CHAPTER 8

Applications of the Integral

We are experts in one application of the integral—to find the area under a curve. The curve is the graph of $y = v(x)$, extending from $x = a$ at the left to $x = b$ at the right. The area between the curve and the x axis is the definite integral.

I think of that integral in the following way. The region is made up of ***thin strips***. Their width is dx and their height is $v(x)$. The area of a strip is $v(x)$ times dx. The area of all the strips is $\int_a^b v(x)\,dx$. Strictly speaking, the area of one strip is meaningless—genuine rectangles have width Δx. My point is that the picture of thin strips gives the correct approach.

We know what function to integrate (from the picture). We also know how (from this course or a calculator). The new applications to volume and length and surface area cut up the region in new ways. Again the small pieces tell the story. In this chapter, *what* to integrate is more important than *how*.

8.1 Areas and Volumes by Slices

This section starts with areas between curves. Then it moves to ***volumes***, where the strips become *slices*. We are weighing a loaf of bread by adding the weights of the slices. The discussion is dominated by examples and figures—the theory is minimal. The real problem is to set up the right integral. At the end we look at a different way of cutting up volumes, into thin shells. *All formulas are collected into a final table.*

Figure 8.1 shows ***the area between two curves***. The upper curve is the graph of $y = v(x)$. The lower curve is the graph of $y = w(x)$. The strip height is $v(x) - w(x)$, from one curve down to the other. The width is dx (speaking informally again). The total area is the integral of "top minus bottom":

$$\textbf{\textit{area between two curves}} = \int_a^b \Big[v(x) - w(x)\Big]\,dx. \tag{1}$$

EXAMPLE 1 The upper curve is $y = 6x$ (straight line). The lower curve is $y = 3x^2$ (parabola). The area lies between the points where those curves intersect.

To find the intersection points, solve $v(x) = w(x)$ ***or*** $6x = 3x^2$.

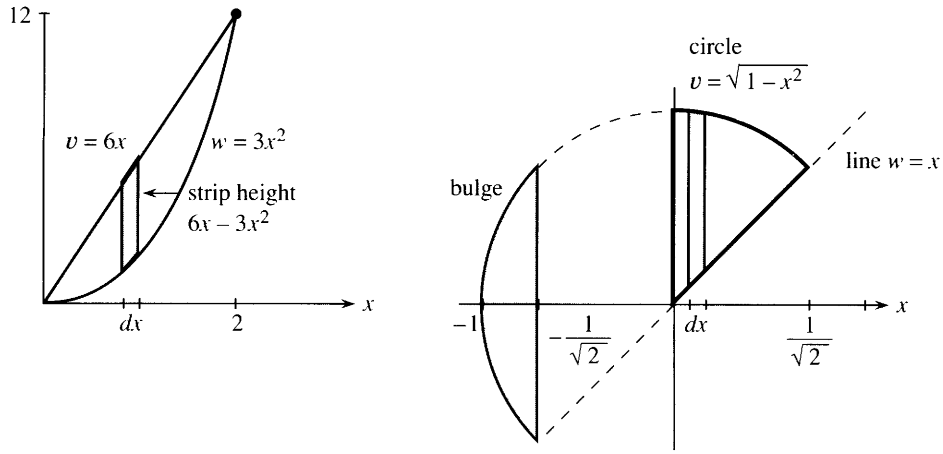


Fig. 8.1 Area between curves = integral of $v - w$. Area in Example 2 starts with $x \geqslant 0$.

One crossing is at $x = 0$, the other is at $x = 2$. The area is an integral from 0 to 2:

$$\text{area} = \int_a^b (v - w)\, dx = \int_0^2 (6x - 3x^2)\, dx = 3x^2 - x^3 \Big]_0^2 = 4.$$

EXAMPLE 2 Find the area between the circle $v = \sqrt{1 - x^2}$ and the 45° line $w = x$.

First question: Which area and what limits? Start with the pie-shaped wedge in Figure 8.1b. The area begins at the y axis and ends where the circle meets the line. At the intersection point we have $v(x) = w(x)$:

$$\text{from } \sqrt{1 - x^2} = x \text{ squaring gives } 1 - x^2 = x^2 \text{ and then } 2x^2 = 1.$$

Thus $x^2 = \frac{1}{2}$. The endpoint is at $x = 1/\sqrt{2}$. Now integrate the strip height $v - w$:

$$\int_0^{1/\sqrt{2}} (\sqrt{1 - x^2} - x)\, dx = \frac{1}{2}\sin^{-1} x + \frac{1}{2} x\sqrt{1 - x^2} - \frac{1}{2}x^2 \Big]_0^{1/\sqrt{2}}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{4} - \frac{1}{4} = \frac{1}{2}\left(\frac{\pi}{4}\right).$$

The area is $\pi/8$ (one eighth of the circle). To integrate $\sqrt{1 - x^2}\, dx$ we apply the techniques of Chapter 7: Set $x = \sin\theta$, convert to $\int \cos^2\theta\, d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta)$, convert back using $\theta = \sin^{-1} x$. It is harder than expected, for a familiar shape.

Remark Suppose the problem is to find the *whole area* between the circle and the line. The figure shows $v = w$ at two points, which are $x = 1/\sqrt{2}$ (already used) and also $x = -1/\sqrt{2}$. Instead of starting at $x = 0$, which gave $\frac{1}{8}$ of a circle, we now include the area to the left.

Main point: *Integrating from $x = -1/\sqrt{2}$ to $x = 1/\sqrt{2}$ will give the wrong answer.* It misses the part of the circle that bulges out over itself, at the far left. In that part, the strips have height $2v$ instead of $v - w$. The figure is essential, to get the correct area of this half-circle.

HORIZONTAL STRIPS INSTEAD OF VERTICAL STRIPS

There is more than one way to slice a region. *Vertical slices give x integrals.* ***Horizontal slices give y integrals.*** We have a free choice, and sometimes the y integral is better.

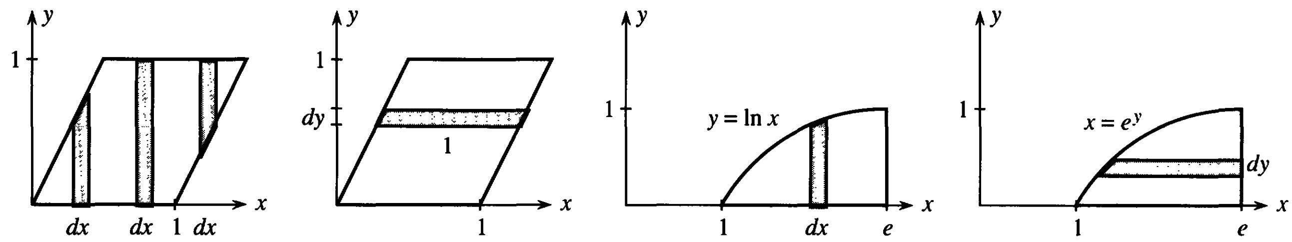


Fig. 8.2 Vertical slices (x integrals) vs. horizontal slices (y integrals).

Figure 8.2 shows a unit parallelogram, with base 1 and height 1. To find its area from vertical slices, three separate integrals are necessary. You should see why! With horizontal slices of length 1 and thickness dy, the area is just $\int_0^1 dy = 1$.

EXAMPLE 3 Find the area under $y = \ln x$ (or beyond $x = e^y$) out to $x = e$.

The x integral from vertical slices is in Figure 8.2c. The y integral is in 8.2d. The area is a choice between two equal integrals (I personally would choose y):

$$\int_{x=1}^{e} \ln x \, dx = \Big[x \ln x - x\Big]_1^e = 1 \qquad \text{or} \qquad \int_{y=0}^{1} (e - e^y)\, dy = \Big[ey - e^y\Big]_0^1 = 1.$$

VOLUMES BY SLICES

For the first time in this book, we now look at *volumes*. ***The regions are three-dimensional solids.*** There are three coordinates x, y, z—and many ways to cut up a solid.

Figure 8.3 shows one basic way—using ***slices***. The slices have thickness dx, like strips in the plane. Instead of the height y of a strip, we now have ***the area A of a cross-section***. This area is different for different slices: A depends on x. The volume of the slice is its area times its thickness: $dV = A(x)\,dx$. *The volume of the whole solid is the integral*:

$$\textbf{\textit{volume = integral of area times thickness}} = \int A(x)\,dx. \tag{2}$$

Note An actual slice does not have the same area on both sides! Its thickness is Δx (not dx). Its volume is approximately $A(x)\,\Delta x$ (but not exactly). In the limit, the thickness approaches zero and the sum of volumes approaches the integral.

