative x. We are defining $\ln x$ only for $x > 0$.†

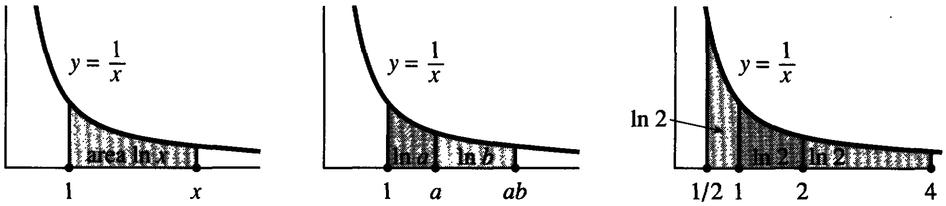


Fig. 6.11 ***Logarithm as area.*** Neighbors $\ln a + \ln b = \ln ab$. Equal areas: $-\ln \frac{1}{2} = \ln 2 = \frac{1}{2} \ln 4$.

With this new approach, $\ln x$ has a direct definition. *It is an integral* (or an area). Its two key properties must follow from this definition. That step is a beautiful application of the theory behind integrals.

***Property* 1**: $\ln ab = \ln a + \ln b$. The areas from 1 to a and from a to ab combine into a single area (1 to ab in the middle figure):

$$\text{Neighboring areas: } \int_1^a \frac{1}{x}\,dx + \int_a^{ab} \frac{1}{x}\,dx = \int_1^{ab} \frac{1}{x}\,dx. \tag{2}$$

The right side is $\ln ab$, from definition (1). The first term on the left is $\ln a$. The problem is to show that the second integral (a to ab) is $\ln b$:

$$\int_a^{ab} \frac{1}{x}\,dx \stackrel{(?)}{=} \int_1^b \frac{1}{u}\,du = \ln b. \tag{3}$$

We need $u = 1$ when $x = a$ (the lower limit) and $u = b$ when $x = ab$ (the upper limit). The choice $u = x/a$ satisfies these requirements. Substituting $x = au$ and $dx = a\,du$ yields $dx/x = du/u$. Equation (3) gives $\ln b$, and equation (2) is $\ln a + \ln b = \ln ab$.

***Property* 2**: $\ln b^n = n \ln b$. These are the left and right sides of

$$\int_1^{b^n} \frac{1}{x}\,dx \stackrel{(?)}{=} n \int_1^b \frac{1}{u}\,du. \tag{4}$$

This comes from the substitution $x = u^n$. The lower limit $x = 1$ corresponds to $u = 1$, and $x = b^n$ corresponds to $u = b$. The differential dx is $nu^{n-1}du$. Dividing by $x = u^n$ leaves $dx/x = n\,du/u$. Then equation (4) becomes $\ln b^n = n \ln b$.

Everything comes logically from the definition as an area. Also definite integrals:

EXAMPLE 1 Compute $\int_x^{3x} \frac{1}{t}\,dt$. Solution: $\ln 3x - \ln x = \ln \frac{3x}{x} = \ln 3$.

EXAMPLE 2 Compute $\int_{.1}^{1} \frac{1}{x}\,dx$. Solution: $\ln 1 - \ln .1 = \ln 10$. (Why?)

†The logarithm of -1 is πi (an imaginary number). That is because $e^{\pi i} = -1$. The logarithm of i is also imaginary—it is $\frac{1}{2}\pi i$. In general, logarithms are complex numbers.

EXAMPLE 3 Compute $\int_1^{e^2} \frac{1}{u} du$. Solution: $\ln e^2 = 2$. The area from 1 to e^2 is 2.

Remark While working on the theory this is a chance to straighten out old debts. The book has discussed and computed (and even differentiated) the functions e^x and b^x and x^n, without defining them properly. When the exponent is an irrational number like π, ***how do we multiply e by itself π times***? One approach (not taken) is to come closer and closer to π by rational exponents like 22/7. Another approach (taken now) is to determine the number $e^\pi = 23.1\ldots$ by its logarithm.† Start with e itself:

e is (by definition) the number whose logarithm is 1

e^π is (by definition) the number whose logarithm is π.

