# CHAPTER 3 APPLICATIONS OF THE DERIVATIVE

## 3.1 Linear Approximation (page 95)

On the graph, a linear approximation is given by the **tangent** line. At $x = a$, the equation for that line is $Y = f(a) + \mathbf{f'(a)(x-a)}$. Near $x = a = 10$, the linear approximation to $y = x^3$ is $Y = 1000 + \mathbf{300(x-10)}$. At $x = 11$ the exact value is $(11)^3 = \mathbf{1331}$. The approximation is $Y = \mathbf{1300}$. In this case $\Delta y = \mathbf{331}$ and $dy = \mathbf{300}$. If we know $\sin x$, then to estimate $\sin(x + \Delta x)$ we add $\mathbf{(\cos x)\Delta x}$.

In terms of $x$ and $\Delta x$, linear approximation is $f(x + \Delta x) \approx f(x) + \mathbf{f'(x)\Delta x}$. The error is of order $(\Delta x)^p$ or $(x-a)^p$ with $p = \mathbf{2}$. The differential $dy$ equals $\mathbf{dy/dx}$ times the differential $\mathbf{dx}$. Those movements are along the **tangent** line, where $\Delta y$ is along the **curve**.

**1** $Y = x$ **3** $Y = 1 + 2(x - \frac{\pi}{4})$ **5** $Y = 2\pi(x - 2\pi)$ **7** $2^6 + 6 \cdot 2^5 \cdot .001$ **9** 1
**11** $1 - 1(-.02) = 1.02$ **13** Error .000301 vs. $\frac{1}{2}$ (.0001)6 **15** $.0001 - \frac{1}{3}10^{-8}$ vs. $\frac{1}{2}(.0001)(2)$
**17** Error .59 vs. $\frac{1}{2}(.01)(90)$ **19** $\frac{d}{dx}\sqrt{1-x} = \frac{-1}{2\sqrt{1-x}} = -\frac{1}{2}$ at $x = 0$
**21** $\frac{d}{du}\sqrt{c^2+u} = \frac{1}{2\sqrt{c^2+u}} = \frac{1}{2c}$ at $u = 0, c + \frac{u}{2c} = c + \frac{x^2}{2c}$ **23** $dV = 3(10)^2(.1)$
**25** $A = 4\pi r^2, dA = 8\pi r\, dr$ **27** $V = \pi r^2 h, dV = 2\pi rh\, dr$ (plus $\pi r^2\, dh$) **29** $1 + \frac{1}{2}x$ **31** 32nd root