For a cylinder all slices are the same. Figure 8.3b shows a cylinder—*not circular*. The area is a fixed number A, so integration is trivial. *The volume is A times h.* The

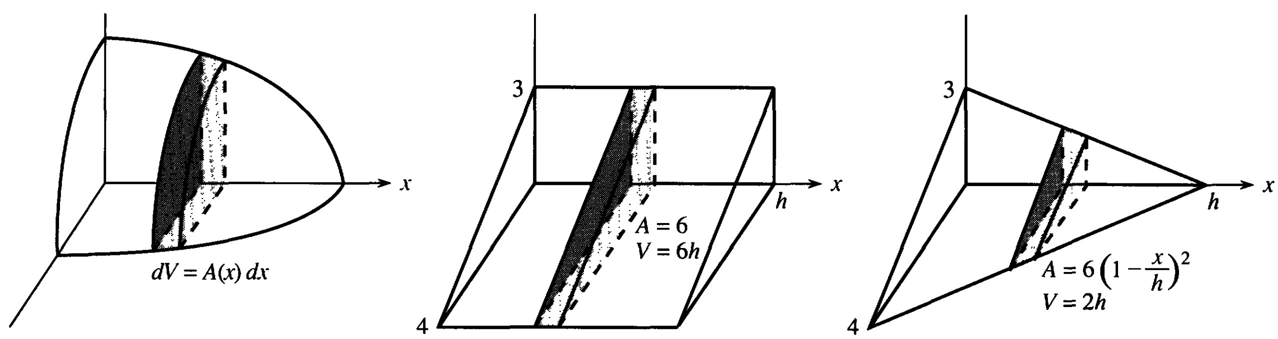


Fig. 8.3 Cross-sections have area $A(x)$. Volumes are $\int A(x)\,dx$.

letter h, which stands for *height*, reminds us that the cylinder often stands on its end. Then the slices are horizontal and the y integral or z integral goes from 0 to h.

When the cross-section is a circle, the cylinder has volume $\pi r^2 h$.

EXAMPLE 4 The *triangular wedge* in Figure 8.3b has constant cross-sections with area $A = \frac{1}{2}(3)(4) = 6$. The volume is $6h$.

EXAMPLE 5 For the *triangular pyramid* in Figure 8.3c, the area $A(x)$ drops from 6 to 0. It is a general rule for pyramids or cones that their volume has an extra factor $\frac{1}{3}$ (compared to cylinders). The volume is now $2h$ instead of $6h$. For a cone with base area πr^2, the volume is $\frac{1}{3}\pi r^2 h$. *Tapering the area to zero leaves only $\frac{1}{3}$ of the volume.*

Why the $\frac{1}{3}$? Triangles sliced from the pyramid have shorter sides. Starting from 3 and 4, the side lengths $3(1 - x/h)$ and $4(1 - x/h)$ drop to zero at $x = h$. *The area is* $A = 6(1 - x/h)^2$. Notice: The side lengths go down linearly, the area drops quadratically. The factor $\frac{1}{3}$ really comes from integrating x^2 to get $\frac{1}{3}x^3$:

$$\int_0^h A(x)\,dx = \int_0^h 6\left(1 - \frac{x}{h}\right)^2 dx = -2h\left(1 - \frac{x}{h}\right)^3\Big]_0^h = 2h.$$

EXAMPLE 6 A half-sphere of radius R has known volume $\frac{1}{2}(\frac{4}{3}\pi R^3)$. Its cross-sections are *semicircles*. The key relation is $x^2 + r^2 = R^2$, for the right triangle in Figure 8.4a. The area of the semicircle is $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(R^2 - x^2)$. So we integrate $A(x)$:

$$\text{volume} = \int_{-R}^{R} A(x)\,dx = \tfrac{1}{2}\pi(R^2 x - \tfrac{1}{3}x^3)\Big]_{-R}^{R} = \tfrac{2}{3}\pi R^3.$$

EXAMPLE 7 Find the volume of the same half-sphere using horizontal slices (Figure 8.4b). The sphere still has radius R. The new right triangle gives $y^2 + r^2 = R^2$. Since we have full circles the area is $\pi r^2 = \pi(R^2 - y^2)$. Notice that this is $A(y)$ not $A(x)$. But the y integral starts at zero:

$$\text{volume} = \int_0^R A(y)\,dy = \pi(R^2 y - \tfrac{1}{3}y^3)\Big]_0^R = \tfrac{2}{3}\pi R^3 \text{ (as before)}.$$

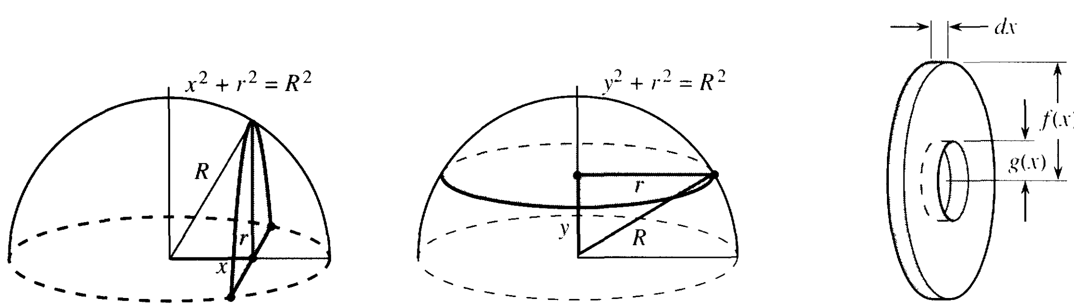


Fig. 8.4 A half-sphere sliced vertically or horizontally. Washer area $\pi f^2 - \pi g^2$.

SOLIDS OF REVOLUTION

Cones and spheres and circular cylinders are "solids of revolution." Rotating a horizontal line around the x axis gives a cylinder. Rotating a sloping line gives a cone. Rotating a semicircle gives a sphere. If a circle is moved away from the axis, rotation produces a torus (a doughnut). The rotation of any curve $y = f(x)$ produces a ***solid of revolution***.

The volume of that solid is made easier because ***every cross-section is a circle***. All slices are pancakes (or pizzas). Rotating the curve $y=f(x)$ around the x axis gives disks of radius y, so the area is $A=\pi y^2=\pi[f(x)]^2$. We add the slices:

$$\textbf{\textit{volume of solid of revolution}} = \int_a^b \pi y^2\,dx = \int_a^b \pi\big[f(x)\big]^2\,dx.$$

EXAMPLE 8 Rotating $y=\sqrt{x}$ with $A=\pi(\sqrt{x})^2$ produces a "headlight" (Figure 8.5a):

$$\text{volume of headlight} = \textstyle\int_0^2 A\,dx = \int_0^2 \pi x\,dx = \tfrac{1}{2}\pi x^2\Big]_0^2 = 2\pi.$$

If the same curve is rotated around the y axis, it makes a champagne glass. *The slices are horizontal.* The area of a slice is πx^2 not πy^2. When $y=\sqrt{x}$ this area is πy^4. Integrating from $y=0$ to $\sqrt{2}$ gives the champagne volume $\pi(\sqrt{2})^5/5$.

$$\textbf{\textit{revolution around the y axis}}: \quad \textbf{\textit{volume}} = \int \pi x^2\,dy.$$

EXAMPLE 9 The headlight has a hole down the center (Figure 8.5b). Volume = ?

The hole has radius 1. *All* of the $\sqrt{x}$ solid is removed, up to the point where $\sqrt{x}$ reaches 1. After that, from $x=1$ to $x=2$, each cross-section is a disk with a hole. The disk has radius $f=\sqrt{x}$ and the hole has radius $g=1$. *The slice is a flat ring or a "**washer**."* Its area is the full disk minus the area of the hole:

$$\textbf{\textit{area of washer}} = \pi f^2 - \pi g^2 = \pi(\sqrt{x})^2 - \pi(1)^2 = \pi x - \pi.$$

This is the area $A(x)$ in the *method of washers*. Its integral is the volume:

$$\textstyle\int_1^2 A\,dx = \int_1^2 (\pi x - \pi)\,dx = \Big[\tfrac{1}{2}\pi x^2 - \pi x\Big]_1^2 = \tfrac{1}{2}\pi.$$

Please notice: *The washer area is not* $\pi(f-g)^2$. It is $A=\pi f^2-\pi g^2$.

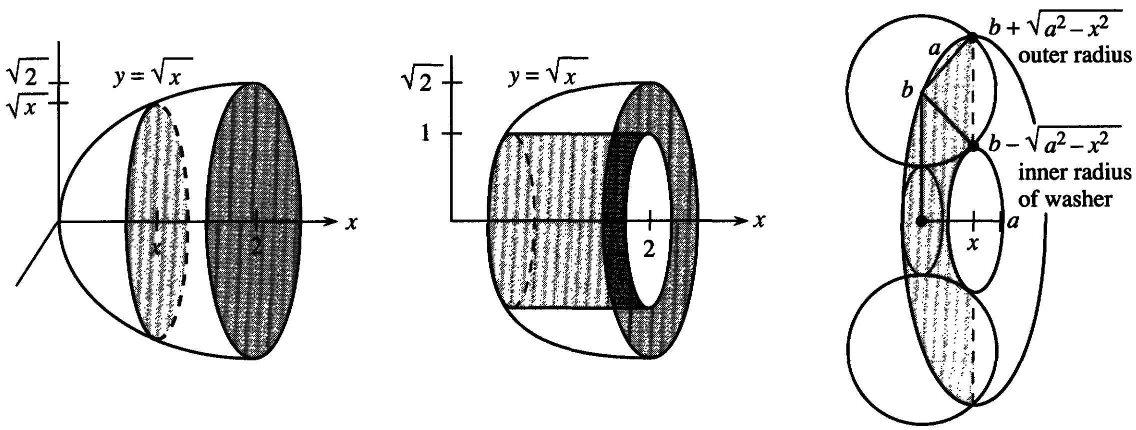


Fig. 8.5 $y=\sqrt{x}$ revolved; $y=1$ revolved inside it; circle revolved to give torus.