When the area in Figure 6.12 ***reaches*** 1, ***the basepoint is*** e. When the area reaches π, the basepoint is e^π. We are constructing the inverse function (which is e^x). But how do we know that the area reaches π or 1000 or -1000 at exactly one point? (The area is 1000 far out at e^{1000}. The area is -1000 very near zero at e^{-1000}.) To define e we have to know that somewhere the area equals 1!

For a proof in two steps, go back to Figure 6.11c. The area from 1 to 2 is *more than* $\frac{1}{2}$ (because $1/x$ is more than $\frac{1}{2}$ on that interval of length one). The combined area from 1 to 4 is more than 1. ***We come to*** area = 1 ***before reaching*** 4. (Actually at $e = 2.718\ldots$.) Since $1/x$ is positive, the area is increasing and never comes back to 1.

To double the area we have to square the distance. The logarithm creeps upwards:

$$\ln x \to \infty \quad \text{but} \quad \frac{\ln x}{x} \to 0. \tag{5}$$

The logarithm grows slowly because e^x ***grows so fast*** (and vice versa—they are inverses). Remember that e^x goes past every power x^n. Therefore $\ln x$ is passed by every root $x^{1/n}$. Problems 60 and 61 give two proofs that $(\ln x)/x^{1/n}$ approaches zero.

We might compare $\ln x$ with $\sqrt{x}$. At $x = 10$ they are close (2.3 versus 3.2). But out at $x = e^{10}$ the comparison is 10 against e^5, and $\ln x$ loses to $\sqrt{x}$.

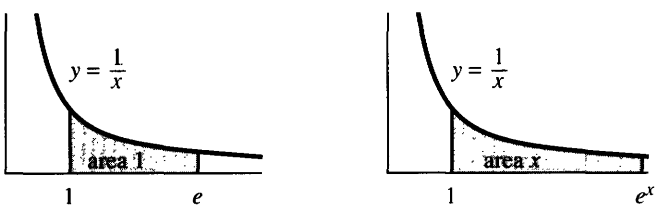


Fig. 6.12 Area is logarithm of basepoint.

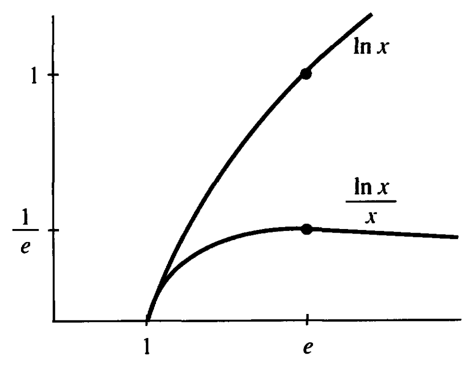


Fig. 6.13 $\ln x$ grows more slowly than x.

†Chapter 9 goes on to *imaginary exponents*, and proves the remarkable formula $e^{\pi i} = -1$.

APPROXIMATION OF LOGARITHMS

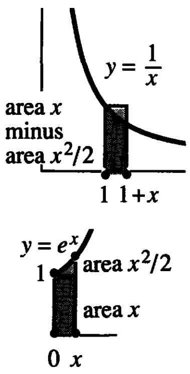


Fig. 6.14

The limiting cases $\ln 0 = -\infty$ and $\ln \infty = +\infty$ are important. More important are logarithms near the starting point $\ln 1 = 0$. Our question is: ***What is*** $\ln(1+x)$ ***for*** x ***near zero***? The exact answer is an area. The approximate answer is much simpler. If x (positive or negative) is small, then

$$\ln(1+x) \approx x \quad \text{and} \quad e^x \approx 1 + x. \tag{6}$$

The calculator gives $\ln 1.01 = .0099503$. This is close to $x = .01$. Between 1 and $1 + x$ the area under the graph of $1/x$ is nearly a rectangle. Its base is x and its height is 1. So the curved area $\ln(1+x)$ is close to the rectangular area x. Figure 6.14 shows how a small triangle is chopped off at the top.