**2** $f(x) = \frac{1}{x}$ and $a = 2 : Y = f(a) + f'(a)(x-a) = \frac{1}{2} - \frac{1}{4}(x-2)$. Tangent line is $Y = 1 - \frac{1}{4}x$.
**4** $f(x) = \sin x$ and $a = \frac{\pi}{2}$: $Y = \sin\frac{\pi}{2} + \cos\frac{\pi}{2}(x - \frac{\pi}{2}) = 1$. Level tangent line.
**6** $f(x) = \sin^2 x$ and $a = 0 : Y = \sin^2 0 + 2\sin 0\cos 0(x-0) = 0$. Tangent line is $x$ axis.
**8** $f(x) = \sin x, a = 0, Y = \sin 0 + \cos 0(.02 - 0) = .02$. Compare with $\sin .02 = .019999$
**10** $f(x) = x^{1/4}, Y = 16^{1/4} + \frac{1}{4}16^{-3/4}(15.99 - 16) = 2 + \frac{1}{4}\frac{1}{8}(-.01) = 1.9996875$. Compare $15.99^{1/4} = 1.9996874$.
**12** $f(x) = \sin x, a = \pi, Y = \sin\pi + \cos\pi(3.14 - \pi) = -1(.00159)$. Compare $\sin 3.14 = -.00159$.
**14** Actual error: $\cos(.01) - 1 = -4.99996\ 10^{-5}$; predicted error for $f = \cos x, f'' = -\cos x$ near $x = 0$ : $\frac{1}{2}(\Delta x)^2 f''(0) = \frac{1}{2}(.0001)(-1) = -5\ 10^{-5}$
**16** Actual error: $(1.01)^{-3} - (1 - .03) = .00059$; predicted error for $f = \frac{1}{x^3}, f'' = \frac{12}{x^5} : \frac{1}{2}(.01)^2\frac{12}{1^5} = .00060$.
**18** Actual error: $\sqrt{8.99} - (3 + \frac{1}{6}(-.01)) = -4.6322\ 10^{-7}$; predicted error for $f = \sqrt{x}, f'' = \frac{-1}{4x^{3/2}}$ near $x = 9 : \frac{1}{2}(.01)^2(\frac{-1}{4(9)^{3/2}}) = -4.6296\ 10^{-7}$.
**20** $(1-u)^{-1/2}$ has derivative $-\frac{1}{2}(1-u)^{-3/2}(-1)$. At $u = 0$ these equal 1 and $\frac{1}{2}$. Then $\frac{1}{\sqrt{1-u}} \approx 1 + \frac{1}{2}u$ or $\frac{1}{\sqrt{1-x^2}} \approx 1 + \frac{1}{2}x^2$.
**22** $df = -\sin x\, dx$ and $df = \frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}dx = \frac{-2}{(x-1)^2}dx$ (by the quotient rule) and $df = 2(x^2+1)(2x)dx = 4x(x^2+1)dx$ by the power rule. Notice $dx$ in the formula for $df$.
**24** $A = 6x^2$ so $dA = 12x\, dx$. **26** $V = \pi r^2 h$ so $dV = \pi r^2 dh = \pi(2)^2(0.5) = \pi(.2)$.
**28** With $\frac{v}{c}$ as $x$ in Problem 20, the correction $\Delta m$ is $m_0$ times $\frac{1}{2}x^2 = \frac{1}{2}(\frac{v}{c})^2$. Then $\Delta m$ times $c^2$ is $\frac{1}{2}m_0 v^2 =$ energy equivalent to change in mass.
**30** $\sqrt{1+x}$ has derivatives $\frac{1}{2}(1+x)^{-1/2}$ and $-\frac{1}{4}(1+x)^{-3/2}$. At $x = 0$ the second derivative is $-\frac{1}{4}$. The difference

$y - Y$ between curve and tangent line is about $\frac{1}{2}y''(0)x^2 = -\frac{1}{8}x^2$. This is negative so $Y$ is higher than $y$.

## 3.2 Maximum and Minimum Problems (page 103)

If $df/dx > 0$ in an interval then $f(x)$ is **increasing**. If a maximum or minimum occurs at $x$ then $f'(x) = \mathbf{0}$. Points where $f'(x) = 0$ are called **stationary** points. The function $f(x) = 3x^2 - x$ has a **(minimum)** at $x = \mathbf{\frac{1}{6}}$. A stationary point that is not a maximum or minimum occurs for $f(x) = \mathbf{x^3}$.

Extreme values can also occur when $\mathbf{f'(x)}$ is not defined or at the **endpoints** of the domain. The minima of $|x|$ and $5x$ for $-2 \le x \le 2$ are at $x = \mathbf{0}$ and $x = \mathbf{-2}$, even though $df/dx$ is not zero. $x^*$ is an absolute **maximum** when $f(x^*) \ge f(x)$ for all $x$. A **relative minimum** occurs when $f(x^*) \le f(x)$ for all $x$ near $x^*$.

The minimum of $\frac{1}{2}ax^2 - bx$ is $\mathbf{-b^2/2a}$ at $x = \mathbf{b/a}$.