EXAMPLE 10 (Doughnut sliced into washers) Rotate a circle of radius a around the x axis. The center of the circle stays out at a distance $b>a$. Show that the volume of the doughnut (or torus) is $2\pi^2a^2b$.

The outside half of the circle rotates to give the outside of the doughnut. The inside half gives the hole. The biggest slice (through the center plane) has outer radius $b + a$ and inner radius $b - a$.

Shifting over by x, the outer radius is $f = b + \sqrt{a^2 - x^2}$ and the inner radius is $g = b - \sqrt{a^2 - x^2}$. Figure 8.5c shows a slice (a washer) with area $\pi f^2 - \pi g^2$.

$$\text{area } A = \pi(b + \sqrt{a^2 - x^2})^2 - \pi(b - \sqrt{a^2 - x^2})^2 = 4\pi b\sqrt{a^2 - x^2}.$$

Now integrate over the washers to find the volume of the doughnut:

$$\int_{-a}^{a} A(x)\,dx = 4\pi b \int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = (4\pi b)(\tfrac{1}{2}\pi a^2) = 2\pi^2 a^2 b.$$

That integral $\frac{1}{2}\pi a^2$ is the area of a semicircle. When we set $x = a \sin\theta$ the area is $\int a^2 \cos^2\theta\,d\theta$. Not for the last time do we meet $\cos^2\theta$.

The hardest part is visualizing the washers, because a doughnut usually breaks the other way. A better description is a ***bagel***, sliced the long way to be buttered.

VOLUMES BY CYLINDRICAL SHELLS

Finally we look at a different way of cutting up a solid of revolution. So far it was cut into slices. The slices were perpendicular to the axis of revolution. Now the cuts are *parallel* to the axis, and each piece is a ***thin cylindrical shell***. The new formula gives the same volume, but the integral to be computed might be easier.

Figure 8.6a shows a solid cone. A shell is inside it. The inner radius is x and the outer radius is $x + dx$. *The shell is an outer cylinder minus an inner cylinder*:

$$\text{shell volume } \pi(x + dx)^2 h - \pi x^2 h = \pi x^2 h + 2\pi x(dx)h + \pi(dx)^2 h - \pi x^2 h. \qquad (3)$$

The term that matters is $2\pi x(dx)h$. ***The shell volume is essentially*** $2\pi x$ (the distance around) ***times*** dx (the thickness) ***times*** h (the height). The volume of the solid comes from putting together the thin shells:

$$\textbf{\textit{solid volume}} = \textbf{\textit{integral of shell volumes}} = \int 2\pi x h\,dx. \qquad (4)$$

This is the central formula of the shell method. The rest is examples.

Remark on this volume formula It is completely typical of integration that $(dx)^2$ and $(\Delta x)^2$ disappear. The reason is this. The number of shells grows like $1/\Delta x$. Terms of order $(\Delta x)^2$ add up to a volume of order Δx (approaching zero). ***The linear term involving*** Δx ***or*** dx ***is the one to get right***. Its limit gives the integral $\int 2\pi x h\,dx$. The key is to build the solid out of shells—and to find the area or volume of each piece.

EXAMPLE 11 Find the volume of a cone (base area πr^2, height b) cut into shells.

A tall shell at the center has h near b. A short shell at the outside has h near zero. In between the shell height h decreases linearly, reaching zero at $x = r$. The height in Figure 8.6a is $h = b - bx/r$. Integrating over all shells gives the volume of the cone (with the expected $\frac{1}{3}$):

$$\int_0^r 2\pi x\left(b - b\frac{x}{r}\right)dx = \left[\pi x^2 b - \frac{2\pi x^3 b}{3r}\right]_0^r = \frac{1}{3}\pi r^2 b.$$

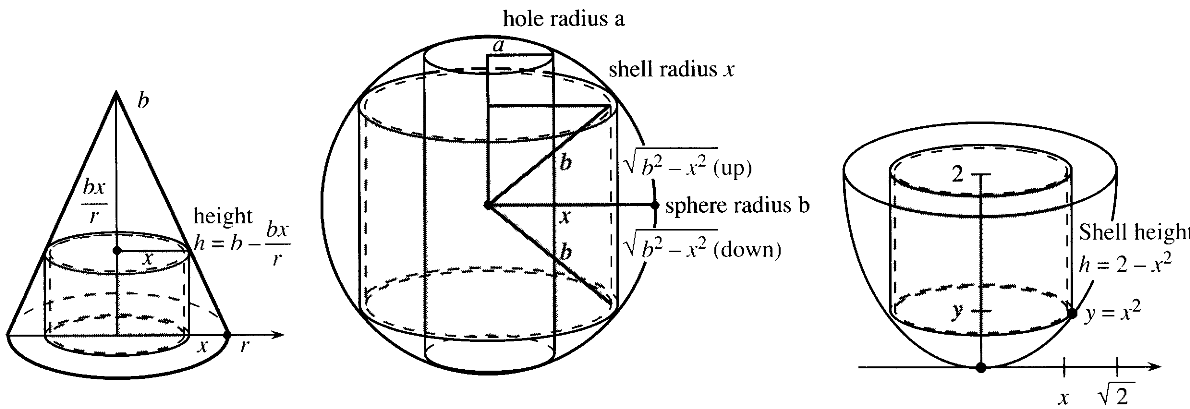


Fig. 8.6 Shells of volume $2\pi xh\,dx$ inside cone, sphere with hole, and paraboloid.

EXAMPLE 12 Bore a hole of radius a through a sphere of radius $b > a$.

The hole removes all points out to $x = a$, where the shells begin. The height of the shell is $h = 2\sqrt{b^2 - x^2}$. (The key is the right triangle in Figure 8.6b. The height upward is $\sqrt{b^2 - x^2}$—this is half the height of the shell.) Therefore the sphere-with-hole has

$$\text{volume} = \int_a^b 2\pi xh\,dx = \int_a^b 4\pi x\sqrt{b^2 - x^2}\,dx.$$

With $u = b^2 - x^2$ we almost see du. Multiplying $du = -2x\,dx$ is an extra factor -2π:

$$\text{volume} = -2\pi \int \sqrt{u}\,du = -2\pi(\tfrac{2}{3}u^{3/2}).$$

We can find limits on u, or we can put back $u = b^2 - x^2$:

$$\text{volume} = -\frac{4\pi}{3}(b^2 - x^2)^{3/2}\Big]_a^b = \frac{4\pi}{3}(b^2 - a^2)^{3/2}.$$

If $a = b$ (the hole is as big as the sphere) this volume is zero. If $a = 0$ (no hole) we have $4\pi b^3/3$ for the complete sphere.

Question What if the sphere-with-hole is cut into slices instead of shells?
Answer Horizontal slices are washers (Problem 66). Vertical slices are not good.

EXAMPLE 13 Rotate the parabola $y = x^2$ around the y axis to form a bowl.

We go out to $x = \sqrt{2}$ (and up to $y = 2$). The shells in Figure 8.6c have height $h = 2 - x^2$. The bowl (or paraboloid) is the same as the headlight in Example 8, but we have shells not slices:

$$\int_0^{\sqrt{2}} 2\pi x(2 - x^2)\,dx = 2\pi x^2 - \frac{2\pi x^4}{4}\Big]_0^{\sqrt{2}} = 2\pi.$$

TABLE OF AREAS AND VOLUMES	***area between curves***: $A = \int (v(x) - w(x))\,dx$
	solid volume cut into slices: $V = \int A(x)\,dx$ or $\int A(y)\,dy$
	solid of revolution: cross-section $A = \pi y^2$ or πx^2
	solid with hole: washer area $A = \pi f^2 - \pi g^2$
	solid of revolution cut into shells: $V = \int 2\pi xh\,dx$.

Which to use, slices or shells? Start with a vertical line going up to $y = \cos x$. Rotating the line around the x axis produces a *slice* (a circular disk). The radius is $\cos x$. Rotating the line around the y axis produces a *shell* (the outside of a cylinder). The height is $\cos x$. See Figure 8.7 for the slice and the shell. For volumes we just integrate $\pi \cos^2 x\, dx$ (the slice volume) or $2\pi x \cos x\, dx$ (the shell volume).

This is the normal choice—slices through the x axis and shells around the y axis. Then $y = f(x)$ gives the disk radius and the shell height. The slice is a washer instead of a disk if there is also an inner radius $g(x)$. No problem—just integrate small volumes.