The difference between .0099503 (actual) and .01 (linear approximation) is $-.0000497$. That is predicted almost exactly by the second derivative: $\frac{1}{2}$ times $(\Delta x)^2$ times $(\ln x)''$ is $\frac{1}{2}(.01)^2(-1) = -.00005$. ***This is the area of the small triangle***!

$$\ln(1+x) \approx \textbf{\textit{rectangular area minus triangular area}} = x - \tfrac{1}{2}x^2.$$

The remaining mistake of .0000003 is close to $\frac{1}{3}x^3$ (Problem 65).

May I switch to e^x? Its slope starts at $e^0 = 1$, so its linear approximation is $1 + x$. Then $\ln(e^x) \approx \ln(1+x) \approx x$. *Two wrongs do make a right*: $\ln(e^x) = x$ exactly.

The calculator gives $e^{.01}$ as 1.0100502 (actual) instead of 1.01 (approximation). The second-order correction is again a small triangle: $\frac{1}{2}x^2 = .00005$. The complete series for $\ln(1+x)$ and e^x are in Sections 10.1 and 6.6:

$$\ln(1+x) = x - x^2/2 + x^3/3 - \ldots \qquad e^x = 1 + x + x^2/2 + x^3/6 + \ldots.$$

DERIVATIVES BASED ON LOGARITHMS

Logarithms turn up as antiderivatives very often. To build up a collection of integrals, we now differentiate $\ln u(x)$ by the chain rule.

6K **The derivative of $\ln x$ is $\frac{1}{x}$.** ***The derivative of*** $\ln u(x)$ ***is*** $\frac{1}{u}\frac{du}{dx}$.

The slope of $\ln x$ was hard work in Section 6.2. With its new definition (the integral of $1/x$) the work is gone. By the Fundamental Theorem, the slope must be $1/x$.

For $\ln u(x)$ the derivative comes from the chain rule. The inside function is u, the outside function is ln. (Keep $u > 0$ to define $\ln u$.) The chain rule gives

$$\frac{d}{dx}\ln cx = \frac{1}{cx}c = \frac{1}{x}(!) \qquad \frac{d}{dx}\ln x^3 = 3x^2/x^3 = \frac{3}{x}$$

$$\frac{d}{dx}\ln(x^2+1) = 2x/(x^2+1) \qquad \frac{d}{dx}\ln\cos x = \frac{-\sin x}{\cos x} = -\tan x$$

$$\frac{d}{dx}\ln e^x = e^x/e^x = 1 \qquad \frac{d}{dx}\ln(\ln x) = \frac{1}{\ln x}\frac{1}{x}.$$

Those are worth another look, especially the first. Any reasonable person would expect the slope of $\ln 3x$ to be $3/x$. ***Not so***. The 3 cancels, and $\ln 3x$ has the same slope as $\ln x$. (The real reason is that $\ln 3x = \ln 3 + \ln x$.) The antiderivative of $3/x$ is not $\ln 3x$ but $3 \ln x$, which is $\ln x^3$.

Before moving to integrals, here is a new method for derivatives: ***logarithmic differentiation*** or **LD**. It applies to *products* and *powers*. The product and power rules are always available, but sometimes there is an easier way.

Main idea: The logarithm of a product $p(x)$ is a *sum of logarithms*. Switching to $\ln p$, the sum rule just adds up the derivatives. But there is a catch at the end, as you see in the example.

EXAMPLE 4 Find dp/dx if $p(x) = x^x\sqrt{x-1}$. Here $\ln p(x) = x \ln x + \frac{1}{2}\ln(x-1)$.

Take the derivative of $\ln p$: $\dfrac{1}{p}\dfrac{dp}{dx} = x \cdot \dfrac{1}{x} + \ln x + \dfrac{1}{2(x-1)}$.

Now multiply by $p(x)$: $\dfrac{dp}{dx} = p\left(1 + \ln x + \dfrac{1}{2(x-1)}\right)$.