**1** $x = -2$ : abs min **3** $x = -1$ : rel max, $x = 0$: abs min, $x = 4$ : abs max
**5** $x = -1$ : abs max, $x = 0, 1$ : abs min, $x = \frac{1}{2}$ : rel max **7** $x = -3$ : abs min, $x = 0$ : rel max, $x = 1$ : rel min
**9** $x = 1, 9$ : abs min, $x = 5$ : abs max **11** $x = \frac{1}{3}$ : rel max, $x = 1$ : rel min, $x = 0$ : stationary (not min or max)
**13** $x = 0, 1, 2, \cdots$ : abs min, $x = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$ : abs max **15** $-1 \le x \le 1$ : abs min ($-1$ and $1$ are rough pts)
**17** $x = 0$ : rel min, $x = \frac{1}{3}$ : abs max, $x = 4$ : abs min
**19** $x = 0$ : abs min, $x = \pi$ : stationary (not min or max), $x = 2\pi$ : abs max
**21** $\theta = 0$ : rel min, $\tan\theta = -\frac{4}{3}$ ($\sin\theta = \frac{4}{5}$ and $\cos\theta = -\frac{3}{5}$ abs max, $\sin\theta = -\frac{4}{5}$ and $\cos\theta = \frac{3}{5}$ abs min), $\theta = 2\pi$ : rel max
**23** $h = \frac{1}{3}(62''$ or 158 cm); cube **25** $\frac{v}{av^2+b}; 2\sqrt{ab}$ gallons/mile, $\frac{1}{2\sqrt{ab}}$ miles/gallon at $v = \sqrt{\frac{b}{a}}$
**27** (b) $\theta = \frac{3\pi}{8} = 67.5°$ **29** $x = \frac{a}{\sqrt{3}}$; compare Example 7; $\frac{a}{b} = \sqrt{3}$
**31** $R(x) - C(x)$; $\frac{R(x)-C(x)}{x}$; $\frac{dR}{dx} - \frac{dC}{dx}$; profit **33** $x = \frac{d-a}{2(b-e)}$; zero **35** $x = 2$
**37** $V = x(6 - \frac{3x}{2})(12 - 2x); x \approx 1.6$ **39** $A = \pi r^2 + x^2, x = \frac{1}{4}(4 - 2\pi r); r_{\min} = \frac{2}{2+\pi}$
**41** max area 2500 vs $\frac{10000}{\pi} = 3185$ **43** $x = 2, y = 3$ **45** $P(x) = 12 - x$; thin rectangle up $y$ axis
**47** $h = \frac{H}{3}, r = \frac{2R}{3}, V = \frac{4\pi R^2 H}{27} = \frac{4}{9}$ of cone volume
**49** $r = \frac{HR}{2(H-R)}$; best cylinder has *no* height, area $2\pi R^2$ from top and bottom (?)
**51** $r = 2, h = 4$ **53** 25 and 0 **55** 8 and $-\infty$
**57** $\sqrt{r^2 + x^2} + \sqrt{q^2 + (s - x)^2}$; $\frac{df}{dx} = \frac{x}{\sqrt{r^2+x^2}} - \frac{s-x}{\sqrt{q^2+(s-x)^2}} = 0$ when $\sin a = \sin c$
**59** $y = x^2 = \frac{3}{2}$ **61** $(1, -1), (\frac{13}{5}, -\frac{1}{5})$ **63** $m = 1$ gives nearest line **65** $m = \frac{1}{3}$ **67** equal; $x = \frac{1}{2}$
**69** $\frac{1}{x}x^2$ **71** True (use sign change of $f''$)
**73** Radius $R$, swim $2R\cos\theta$, run $2R\theta$, time $\frac{2R\cos\theta}{v} + \frac{2R\theta}{10v}$; *max* when $\sin\theta = \frac{1}{10}$, *min* all run