What if you use slices for rotation around the y axis? The disks are in Figure 8.7b, and *their radius is* x. This is $x = \cos^{-1} y$ in the example. It is $x = f^{-1}(y)$ in general. You have to solve $y = f(x)$ to find x in terms of y. Similarly for shells around the x axis: *The length of the shell is* $x = f^{-1}(y)$. Integrating may be difficult or impossible.

When $y = \cos x$ is rotated around the x axis, here are the choices for volume:

$$(\textit{good by slices}) \int \pi \cos^2 x\, dx \qquad (\textit{bad by shells}) \int 2\pi y \cos^{-1} y\, dy.$$

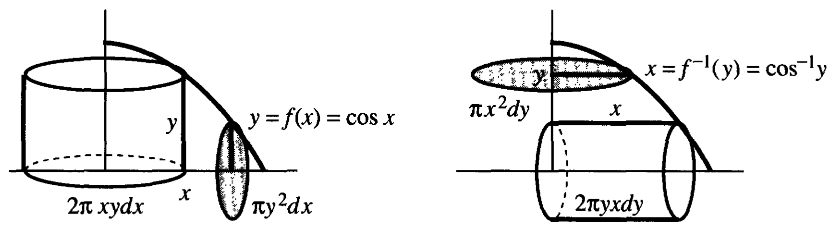


Fig. 8.7 Slices through x axis and shells around y axis (*good*). The opposite way needs $f^{-1}(y)$.

8.1 EXERCISES

Read-through questions

The area between $y = x^3$ and $y = x^4$ equals the integral of __a__. If the region ends where the curves intersect, we find the limits on x by solving __b__. Then the area equals __c__. When the area between $y = \sqrt{x}$ and the y axis is sliced horizontally, the integral to compute is __d__.

In three dimensions the volume of a slice is its thickness dx times its __e__. If the cross-sections are squares of side $1 - x$, the volume comes from $\int$ __f__. From $x = 0$ to $x = 1$, this gives the volume __g__ of a square __h__. If the cross-sections are circles of radius $1 - x$, the volume comes from $\int$ __i__. This gives the volume __j__ of a circular __k__.

For a solid of revolution, the cross-sections are __l__. Rotating the graph of $y = f(x)$ around the x axis gives a solid volume $\int$ __m__. Rotating around the y axis leads to $\int$ __n__. Rotating the area between $y = f(x)$ and $y = g(x)$ around the x axis, the slices look like __o__. Their areas are __p__ so the volume is $\int$ __q__.

Another method is to cut the solid into thin cylindrical __r__. Revolving the area under $y = f(x)$ around the y axis, a shell has height __s__ and thickness dx and volume __t__. The total volume is $\int$ __u__.

Find where the curves in 1–12 intersect, draw rough graphs, and compute the area between them.

1 $y = x^2 - 3$ and $y = 1$

2 $y = x^2 - 2$ and $y = 0$

3 $y^2 = x$ and $x = 9$

4 $y^2 = x$ and $x = y + 2$

5 $y = x^4 - 2x^2$ and $y = 2x^2$

6 $x = y^5$ and $y = x^4$

7 $y = x^2$ and $y = -x^2 + 18x$

8 $y = 1/x$ and $y = 1/x^2$ and $x = 3$

9 $y = \cos x$ and $y = \cos^2 x$

10 $y = \sin \pi x$ and $y = 2x$ and $x = 0$

11 $y = e^x$ and $y = e^{2x-1}$ and $x = 0$

12 $y = e$ and $y = e^x$ and $y = e^{-x}$

13 Find the area inside the three lines $y = 4 - x$, $y = 3x$, and $y = x$.

14 Find the area bounded by $y = 12 - x$, $y = \sqrt{x}$, and $y = 1$.

15 Does the parabola $y = 1 - x^2$ out to $x = 1$ sit inside or outside the unit circle $x^2 + y^2 = 1$? Find the area of the "skin" between them.

16 Find the area of the largest triangle with base on the x axis that fits (a) inside the unit circle (b) inside that parabola.

17 Rotate the ellipse $x^2/a^2 + y^2/b^2 = 1$ around the x axis to find the volume of a football. What is the volume around the y axis? If $a = 2$ and $b = 1$, locate a point (x, y, z) that is in one football but not the other.

18 What is the volume of the loaf of bread which comes from rotating $y = \sin x$ $(0 \leqslant x \leqslant \pi)$ around the x axis?

19 What is the volume of the flying saucer that comes from rotating $y = \sin x$ $(0 \leqslant x \leqslant \pi)$ around the y axis?

20 What is the volume of the galaxy that comes from rotating $y = \sin x$ $(0 \leqslant x \leqslant \pi)$ around the x axis and then rotating the whole thing around the y axis?

Draw the region bounded by the curves in 21–28. Find the volume when the region is rotated (a) around the x axis (b) around the y axis.

21 $x + y = 8,\ x = 0,\ y = 0$

22 $y - e^x = 1,\ x = 1,\ y = 0,\ x = 0$

23 $y = x^4,\ y = 1,\ x = 0$

24 $y = \sin x,\ y = \cos x,\ x = 0$

25 $xy = 1,\ x = 2,\ y = 3$

26 $x^2 - y^2 = 9,\ x + y = 9$ (rotate the region where $y \geqslant 0$)

27 $x^2 = y^3,\ x^3 = y^2$

28 $(x - 2)^2 + (y - 1)^2 = 1$

In 29–34 find the volume and draw a typical slice.

29 A cap of height h is cut off the top of a sphere of radius R. Slice the sphere horizontally starting at $y = R - h$.

30 A pyramid P has height 6 and square base of side 2. Its volume is $\frac{1}{3}(6)(2)^2 = 8$.

(a) Find the volume up to height 3 by horizontal slices. What is the length of a side at height y?

(b) Recompute by removing a smaller pyramid from P.

31 The base is a disk of radius a. Slices perpendicular to the base are squares.

32 The base is the region under the parabola $y = 1 - x^2$. Slices perpendicular to the x axis are squares.

33 The base is the region under the parabola $y = 1 - x^2$. Slices perpendicular to the y axis are squares.

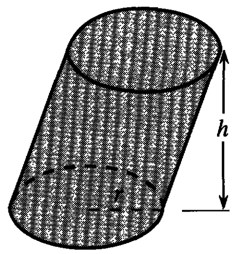
34 The base is the triangle with corners $(0, 0)$, $(1, 0)$, $(0, 1)$. Slices perpendicular to the x axis are semicircles.

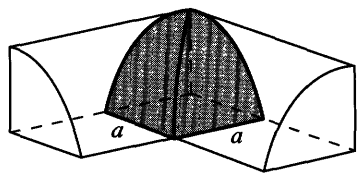
35 Cavalieri's principle for areas: If two regions have strips of equal length, then the regions have the same area. Draw a parallelogram and a curved region, both with the same strips as the unit square. Why are the areas equal?

36 Cavalieri's principle for volumes: If two solids have slices of equal area, the solids have the same volume. Find the volume of the tilted cylinder in the figure.

37 Draw another region with the same slice areas as the tilted cylinder. When all areas $A(x)$ are the same, the volumes $\int$ ________ are the same.

38 Find the volume common to two circular cylinders of radius a. One eighth of the region is shown (axes are perpendicular and horizontal slices are squares).





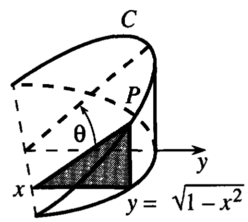
39 A wedge is cut out of a cylindrical tree (see figure). One cut is along the ground to the x axis. The second cut is at angle θ, also stopping at the x axis.

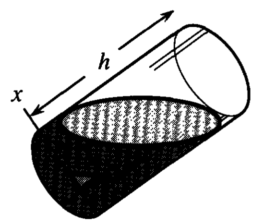
(a) The curve C is part of a (circle) (ellipse) (parabola).

(b) The height of point P in terms of x is ________.

(c) The area $A(x)$ of the triangular slice is ________.

(d) The volume of the wedge is ________.





40 The same wedge is sliced perpendicular to the y axis.

(a) The slices are now (triangles) (rectangles) (curved).

(b) The slice area is ________ (slice height $y \tan \theta$).

(c) The volume of the wedge is the integral ________.

(d) Change the radius from 1 to r. The volume is multiplied by ________.

41 A cylinder of radius r and height h is half full of water. Tilt it so the water just covers the base.

(a) Find the volume of water by common sense.

(b) Slices perpendicular to the x axis are (rectangles) (trapezoids) (curved). I had to tilt an actual glass.

*42 Find the area of a slice in Problem 41. (The tilt angle has $\tan \theta = 2h/r$.) Integrate to find the volume of water.

The slices in 43–46 are washers. Find the slice area and volume.

43 The rectangle with sides $x = 1$, $x = 3$, $y = 2$, $y = 5$ is rotated around the x axis.

44 The same rectangle is rotated around the y axis.

45 The same rectangle is rotated around the line $y = 1$.

46 Draw the triangle with corners (1, 0), (1, 1), (0, 1). After rotation around the x axis, describe the solid and find its volume.