The catch is that last step. Multiplying by p complicates the answer. This can't be helped—logarithmic differentiation contains no magic. The derivative of $p = fg$ is the same as from the product rule: $\ln p = \ln f + \ln g$ gives

$$\frac{p'}{p} = \frac{f'}{f} + \frac{g'}{g} \quad \text{and} \quad p' = p\left(\frac{f'}{f} + \frac{g'}{g}\right) = f'g + fg'. \tag{7}$$

For $p = xe^x \sin x$, with three factors, the sum has three terms:

$$\ln p = \ln x + x + \ln \sin x \text{ and } p' = p\left[\frac{1}{x} + 1 + \frac{\cos x}{\sin x}\right].$$

We multiply** p **times** p'/p* (the derivative of $\ln p$). ***Do the same for powers:

EXAMPLE 5 $p = x^{1/x} \Rightarrow \ln p = \dfrac{1}{x}\ln x \Rightarrow \dfrac{dp}{dx} = p\left[\dfrac{1}{x^2} - \dfrac{\ln x}{x^2}\right]$.

EXAMPLE 6 $p = x^{\ln x} \Rightarrow \ln p = (\ln x)^2 \Rightarrow \dfrac{dp}{dx} = p\left[\dfrac{2 \ln x}{x}\right]$.

EXAMPLE 7 $p = x^{1/\ln x} \Rightarrow \ln p = \dfrac{1}{\ln x}\ln x = 1 \Rightarrow \dfrac{dp}{dx} = 0$ (!)

INTEGRALS BASED ON LOGARITHMS

Now comes an important step. Many integrals produce logarithms. The foremost example is $1/x$, whose integral is $\ln x$. In a certain way that is the only example, but its range is enormously extended by the chain rule. The derivative of $\ln u(x)$ is u'/u, so the integral goes from u'/u back to $\ln u$:

$$\int \frac{du/dx}{u(x)}\,dx = \ln u(x) \quad \text{or equivalently} \quad \int \frac{du}{u} = \ln u.$$

***Try to choose** $u(x)$ **so that the integral contains** du/dx **divided by** u.*

EXAMPLES $\displaystyle\int \frac{dx}{x+7} = \ln|x+7| \qquad \int \frac{dx}{cx+7} = \frac{1}{c}\ln|cx+7|$

Final remark When u is negative, $\ln u$ cannot be the integral of $1/u$. The logarithm is not defined when $u<0$. But the integral can go forward by switching to $-u$:

$$\int \frac{du/dx}{u}\,dx=\int \frac{-du/dx}{-u}\,dx=\ln(-u). \tag{8}$$

Thus $\ln(-u)$ succeeds when $\ln u$ fails.† ***The forbidden case is*** $u=0$. The integrals $\ln u$ and $\ln(-u)$, on the plus and minus sides of zero, can be combined as $\ln|u|$. Every integral that gives a logarithm allows $u<0$ by changing to the absolute value $|u|$:

$$\int_{-e}^{-1}\frac{dx}{x}=\Big[\ln|x|\Big]_{-e}^{-1}=\ln 1-\ln e \qquad \int_2^4\frac{dx}{x-5}=\Big[\ln|x-5|\Big]_2^4=\ln 1-\ln 3.$$

The areas are -1 and $-\ln 3$. The graphs of $1/x$ and $1/(x-5)$ are below the x axis. We do *not* have logarithms of negative numbers, and we will not integrate $1/(x-5)$ from 2 to 6. That crosses the forbidden point $x=5$, with infinite area on both sides.