**2** $f'(x) = 3x^2 - 12 = 0$ gives stationary points $x = 2$ and $x = -2$. The graph comes up to a local maximum at $x = -2$ and falls to a local minimum at $x = 2$. (*You could ask for a sketch of the graph.*)
**4** $f'(x) = 2x - \frac{2}{x^2} = 0$ at $x = 1$. That stationary point (and endpoint) has $f(1) = 3$. Then $f(x)$ increases to 16.5 at the endpoint $x = 4$.
**6** $f'(x) = (x - x^2)^{-2}(1 - 2x) = 0$ at $x = \frac{1}{2}$. This stationary point is a minimum; $f(x) \to +\infty$ at the endpoints.
**8** $f'(x) = 2x - 4$ for $0 \le x < 1, f'(x) = 2x$ for $1 < x \le 2$. No stationary points. The endpoints give $f(0) = 0$ and $f(2) = 0$. These are maxima. The graph has a rough point (corner) at $x = 1$, where $f(1) = -3$ and

the slope jumps from negative to positive.

**10** $f'(x) = 1 + \cos x = 0$ at the stationary point $x = \pi$. This is not a minimum or maximum; since $f' = 1 + \cos x$ is never negative, the graph continues upward. The endpoints give $f(0) = 0$ and $f(2\pi) = 2\pi$.

**12** $f'(x) = \frac{1}{(1+x)^2} > 0$ so the endpoints 0 and 100 give $f_{\min} = 0$ and $f_{\max} = \frac{100}{101}$.

**14** Halfway between each prime number and the next is a local maximum of $f(x)$. This is a rough point (corner) where the slope changes from 1 to $-1$. The minimum is $f(x) = 0$ when $x$ is prime. The endpoint $x = 0$ has $f = 2$, a local maximum.

**16** $f'(x) = \sqrt{1-x^2} + x\frac{(-x)}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}} = 0$ at $x = 1/\sqrt{2}$. The endpoints give $f(0) = 0$ and $f(1) = 0$. Then $f(1/\sqrt{2}) = \frac{1}{2}$ must be an absolute maximum.

**18** $f'(x) = \cos x - \sin x = 0$ at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. At those points $f(\frac{\pi}{4}) = \sqrt{2}$, the maximum, and $f(\frac{5\pi}{4}) = -\sqrt{2}$, the minimum. The endpoints give $f(0) = f(2\pi) = 1$.

**20** By the product rule $f'(\theta) = -2\cos\theta\sin^2\theta + \cos^3\theta = 3\cos^3\theta - 2\cos\theta = 0$ when $\cos^2\theta = \frac{2}{3}$ or $\cos\theta = 0$. At those points $f = \frac{2}{3}\sin\theta = \frac{2}{3}\sqrt{\frac{1}{3}}$, (the maximum) or $f = -\frac{2}{3}\sqrt{\frac{1}{3}}$ (the minimum) or $f = 0$ (stationary point). The graph is below the axis, then above.

**22** $f'(x) = \{2x \text{ for } x < 1, 2x - 4 \text{ for } x > 1\} = 0$ at $x = 0$ and $x = 2$. At those stationary points $f = 1$ (both minima). At the rough point $f(1) = 2$, a local maximum.

**24** The derivative $2(x-70) + 2(x-80) + 2(x-120)$ is zero at the average $x = \frac{70+80+120}{3} = 90$. Nervous patient: The derivative $2(x-70) + 2(x-80) + (x-120)$ is zero at the weighted average $\frac{2(70)+2(80)+120}{5} = 84$.

**26** (a) The chauffeur costs \$10 per hour or \$10/$v$ per mile. The gas costs \$1 per gallon or \$5/$(120-2v)$ per mile. The total cost per mile $f(v) = \frac{10}{v} + \frac{5}{120-2v}$ has $f'(v) = -\frac{10}{v^2} + \frac{10}{(120-2v)^2} = 0$ when $v = 120 - 2v$ or $v = 40$. Then $f(40) = \frac{10}{40} + \frac{5}{40} = \frac{3}{8}$ dollars per mile (the minimum). At $v = 0$ the chauffeur costs infinity per mile; at $v = 60$ the gas costs infinity.