47 Bore a hole of radius a down the axis of a cone and through the base of radius b. If it is a 45° cone (height also b), what volume is left? Check $a = 0$ and $a = b$.

48 Find the volume common to two spheres of radius r if their centers are $2(r - h)$ apart. Use Problem 29 on spherical caps.

49 (Shells vs. disks) Rotate $y = 3 - x$ around the x axis from $x = 0$ to $x = 2$. Write down the volume integral by disks and then by shells.

50 (Shells vs. disks) Rotate $y = x^3$ around the y axis from $y = 0$ to $y = 8$. Write down the volume integral by shells and disks and compute both ways.

51 Yogurt comes in a solid of revolution. Rotate the line $y = mx$ around the y axis to find the volume between $y = a$ and $y = b$.

52 Suppose $y = f(x)$ decreases from $f(0) = b$ to $f(1) = 0$. The curve is rotated around the y axis. *Compare shells to disks*:

$$\int_0^1 2\pi x f(x)\,dx = \int_0^b \pi(f^{-1}(y))^2\,dy.$$

Substitute $y = f(x)$ in the second. Also substitute $dy = f'(x)\,dx$. Integrate by parts to reach the first.

53 If a roll of paper with inner radius 2 cm and outer radius 10 cm has about 10 thicknesses per centimeter, approximately how long is the paper when unrolled?

54 Find the approximate volume of your brain. OK to include everything above your eyes (skull too).

Use shells to find the volumes in 55–63. The rotated regions lie between the curve and x axis.

55 $y = 1 - x^2$, $0 \leqslant x \leqslant 1$ (around the y axis)

56 $y = 1/x$, $1 \leqslant x \leqslant 100$ (around the y axis)

57 $y = \sqrt{1 - x^2}$, $0 \leqslant x \leqslant 1$ (around either axis)

58 $y = 1/(1 + x^2)$, $0 \leqslant x \leqslant 3$ (around the y axis)

59 $y = \sin(x^2)$, $0 \leqslant x \leqslant \sqrt{\pi}$ (around the y axis)

60 $y = 1/\sqrt{1 - x^2}$, $0 \leqslant x \leqslant 1$ (around the y axis)

61 $y = x^2$, $0 \leqslant x \leqslant 2$ (around the x axis)

62 $y = e^x$, $0 \leqslant x \leqslant 1$ (around the x axis)

63 $y = \ln x$, $1 \leqslant x \leqslant e$ (around the x axis)

64 The region between $y = x^2$ and $y = x$ is revolved around the y axis. (a) Find the volume by cutting into shells. (b) Find the volume by slicing into washers.

65 The region between $y = f(x)$ and $y = 1 + f(x)$ is rotated around the y axis. The shells have height ________. The volume out to $x = a$ is ________. It equals the volume of a ________ because the shells are the same.

66 A horizontal slice of the sphere-with-hole in Figure 8.6b is a washer. Its area is $\pi x^2 - \pi a^2 = \pi(b^2 - y^2 - a^2)$.

(a) Find the upper limit on y (the top of the hole).

(b) Integrate the area to verify the volume in Example 12.

67 If the hole in the sphere has length 2, show that the volume is $4\pi/3$ regardless of the radii a and b.

***68** An upright cylinder of radius r is sliced by two parallel planes at angle α. One is a height h above the other.

(a) Draw a picture to show that the volume between the planes is $\pi r^2 h$.

(b) Tilt the picture by α, so the base and top are flat. What is the shape of the base? What is its area A? What is the height H of the tilted cylinder?

69 ***True or false***, with a reason.

(a) A cube can only be sliced into squares.

(b) A cube cannot be cut into cylindrical shells.

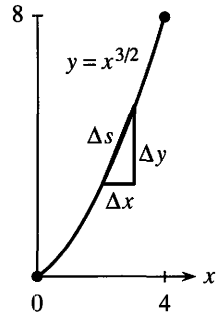
(c) The washer with radii r and R has area $\pi(R - r)^2$.

(d) The plane $w = \frac{1}{2}$ slices a 3-dimensional sphere out of a 4-dimensional sphere $x^2 + y^2 + z^2 + w^2 = 1$.

8.2 Length of a Plane Curve

The graph of $y = x^{3/2}$ is a curve in the x-y plane. ***How long is that curve***? A definite integral needs endpoints, and we specify $x = 0$ and $x = 4$. The first problem is to know what "length function" to integrate.

The distance along a curve is the *arc length*. To set up an integral, we break the problem into small pieces. Roughly speaking, ***small pieces of a smooth curve are nearly straight***. We know the exact length Δs of a straight piece, and Figure 8.8 shows how it comes close to a curved piece.



$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$
$$(ds)^2 = (dx)^2 + \left(\frac{dy}{dx}\right)^2 (dx)^2$$
$$ds = \sqrt{1 + (dy/dx)^2}\, dx$$

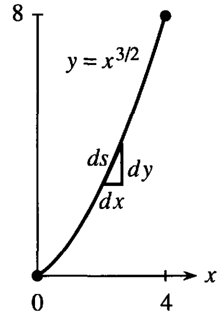


Fig. 8.8 Length Δs of short straight segment. Length ds of very short curved segment.

Here is the unofficial reasoning that gives the length of the curve. A straight piece has $(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$. Within that right triangle, the height Δy is the slope $(\Delta y/\Delta x)$ times Δx. This secant slope is close to the slope of the curve. Thus Δy is approximately $(dy/dx)\,\Delta x$.

$$\Delta s \approx \sqrt{(\Delta x)^2 + (dy/dx)^2(\Delta x)^2} = \sqrt{1 + (dy/dx)^2}\,\Delta x. \tag{1}$$

Now add these pieces and make them smaller. The infinitesimal triangle has $(ds)^2 = (dx)^2 + (dy)^2$. Think of ds as $\sqrt{1 + (dy/dx)^2}\,dx$ and integrate:

$$\textbf{\textit{length of curve}} = \int ds = \int \sqrt{1 + (dy/dx)^2}\,dx. \tag{2}$$

EXAMPLE 1 Keep $y = x^{3/2}$ and $dy/dx = \frac{3}{2}x^{1/2}$. Watch out for $\frac{3}{2}$ and $\frac{9}{4}$:

$$\text{length} = \int_0^4 \sqrt{1 + \tfrac{9}{4}x}\,dx = (\tfrac{2}{3})(\tfrac{4}{9})(1 + \tfrac{9}{4}x)^{3/2}\Big]_0^4 = \tfrac{8}{27}(10^{3/2} - 1^{3/2}). \tag{3}$$

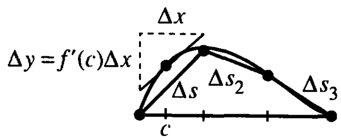
This answer is just above 9. A straight line from (0, 0) to (4, 8) has exact length $\sqrt{80}$. Note $4^2 + 8^2 = 80$. Since $\sqrt{80}$ is just below 9, the curve is surprisingly straight.

You may not approve of those numbers (or the reasoning behind them). We can fix the reasoning, but nothing can be done about the numbers. This example $y = x^{3/2}$ had to be chosen carefully to make the integration possible at all. The length integral is difficult because of the square root. In most cases we integrate numerically.

EXAMPLE 2 The straight line $y = 2x$ from $x = 0$ to $x = 4$ has $dy/dx = 2$:

$$\text{length} = \int_0^4 \sqrt{1+4}\,dx = 4\sqrt{5} = \sqrt{80} \text{ as before} \qquad \text{(just checking)}.$$

We return briefly to the reasoning. The curve is the graph of $y = f(x)$. Each piece contains at least one point where secant slope equals tangent slope: $\Delta y/\Delta x = f'(c)$. The Mean Value Theorem applies when the slope is continuous—this is required for a smooth curve. The straight length Δs is exactly $\sqrt{(\Delta x)^2 + (f'(c)\,\Delta x)^2}$. Adding the n pieces gives the length of the broken line (close to the curve):



$$\sum_1^n \Delta s_i = \sum_1^n \sqrt{1 + [f'(c_i)]^2}\,\Delta x_i.$$

As $n \to \infty$ and $\Delta x_{\max} \to 0$ this approaches the integral that gives arc length.

8A The length of the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$s = \int ds = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx = \int_a^b \sqrt{1 + (dy/dx)^2}\,dx. \qquad (4)$$

EXAMPLE 3 Find the length of the first quarter of the circle $y = \sqrt{1 - x^2}$.

Here $dy/dx = -x/\sqrt{1 - x^2}$. From Figure 8.9a, the integral goes from $x = 0$ to $x = 1$:

$$\text{length} = \int_0^1 \sqrt{1 + (dy/dx)^2}\,dx = \int_0^1 \sqrt{1 + \frac{x^2}{1 - x^2}}\,dx = \int_0^1 \frac{dx}{\sqrt{1 - x^2}}.$$

The antiderivative is $\sin^{-1} x$. It equals $\pi/2$ at $x = 1$. This length $\pi/2$ is a quarter of the full circumference 2π.