The ratio du/u leads to important integrals. When $u=\cos x$ or $u=\sin x$, we are integrating the ***tangent*** and ***cotangent***. When there is a possibility that $u<0$, write the integral as $\ln|u|$.

$$\int \tan x\,dx=\int\frac{\sin x}{\cos x}dx=-\ln|\cos x| \qquad \int\frac{x\,dx}{x^2+7}=\frac{1}{2}\ln(x^2+7)$$

$$\int \cot x\,dx=\int\frac{\cos x}{\sin x}dx=\ln|\sin x| \qquad \int\frac{dx}{x\ln x}=\ln|\ln x|$$

Now we report on the ***secant*** and ***cosecant***. The integrals of $1/\cos x$ and $1/\sin x$ also surrender to an attack by logarithms — based on a crazy trick:

$$\int \sec x\,dx=\int \sec x\left(\frac{\sec x+\tan x}{\sec x+\tan x}\right)dx=\ln|\sec x+\tan x|. \tag{9}$$

$$\int \csc x\,dx=\int \csc x\left(\frac{\csc x-\cot x}{\csc x-\cot x}\right)dx=\ln|\csc x-\cot x|. \tag{10}$$

Here $u=\sec x+\tan x$ is in the denominator; $du/dx=\sec x\tan x+\sec^2 x$ is above it. The integral is $\ln|u|$. Similarly (10) contains du/dx over $u=\csc x-\cot x$.

In closing we integrate $\ln x$ itself. The derivative of $x\ln x$ is $\ln x+1$. To remove the extra 1, subtract x from the integral: $\int \ln x\,dx=x\ln x-x$.

In contrast, the area under $1/(\ln x)$ has no elementary formula. Nevertheless it is the key to the greatest approximation in mathematics—the ***prime number theorem.*** *The area* $\int_a^b dx/\ln x$ *is approximately the number of primes between a and b.* Near e^{1000}, about 1/1000 of the integers are prime.

6.4 EXERCISES

Read-through questions

The natural logarithm of x is $\int_1^x$ __a__. This definition leads to $\ln xy=$ __b__ and $\ln x^n=$ __c__. Then e is the number whose logarithm (area under $1/x$ curve) is __d__. Similarly e^x is now defined as the number whose natural logarithm is __e__. As $x\to\infty$, $\ln x$ approaches __f__. But the ratio $(\ln x)/\sqrt{x}$ approaches __g__. The domain and range of $\ln x$ are __h__.

The derivative of $\ln x$ is __i__. The derivative of $\ln(1+x)$

†The integral of $1/x$ (odd function) is $\ln|x|$ (even function). Stay clear of $x=0$.

is __j__. The tangent approximation to $\ln(1+x)$ at $x=0$ is __k__. The quadratic approximation is __l__. The quadratic approximation to e^x is __m__.

The derivative of $\ln u(x)$ by the chain rule is __n__. Thus $(\ln \cos x)' =$ __o__. An antiderivative of $\tan x$ is __p__. The product $p = x\,e^{5x}$ has $\ln p =$ __q__. The derivative of this equation is __r__. Multiplying by p gives $p' =$ __s__, which is LD or logarithmic differentiation.

The integral of $u'(x)/u(x)$ is __t__. The integral of $2x/(x^2+4)$ is __u__. The integral of $1/cx$ is __v__. The integral of $1/(ct+s)$ is __w__. The integral of $1/\cos x$, after a trick, is __x__. We should write $\ln |x|$ for the antiderivative of $1/x$, since this allows __y__. Similarly $\int du/u$ should be written __z__.

Find the derivative dy/dx in 1–10.

1 $y = \ln(2x)$

2 $y = \ln(2x+1)$

3 $y = (\ln x)^{-1}$

4 $y = (\ln x)/x$

5 $y = x \ln x - x$

6 $y = \log_{10} x$

7 $y = \ln(\sin x)$

8 $y = \ln(\ln x)$

9 $y = 7 \ln 4x$

10 $y = \ln((4x)^7)$

Find the indefinite (or definite) integral in 11–24.