**28** When the length of day has its maximum and minimum, its derivative is zero (no change in the length of day). In reality the time unit of days is discrete not continuous; then $\Delta f$ is small instead of $df = 0$.

**30** $f'(t) = \frac{(1+3t^2)3-(1+3t)(6t)}{(1+3t^2)^2} = \frac{-9t^2-6t+3}{(1+3t^2)^2}$. Factoring out $-3$, the equation $3t^2+2t-1 = 0$ gives $t = \frac{-2+\sqrt{16}}{6} = \frac{1}{3}$. At that point $f_{\max} = \frac{2}{4/3} = \frac{3}{2}$. The endpoints $f(0) = 1$ and $f(\infty) = 0$ are minima.

**32** We receive $R(x) = ax + bx^2$ when the price per pizza is $p(x) = a + bx$. In reverse: When the price is $p$ we sell $x = \frac{p-a}{b}$ pizzas. We expect $b < 0$ because additional pizzas are cheaper.

**34** The profit crosses zero when $3x - x^2 = 1 + x^2$ or $2x^2 - 3x + 1 = 0$ or $(2x-1)(x-1) = 0$. The profit is positive between the roots $x = \frac{1}{2}$ and $x = 1$. The largest profit is when $3 - 2x = 2x$ or $x = \frac{3}{4}$.
For cost $2 + x^2$, the equation $3x - x^2 = 2 + x^2$ or $2x^2 - 3x + 2 = 0$ has no real roots ($b^2 - 4ac$ is $-7$ in the quadratic formula). So the profit never crosses zero.

**36** Volume of popcorn box $= x(6-x)(12-x) = 72x - 18x^2 + x^3$. Then $\frac{dV}{dx} = 72 - 36x + 3x^2$. Dividing by 3 gives $x^2 - 12x + 24 = 0$ or $x = 6 \pm \sqrt{36-24} = 6 \pm \sqrt{12}$ at stationary points. **Maximum volume is at** $\mathbf{x = 6 - \sqrt{12}}$. ($V$ has a minimum at $x = 6 + \sqrt{12}$, when the box has negative width.)

**38** Classic: The side lengths are $12 - 2x$ so the volume is $V(x) = x(12-2x)^2$. By the product rule $\frac{dV}{dx} = (12-2x)^2 + x(12-2x)(-4)$. Factor out $12 - 2x$, which is zero when $x = 6$ (no volume). Then $\frac{dV}{dx} = 0$ for $12 - 2x - 4x = 0$ or $x = 2$. The maximum volume is $V = 2(8)^2 = 128$ cubic inches.

**40** Let $x$ be the length of the sides perpendicular to the wall. The side parallel to the wall uses the remaining $200 - 2x$ feet. The area is $A = x(200-2x)$ and $\frac{dA}{dx} = 200 - 4x = 0$ at $x = 50$. Then $\mathbf{A = 5000}$ **square feet.** Alternative: $x$ is the length parallel to the wall and the other sides have length $\frac{1}{2}(200-x)$. The maximum of $A = \frac{1}{2}x(200-x)$ is still 5000 square feet.

**42** Let the sides perpendicular to the existing fence have length $x$. This leaves $300 - 2x$ meters of fence so the other sides are $150 - x$ meters. (Note: $x > 50$ is not allowed or this length would be below the existing 100 meters.) The area $A = x(150-x)$ has $\frac{dA}{dx} = 150 - 2x = 0$ when $x = 75$ meters which is not allowed.

The best choice $\mathbf{x = 50}$ gives $\mathbf{A = 5000\ m^2}$. **(Maximum at endpoint.)**

**44** The triangle has corners at (0,0), (4,0), and (0,6). The biggest rectangle inside it has corners at (2,0) and (0,3). If the rectangle sits straight up, its area is 2 times 3 equals 6. If the rectangle sits parallel to the hypotenuse, its area is $\mathbf{\sqrt{13}}$ **times** $\mathbf{\frac{6}{\sqrt{13}}}$ **equals 6** (same maximum in new orientation!)