EXAMPLE 4 Compute the distance around a quarter of the *ellipse* $y^2 + 2x^2 = 2$.

The equation is $y = \sqrt{2 - 2x^2}$ and the slope is $dy/dx = -2x/\sqrt{2 - 2x^2}$. So $\int ds$ is

$$\int_0^1 \sqrt{1 + \frac{4x^2}{2 - 2x^2}}\,dx = \int_0^1 \sqrt{\frac{2 + 2x^2}{2 - 2x^2}}\,dx = \int_0^1 \sqrt{\frac{1 + x^2}{1 - x^2}}\,dx. \qquad (5)$$

That integral can't be done in closed form. *The length of an ellipse can only be computed numerically.* The denominator is zero at $x = 1$, so a blind application of the trapezoidal rule or Simpson's rule would give length $= \infty$. The midpoint rule gives length $= 1.91$ with thousands of intervals.

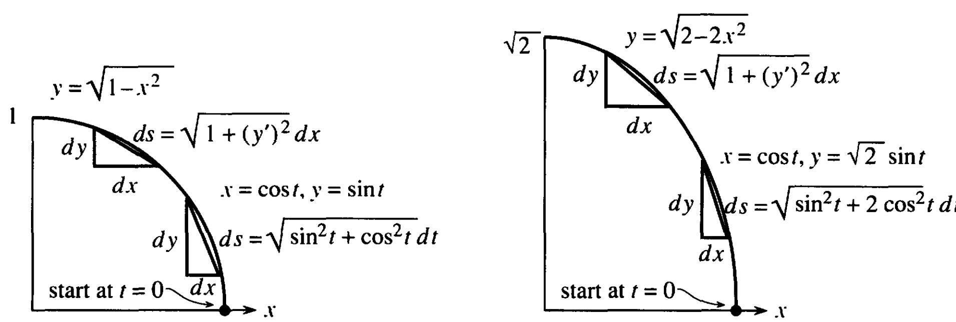


Fig. 8.9 Circle and ellipse, directly by $y = f(x)$ or parametrically by $x(t)$ and $y(t)$.

LENGTH OF A CURVE FROM PARAMETRIC EQUATIONS: $x(t)$ AND $y(t)$

We have met the unit circle in two forms. One is $x^2 + y^2 = 1$. The other is $x = \cos t$, $y = \sin t$. Since $\cos^2 t + \sin^2 t = 1$, this point goes around the correct circle. One advantage of the "***parameter***" t is to give extra information—it tells *where* the point is and also *when*. In Chapter 1, the parameter was the time and also the angle—because we moved around the circle with speed 1.

Using t is a natural way to give the position of a particle or a spacecraft. We can recover the velocity if we know x and y at every time t. An equation $y=f(x)$ tells the shape of the path, not the speed along it.

Chapter 12 deals with parametric equations for curves. Here we concentrate on the *path length*—which allows you to see the idea of a parameter t without too much detail. We give x as a function of t and y as a function of t. The curve is still approximated by straight pieces, and each piece has $(\Delta s)^2=(\Delta x)^2+(\Delta y)^2$. But instead of using $\Delta y \approx (dy/dx)\,\Delta x$, we approximate Δx and Δy separately:

$$\Delta x \approx (dx/dt)\,\Delta t, \qquad \Delta y \approx (dy/dt)\,\Delta t, \qquad \Delta s \approx \sqrt{(dx/dt)^2+(dy/dt)^2}\,\Delta t.$$

8B The length of a parametric curve is an integral with respect to t:

$$\int ds = \int (ds/dt)\,dt = \int \sqrt{(dx/dt)^2+(dy/dt)^2}\,dt. \tag{6}$$

EXAMPLE 5 Find the length of the quarter-circle using $x=\cos t$ and $y=\sin t$:

$$\int_0^{\pi/2}\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt=\int_0^{\pi/2}\sqrt{\sin^2 t+\cos^2 t}\,dt=\int_0^{\pi/2}dt=\frac{\pi}{2}.$$

The integral is simpler than $1/\sqrt{1-x^2}$, and there is one new advantage. *We can integrate around a whole circle with no trouble.* Parametric equations allow a path to close up or even cross itself. The time t keeps going and the point $(x(t), y(t))$ keeps moving. In contrast, curves $y=f(x)$ are limited to one y for each x.

EXAMPLE 6 Find the length of the quarter-ellipse: $x=\cos t$ and $y=\sqrt{2}\sin t$:

On this path y^2+2x^2 is $2\sin^2 t+2\cos^2 t=2$ (same ellipse). The ***non***-parametric equation $y=\sqrt{2-2x^2}$ comes from eliminating t. We keep t:

$$\text{length}=\int_0^{\pi/2}\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt=\int_0^{\pi/2}\sqrt{\sin^2 t+2\cos^2 t}\,dt. \tag{7}$$

This integral (7) must equal (5). If one cannot be done, neither can the other. They are related by $x=\cos t$, but (7) does not blow up at the endpoints. The trapezoidal rule gives 1.9101 with less than 100 intervals. Section 5.8 mentioned that calculators *automatically* do a substitution that makes (5) more like (7).

EXAMPLE 7 The path $x=t^2$, $y=t^3$ goes from (0, 0) to (4, 8). Stop at $t=2$.

To find this path without the parameter t, first solve for $t=x^{1/2}$. Then substitute into the equation for y: $y=t^3=x^{3/2}$. ***The non-parametric form*** (with t eliminated) ***is the same curve $y=x^{3/2}$ as in Example*** **1**.

The length from the t-integral equals the length from the x-integral. This is Problem 22.

EXAMPLE 8 ***Special choice of parameter***: t is x. The curve becomes $x=t$, $y=t^{3/2}$.

If $x=t$ then $dx/dt=1$. The square root in (6) is the same as the square root in (4). Thus the non-parametric form $y=f(x)$ is a special case of the parametric form—just take $t=x$.

Compare $x=t$, $y=t^{3/2}$ with $x=t^2$, $y=t^3$. ***Same curve, same length, different speed.***

EXAMPLE 9 Define "*speed*" by $\dfrac{\text{short distance}}{\text{short time}} = \dfrac{ds}{dt}$. It is $\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$.

When a ball is thrown straight upward, dx/dt is zero. But the speed is not dy/dt. It is $|dy/dt|$. The speed is positive downward as well as upward.

8.2 EXERCISES

Read-through questions

The length of a straight segment (Δx across, Δy up) is $\Delta s =$ __a__. Between two points of the graph of $y(x)$, Δy is approximately dy/dx times __b__. The length of that piece is approximately $\sqrt{(\Delta x)^2 + \text{\_\_c\_\_}}$. An infinitesimal piece of the curve has length $ds =$ __d__. Then the arc length integral is $\int$ __e__.

For $y = 4 - x$ from $x = 0$ to $x = 3$ the arc length is $\int$ __f__ $=$ __g__. For $y = x^3$ the arc length integral is __h__.

The curve $x = \cos t$, $y = \sin t$ is the same as __i__. The length of a curve given by $x(t)$, $y(t)$ is $\int \sqrt{\text{\_\_j\_\_}}\, dt$. For example $x = \cos t$, $y = \sin t$ from $t = \pi/3$ to $t = \pi/2$ has length __k__. The speed is $ds/dt =$ __l__. For the special case $x = t$, $y = f(t)$ the length formula goes back to $\int \sqrt{\text{\_\_m\_\_}}\, dx$.

Find the lengths of the curves in Problems 1–8.

1 $y = x^{3/2}$ from (0, 0) to (1, 1)

2 $y = x^{2/3}$ from (0, 0) to (1, 1) (compare with Problem 1 or put $u = \frac{4}{9} + x^{2/3}$ in the length integral)

3 $y = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 1$

4 $y = \frac{1}{3}(x^2 - 2)^{3/2}$ from $x = 2$ to $x = 4$

5 $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$ from $x = 1$ to $x = 3$

6 $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$ from $x = 1$ to $x = 2$

7 $y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}$ from $x = 1$ to $x = 4$

8 $y = x^2$ from (0, 0) to (1, 1)

9 The curve given by $x = \cos^3 t$, $y = \sin^3 t$ is an *astroid* (a hypocycloid). Its non-parametric form is $x^{2/3} + y^{2/3} = 1$. Sketch the curve from $t = 0$ to $t = \pi/2$ and find its length.

10 Find the length from $t = 0$ to $t = \pi$ of the curve given by $x = \cos t + \sin t$, $y = \cos t - \sin t$. Show that the curve is a circle (of what radius?).

11 Find the length from $t = 0$ to $t = \pi/2$ of the curve given by $x = \cos t$, $y = t - \sin t$.

12 What integral gives the length of Archimedes' spiral $x = t \cos t$, $y = t \sin t$?

13 Find the distance traveled in the first second (to $t = 1$) if $x = \frac{1}{2}t^2$, $y = \frac{1}{3}(2t + 1)^{3/2}$.

14 $x = (1 - \frac{1}{2}\cos 2t)\cos t$ and $y = (1 + \frac{1}{2}\cos 2t)\sin t$ lead to $4(1 - x^2 - y^2)^3 = 27(x^2 - y^2)^2$. Find the arc length from $t = 0$ to $\pi/4$.