11 $\displaystyle\int \frac{dt}{3t}$

12 $\displaystyle\int \frac{dx}{1+x}$

13 $\displaystyle\int_0^1 \frac{dx}{3+x}$

14 $\displaystyle\int_0^1 \frac{dt}{3+2t}$

15 $\displaystyle\int_0^2 \frac{x\,dx}{x^2+1}$

16 $\displaystyle\int_0^2 \frac{x^3\,dx}{x^2+1}$

17 $\displaystyle\int_2^e \frac{dx}{x(\ln x)}$

18 $\displaystyle\int_2^e \frac{dx}{x(\ln x)^2}$

19 $\displaystyle\int \frac{\cos x\,dx}{\sin x}$

20 $\displaystyle\int_0^{\pi/4} \tan x\,dx$

21 $\displaystyle\int \tan 3x\,dx$

22 $\displaystyle\int \cot 3x\,dx$

23 $\displaystyle\int \frac{(\ln x)^2\,dx}{x}$

24 $\displaystyle\int \frac{dx}{x(\ln x)(\ln \ln x)}$

25 Graph $y = \ln(1+x)$

26 Graph $y = \ln(\sin x)$

Compute dy/dx by differentiating $\ln y$. This is LD:

27 $y = \sqrt{x^2+1}$

28 $y = \sqrt{x^2+1}\sqrt{x^2-1}$

29 $y = e^{\sin x}$

30 $y = x^{-1/x}$

31 $y = e^{(e^x)}$

32 $y = x^e$

33 $y = x^{(e^x)}$

34 $y = (\sqrt{x})(\sqrt[3]{x})(\sqrt[6]{x})$

35 $y = x^{-1/\ln x}$

36 $y = e^{-\ln x}$

Evaluate 37–42 by any method.

37 $\displaystyle\int_5^{10} \frac{dt}{t} - \int_{5x}^{10x} \frac{dt}{t}$

38 $\displaystyle\int_1^{e^\pi} \frac{dx}{x} + \int_{-2}^{-1} \frac{dx}{x}$

39 $\displaystyle\frac{d}{dx}\int_x^1 \frac{dt}{t}$

40 $\displaystyle\frac{d}{dx}\int_x^{x^2} \frac{dt}{t}$

41 $\displaystyle\frac{d}{dx} \ln(\sec x + \tan x)$

42 $\displaystyle\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$

Verify the derivatives 43–46, which give useful antiderivatives:

43 $\displaystyle\frac{d}{dx} \ln(x + \sqrt{x^2+1}) = \frac{1}{\sqrt{1+x^2}}$

44 $\displaystyle\frac{d}{dx} \ln\left(\frac{x-a}{x+a}\right) = \frac{2a}{(x^2-a^2)}$

45 $\displaystyle\frac{d}{dx} \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) = \frac{-1}{x\sqrt{1-x^2}}$

46 $\displaystyle\frac{d}{dx} \ln(x + \sqrt{x^2-a^2}) = \frac{1}{\sqrt{x^2-a^2}}$

Estimate 47–50 to linear accuracy, then quadratic accuracy, by $e^x \approx 1 + x + \frac{1}{2}x^2$. Then use a calculator.

47 $\ln(1.1)$ 48 $e^{.1}$ 49 $\ln(.99)$ 50 e^2

51 Compute $\displaystyle\lim_{x\to 0} \frac{\ln(1+x)}{x}$

52 Compute $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x}$

53 Compute $\displaystyle\lim_{x\to 0} \frac{\log_b(1+x)}{x}$

54 Compute $\displaystyle\lim_{x\to 0} \frac{b^x - 1}{x}$

55 Find the area of the "hyperbolic quarter-circle" enclosed by $x = 2$ and $y = 2$ above $y = 1/x$.

56 Estimate the area under $y = 1/x$ from 4 to 8 by four upper rectangles and four lower rectangles. Then average the answers (trapezoidal rule). What is the exact area?

57 Why is $\displaystyle\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ near $\ln n$? Is it above or below?

58 Prove that $\ln x \leqslant 2(\sqrt{x} - 1)$ for $x > 1$. Compare the integrals of $1/t$ and $1/\sqrt{t}$, from 1 to x.

59 Dividing by x in Problem 58 gives $(\ln x)/x \leqslant 2(\sqrt{x} - 1)/x$. Deduce that $(\ln x)/x \to 0$ as $x \to \infty$. Where is the maximum of $(\ln x)/x$?