Find the arc lengths in 15–18 by numerical integration.

15 One arch of $y = \sin x$, from $x = 0$ to $x = \pi$.

16 $y = e^x$ from $x = 0$ to $x = 1$.

17 $y = \ln x$ from $x = 1$ to $x = e$.

18 $x = \cos t$, $y = 3 \sin t$, $0 \leqslant x \leqslant 2\pi$.

19 Draw a rough picture of $y = x^{10}$. Without computing the length of $y = x^n$ from (0, 0) to (1, 1), find the limit as $n \to \infty$.

20 Which is longer between (1, 1) and $(2, \frac{1}{2})$, the hyperbola $y = 1/x$ or the graph of $x + 2y = 3$?

21 Find the speed ds/dt on the circle $x = 2 \cos 3t$, $y = 2 \sin 3t$.

22 Examples 1 and 7 were $y = x^{3/2}$ and $x = t^2$, $y = t^3$:

$$\text{length} = \int_0^4 \sqrt{1 + \tfrac{9}{4}x}\, dx, \quad \text{length} = \int_0^2 \sqrt{4t^2 + 9t^4}\, dt.$$

Show by substituting $x =$ ________ that these integrals agree.

23 Instead of $y = f(x)$ a curve can be given as $x = g(y)$. Then

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dx/dy)^2 + 1}\, dy.$$

Draw $x = 5y$ from $y = 0$ to $y = 1$ and find its length.

24 The length of $x = y^{3/2}$ from (0, 0) to (1, 1) is $\int ds = \int \sqrt{\frac{9}{4}y + 1}\, dy$. Compare with Problem 1: Same length? Same curve?

25 Find the length of $x = \frac{1}{2}(e^y + e^{-y})$ from $y = -1$ to $y = 1$ and draw the curve.

26 The length of $x = g(y)$ is a special case of equation (6) with $y = t$ and $x = g(t)$. The length integral becomes ________.

27 Plot the point $x = 3 \cos t$, $y = 4 \sin t$ at the five times $t = 0$, $\pi/2$, π, $3\pi/2$, 2π. The equation of the curve is $(x/3)^2 + (y/4)^2 = 1$, not a circle but an ________. This curve cannot be written as $y = f(x)$ because ________.

28 (a) Find the length of $x = \cos^2 t$, $y = \sin^2 t$, $0 \leqslant y \leqslant \pi$.
(b) Why does this path stay on the line $x + y = 1$?
(c) Why isn't the path length equal to $\sqrt{2}$?

29 (important) The line $y=x$ is close to a staircase of pieces that go *straight across or straight up*. With 100 pieces of length $\Delta x = 1/100$ or $\Delta y = 1/100$, find the length of carpet on the staircase. (The length of the 45° line is $\sqrt{2}$. The staircase can be close when its length is not close.)

30 The area of an ellipse is πab. The area of a strip around it (width Δ) is $\pi(a+\Delta)(b+\Delta) - \pi ab \approx \pi(a+b)\Delta$. The distance around the ellipse seems to be $\pi(a+b)$. But this distance is impossible to find—what is wrong?

31 The point $x = \cos t$, $y = \sin t$, $z = t$ moves on a *space curve*.

(a) In three-dimensional space $(ds)^2$ equals $(dx)^2 +$ ________. In equation (6), ds is now ________ dt.

(b) This particular curve has $ds =$ ________. Find its length from $t=0$ to $t=2\pi$.

(c) Describe the curve and its shadow in the xy plane.

32 Explain in 50 words the difference between a non-parametric equation $y=f(x)$ and two parametric equations $x=x(t)$, $y=y(t)$.

33 Write down the integral for the length L of $y=x^2$ from (0, 0) to (1, 1). Show that $y=\frac{1}{2}x^2$ from (0, 0) to (2, 2) is exactly twice as long. If possible give a reason using the graphs.

34 (for professors) Compare the lengths of the parabola $y=x^2$ and the line $y=bx$ from (0, 0) to (b, b^2). Does the difference approach a limit as $b \to \infty$?

8.3 Area of a Surface of Revolution

This section starts by constructing surfaces. ***A curve $y=f(x)$ is revolved around an axis.*** That produces a "*surface of revolution,*" which is symmetric around the axis. If we revolve a sloping line, the result is a cone. When the line is parallel to the axis we get a cylinder (a pipe). By revolving a curve we might get a lamp or a lamp shade (or even the light bulb).

Section 8.1 computed the volume inside that surface. ***This section computes the surface area.*** Previously we cut the solid into slices or shells. Now we need a good way to cut up the surface.

The key idea is *to revolve short straight line segments*. Their slope is $\Delta y/\Delta x$. They can be the same pieces of length Δs that were used to find length—now we compute area. When revolved, a straight piece produces a "***thin band***" (Figure 8.10). The curved surface, from revolving $y=f(x)$, is close to the bands. The first step is to compute ***the surface area of a band.***

A small comment: Curved surfaces can also be cut into tiny patches. Each patch is nearly flat, like a little square. The sum of those patches leads to a double integral (with $dx\,dy$). Here the integral stays one-dimensional (dx or dy or dt). Surfaces of revolution are special—we approximate them by bands that go all the way around. A band is just a belt with a slope, and its slope has an effect on its area.

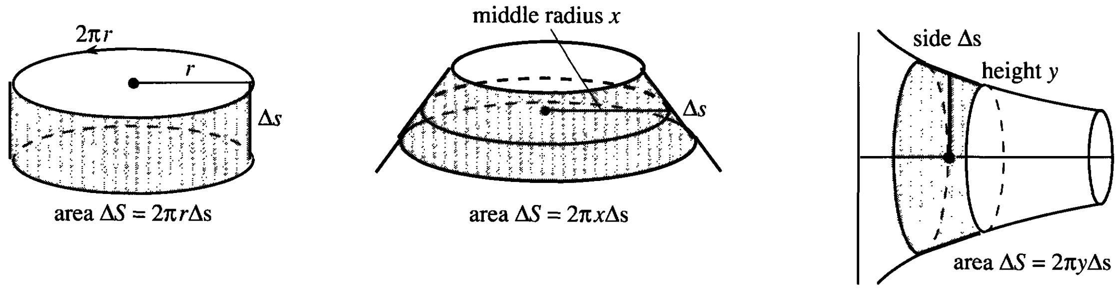


Fig. 8.10 Revolving a straight piece and a curve around the y axis and x axis.

Revolve a small straight piece (*length* Δs *not* Δx). The center of the piece goes around a circle of radius r. The band is ***a slice of a cone***. When we flatten it out (Problems 11–13) we discover its area. The area is the *side length* Δs times the *middle circumference* $2\pi r$:

$$\textbf{\textit{The surface area of a band is }} 2\pi r\Delta s = 2\pi r\sqrt{1+(\Delta y/\Delta x)^2}\ \Delta x.$$

For revolution around the y axis, the radius is $r = x$. For revolution around the x axis, the radius is the height: $r = y = f(x)$. Figure 8.10 shows both bands—the problem tells us which to use. The sum of band areas $2\pi r\ \Delta s$ is close to the area S of the curved surface. In the limit we integrate $2\pi r\ ds$:

> **8C** The surface area generated by revolving the curve $y = f(x)$ between $x = a$ and $x = b$ is
>
> $$S = \int_a^b 2\pi y\sqrt{1+(dy/dx)^2}\ dx \quad \text{around the } x \text{ axis} \quad (r = y) \qquad (1)$$
>
> $$S = \int_a^b 2\pi x\sqrt{1+(dy/dx)^2}\ dx \quad \text{around the } y \text{ axis} \quad (r = x). \qquad (2)$$

EXAMPLE 1 Revolve a complete semicircle $y = \sqrt{R^2 - x^2}$ around the x axis.

The surface of revolution is a ***sphere***. Its area (known!) is $4\pi R^2$. The limits on x are $-R$ and R. The slope of $y = \sqrt{R^2 - x^2}$ is $dy/dx = -x/\sqrt{R^2 - x^2}$:

$$\text{area } S = \int_{-R}^{R} 2\pi\sqrt{R^2 - x^2}\sqrt{1 + \frac{x^2}{R^2 - x^2}}\ dx = \int_{-R}^{R} 2\pi R\ dx = 4\pi R^2.$$

EXAMPLE 2 Revolve a piece of the straight line $y = 2x$ around the x axis.

The surface is a ***cone*** with $(dy/dx)^2 = 4$. The band from $x = 0$ to $x = 1$ has area $2\pi\sqrt{5}$:

$$S = \int 2\pi y\ ds = \int_0^1 2\pi(2x)\ \sqrt{1+4}\ dx = 2\pi\sqrt{5}.$$

This answer must agree with the formula $2\pi r\ \Delta s$ (which it came from). The line from $(0, 0)$ to $(1, 2)$ has length $\Delta s = \sqrt{5}$. Its midpoint is $(\frac{1}{2}, 1)$. Around the x axis, the middle radius is $r = 1$ and the area is $2\pi\sqrt{5}$.