60 Prove that $(\ln x)/x^{1/n}$ also approaches zero. (Start with $(\ln x^{1/n})/x^{1/n} \to 0$.) Where is its maximum?

61 For any power n, Problem 6.2.59 proved $e^x > x^n$ for large x. Then by logarithms, $x > n \ln x$. Since $(\ln x)/x$ goes below $1/n$ and stays below, it converges to ________.

62 Prove that $y \ln y$ approaches zero as $y \to 0$, by changing y to $1/x$. Find the limit of y^y (take its logarithm as $y \to 0$). What is $.1^{.1}$ on your calculator?

63 Find the limit of $\ln x/\log_{10}x$ as $x \to \infty$.

64 We know the integral $\int_1^x t^{h-1}\,dt = [t^h/h]_1^x = (x^h - 1)/h$. Its limit as $h \to 0$ is ________.

65 Find linear approximations near $x = 0$ for e^{-x} and 2^x.

66 The x^3 correction to $\ln(1+x)$ yields $x - \frac{1}{2}x^2 + \frac{1}{3}x^3$. Check that $\ln 1.01 \approx .0099503$ and find $\ln 1.02$.

67 An ant crawls at 1 foot/second along a rubber band whose original length is 2 feet. The band is being stretched at 1 foot/second by pulling the other end. At what time T, *if ever*, does the ant reach the other end?

One approach: The band's length at time t is $t + 2$. Let $y(t)$ be the *fraction* of that length which the ant has covered, and explain

(a) $y' = 1/(t+2)$ (b) $y = \ln(t+2) - \ln 2$ (c) $T = 2e - 2$.

68 If the rubber band is stretched at 8 feet/second, when if ever does the same ant reach the other end?

69 A weaker ant slows down to $2/(t+2)$ feet/second, so $y' = 2/(t+2)^2$. Show that the other end is never reached.

70 The slope of $p = x^x$ comes two ways from $\ln p = x \ln x$:

1 Logarithmic differentiation (**LD**): Compute $(\ln p)'$ and multiply by p.

2 Exponential differentiation (**ED**): Write x^x as $e^{x \ln x}$, take its derivative, and put back x^x.

71 If $p = 2^x$ then $\ln p =$ ________. **LD** gives $p' = (p)(\ln p)' =$ ________. **ED** gives $p = e$________ and then $p' =$ ________.

72 Compute ln 2 by the trapezoidal rule and/or Simpson's rule, to get five correct decimals.

73 Compute ln 10 by either rule with $\Delta x = 1$, and compare with the value on your calculator.

74 Estimate 1/ln 90,000, the fraction of numbers near 90,000 that are prime. (879 of the next 10,000 numbers are actually prime.)

75 Find a pair of positive integers for which $x^y = y^x$. Show how to change this equation to $(\ln x)/x = (\ln y)/y$. So look for two points at the same height in Figure 6.13. Prove that you have discovered all the integer solutions.

***76** Show that $(\ln x)/x = (\ln y)/y$ is satisfied by

$$x = \left(\frac{t+1}{t}\right)^t \text{ and } y = \left(\frac{t+1}{t}\right)^{t+1}$$

with $t \neq 0$. Graph those points to show the curve $x^y = y^x$. It crosses the line $y = x$ at $x =$ ________, where $t \to \infty$.

6.5 Separable Equations Including the Logistic Equation

This section begins with the integrals that solve two basic differential equations:

$$\frac{dy}{dt} = cy \quad \text{and} \quad \frac{dy}{dt} = cy + s. \tag{1}$$

We already know the solutions. What we don't know is how to discover those solutions, when a suggestion "*try* e^{ct}" has not been made. Many important equations, including these, separate into a y-integral and a t-integral. The answer comes directly from the two separate integrations. When a differential equation is reduced that far—to integrals that we know or can look up—it is solved.

One particular equation will be emphasized. The ***logistic equation*** describes the speedup and slowdown of growth. Its solution is an **S-*curve***, which starts slowly, rises quickly, and levels off. (The 1990's are near the middle of the S, if the prediction is correct for the world population.) S-curves are solutions to ***nonlinear*** equations, and we will be solving our first nonlinear model. It is highly important in biology and all life sciences.