EXAMPLE 3 Revolve the same straight line segment around the y axis. Now the radius is x instead of $y = 2x$. The area in Example 2 is cut in half:

$$S = \int 2\pi x\ ds = \int_0^1 2\pi x\sqrt{1+4}\ dx = \pi\sqrt{5}.$$

For surfaces as for arc length, only a few examples have convenient answers. Watermelons and basketballs and light bulbs are in the exercises. Rather than stretching out this section, we give a final area formula and show how to use it.

The formula applies when there is a ***parameter*** t. Instead of $(x, f(x))$ the points on the curve are $(x(t), y(t))$. As t varies, we move along the curve. The length formula $(ds)^2 = (dx)^2 + (dy)^2$ is expressed *in terms of* t.

For the surface of revolution around the x axis, the area becomes a t-integral:

> **8D** The surface area is $\int 2\pi y\ ds = \int 2\pi y(t)\sqrt{(dx/dt)^2 + (dy/dt)^2}\ dt.$ (3)

EXAMPLE 4 The point $x = \cos t$, $y = 5 + \sin t$ travels on a circle with center at (0, 5). Revolving that circle around the x axis produces a doughnut. Find its surface area.

Solution $(dx/dt)^2 + (dy/dt)^2 = \sin^2 t + \cos^2 t = 1$. The circle is complete at $t = 2\pi$:

$$\int 2\pi y\, ds = \int_0^{2\pi} 2\pi(5 + \sin t)\, dt = \Big[2\pi(5t - \cos t)\Big]_0^{2\pi} = 20\pi^2.$$

8.3 EXERCISES

Read-through questions

A surface of revolution comes from revolving a ___a___ around ___b___. This section computes the ___c___. When the curve is a short straight piece (length Δs), the surface is a ___d___. Its area is $\Delta S =$ ___e___. In that formula (Problem 13) r is the radius of ___f___. The line from (0, 0) to (1, 1) has length ___g___, and revolving it produces area ___h___.

When the curve $y = f(x)$ revolves around the x axis, the surface area is the integral ___i___. For $y = x^2$ the integral to compute is ___j___. When $y = x^2$ is revolved around the y axis, the area is $S =$ ___k___. For the curve given by $x = 2t$, $y = t^2$, change ds to ___l___ dt.

Find the surface area when curves 1–6 revolve around the x axis.

1 $y = \sqrt{x}$, $2 \leqslant x \leqslant 6$

2 $y = x^3$, $0 \leqslant x \leqslant 1$

3 $y = 7x$, $-1 \leqslant x \leqslant 1$ (watch sign)

4 $y = \sqrt{4 - x^2}$, $0 \leqslant x \leqslant 2$

5 $y = \sqrt{4 - x^2}$, $-1 \leqslant x \leqslant 1$

6 $y = \cosh x$, $0 \leqslant x \leqslant 1$.

In 7–10 find the area of the surface of revolution around the y axis.

7 $y = x^2$, $0 \leqslant x \leqslant 2$

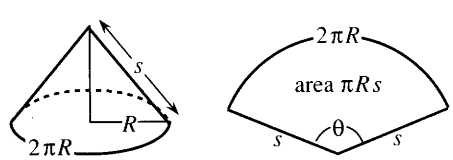
8 $y = \frac{1}{2}x^2 + \frac{1}{2}$, $0 \leqslant x \leqslant 1$

9 $y = x + 1$, $0 \leqslant x \leqslant 3$

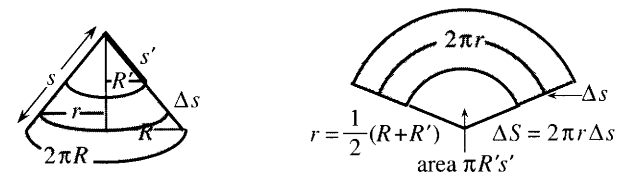
10 $y = x^{1/3}$, $0 \leqslant x \leqslant 1$

11 A cone with base radius R and slant height s is laid out flat. Explain why the angle (in radians) is $\theta = 2\pi R/s$. Then the surface area is a fraction of a circle:

$$\text{area} = \left(\frac{\theta}{2\pi}\right)\pi s^2 = \left(\frac{R}{s}\right)\pi s^2 = \pi R s.$$



12 A band with slant height $\Delta s = s - s'$ and radii R and R' is laid out flat. Explain in one line why its surface area is $\pi R s - \pi R' s'$.



13 By similar triangles $R/s = R'/s'$ or $Rs' = R's$. The middle radius r is $\frac{1}{2}(R + R')$. Substitute for r and Δs in the proposed *area formula* $2\pi r\, \Delta s$, to show that this gives the correct area $\pi R s - \pi R' s'$.

14 Slices of a basketball all have the same area of cover, if they have the same thickness.

(a) Rotate $y = \sqrt{1 - x^2}$ around the x axis. Show that $dS = 2\pi\, dx$.

(b) The area between $x = a$ and $x = a + h$ is ________.

(c) $\frac{1}{4}$ of the Earth's area is above latitude ________.

15 Change the circle in Example 4 to $x = a \cos t$ and $y = b + a \sin t$. Its radius is ________ and its center is ________. Find the surface area of a torus by revolving this circle around the x axis.

16 What part of the circle $x = R \cos t$, $y = R \sin t$ should rotate around the y axis to produce the top half of a sphere? Choose limits on t and verify the area.

17 The base of a lamp is constructed by revolving the quarter-circle $y = \sqrt{2x - x^2}$ ($x = 1$ to $x = 2$) around the y axis. Draw the quarter-circle, find the area integral, and compute the area.

18 The light bulb is a sphere of radius 1/2 with its bottom sliced off to fit onto a cylinder of radius 1/4 and length 1/3. Draw the light bulb and find its surface area (ends of the cylinder not included).

19 The lamp shade is constructed by rotating $y = 1/x$ around the y axis, and keeping the part from $y = 1$ to $y = 2$. Set up the definite integral that gives its surface area.

20 Compute the area of that lamp shade.

21 Explain why the surface area is infinite when $y = 1/x$ is rotated around the x axis ($1 \leqslant x < \infty$). But the volume of "Gabriel's horn" is ________. It can't hold enough paint to paint its surface.

22 A disk of radius 1″ can be covered by four strips of tape (width $\frac{1}{2}$″). If the strips are not parallel, prove that they can't cover the disk. **Hint**: Change to a unit sphere sliced by planes $\frac{1}{2}$″ apart. Problem 14 gives surface area π for each slice.

23 A watermelon (maybe a football) is the result of rotating half of the ellipse $x = \sqrt{2}\cos t$, $y = \sin t$ (which means $x^2 + 2y^2 = 2$). Find the surface area, parametrically or not.

24 Estimate the surface area of an egg.

8.4 Probability and Calculus

Discrete probability usually involves careful counting. Not many samples are taken and not many experiments are made. There is a list of possible outcomes, and a known probability for each outcome. But probabilities go far beyond red cards and black cards. The real questions are much more practical:

1. How often will too many passengers arrive for a flight?
2. How many random errors do you make on a quiz?
3. What is the chance of exactly one winner in a big lottery?

Those are important questions and we will set up models to answer them.

There is another point. Discrete models do not involve calculus. The number of errors or bumped passengers or lottery winners is a small whole number. ***Calculus enters for continuous probability***. Instead of results that exactly equal 1 or 2 or 3, calculus deals with results that fall in a range of numbers. Continuous probability comes up in at least two ways:

(A) An experiment is repeated many times and we take *averages*.
(B) The outcome lies anywhere in an *interval* of numbers.

In the continuous case, the probability p_n of hitting a particular value $x = n$ becomes zero. Instead we have a ***probability density*** $p(x)$—which is a key idea. *The chance that a random X falls between a and b is found by integrating the density $p(x)$*:

$$\text{Prob}\{a \leqslant X \leqslant b\} = \int_a^b p(x)\,dx. \tag{1}$$

Roughly speaking, $p(x)\,dx$ is the chance of falling between x and $x + dx$. Certainly $p(x) \geqslant 0$. If a and b are the extreme limits $-\infty$ and ∞, including all possible outcomes, the probability is necessarily one:

$$\text{Prob}\{-\infty < X < +\infty\} = \int_{-\infty}^{\infty} p(x)\,dx = 1. \tag{2}$$

This is a case where infinite limits of integration are natural and unavoidable. In studying probability they create no difficulty—areas out to infinity are often easier.

Here are typical questions involving continuous probability and calculus:

4. How conclusive is a 53%–47% poll of 2500 voters?
5. Are 16 random football players safe on an elevator with capacity 3600 pounds?
6. How long before your car is in an accident?

It is not so traditional for a calculus course to study these questions. They need extra thought, beyond computing integrals (so this section is harder than average). But probability is more important than some traditional topics, and also more interesting.