In the *American Math Monthly* of May, 1990, Mary Embry-Wardrop finds the maximum area for each orientation of the rectangle. Parallel to the hypotenuse, it can have corners at $(2t, 0)$ and $(0, 3t)$ so one side has length $\sqrt{13}t$. *Draw the figure to see similar triangles.* The other side has length $\frac{12-6t}{\sqrt{13}}$. So the area is $t(12-6t)$ with maximum at $t = 1$ as in boldface above.

**46** The cylinder has radius $r$ and height $h$. Going out $r$ and up $\frac{1}{2}h$ brings us to the sphere: $r^2 + (\frac{1}{2}h)^2 = 1$. The volume of the cylinder is $V = \pi r^2 h = \pi[1 - (\frac{1}{2}h)^2]h$. Then $\frac{dV}{dh} = \pi[1 - (\frac{1}{2}h)^2] + \pi(-\frac{1}{2}h)h = 0$ gives $1 = \frac{3}{4}h^2$. The best $h$ is $\mathbf{\frac{2}{\sqrt{3}}}$, so $V = \pi[1 - \frac{1}{3}]\frac{2}{\sqrt{3}} = \mathbf{\frac{4\pi}{3\sqrt{3}}}$. Note: $r^2 + \frac{1}{3} = 1$ gives $r = \sqrt{\frac{2}{3}}$.

**48** The equation in Problem 47 gives $h = H(1 - \frac{r}{R})$. Then the side area is $A = 2\pi r H(1 - \frac{r}{R})$ and $\frac{dA}{dr} = 2\pi H - 4\pi H\frac{r}{R} = 0$ for $\mathbf{r = \frac{R}{2}}$. In that case $A = 2\pi\frac{R}{2}H(\frac{1}{2}) = \frac{1}{2}\pi RH$.

**50** The triangle with height $y$ and base $1+x$ is similar to a triangle with height 8 and base $x$ (hypotenuse along the ladder). Then $\frac{y}{1+x} = \frac{8}{x}$ gives $y = \frac{8}{x}(1+x)$. The ladder length $L = (1+x)^2 + y^2 = (1+x)^2(1 + \frac{64}{x^2})$ has $\frac{dL}{dx} = (1+x)^2(-\frac{128}{x^3}) + 2(1+x)(1+\frac{64}{x^2}) = (1+x)[-\frac{128}{x^3} - \frac{128}{x^2} + 2 + \frac{128}{x^2}]$. Thus $\frac{dL}{dx} = 0$ when $x^3 = 64$ and $\mathbf{x = 4}$ and $\mathbf{L = 5^3 = 125}$.

**52** The upper triangle has area $x\sqrt{1-x^2}$ (twice a right triangle with side $x$ and hypotenuse 1). Similarly the lower triangle has area $x\sqrt{4-x^2}$. The derivatives are $\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}$ and $\sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} = \frac{4-2x^2}{\sqrt{4-x^2}}$. These add to zero when $\frac{4-2x^2}{\sqrt{4-x^2}} = -\frac{1-2x^2}{\sqrt{1-x^2}}$ or $(1-x^2)(4-2x^2)^2 = (4-x^2)(1-2x^2)^2$ or $16 - 32x^2 + 20x^4 - 4x^8 = 4 - 17x^2 + 20x^4 - 4x^8$ or $12 = 15x^2$. When $\mathbf{x^2 = \frac{4}{5}}$ the areas are $\sqrt{\frac{4}{5}}\sqrt{\frac{1}{5}} + \sqrt{\frac{4}{5}}\sqrt{\frac{19}{5}} = \mathbf{\frac{2}{5}(1+\sqrt{19})}$.

**54** $x^2 + y^2 = x^2 + (10-x)^2$ has derivative $2x - 2(10-x) = 0$ at $\mathbf{x = 5}$. Then $y = 5$ and $x^2 + y^2 = \mathbf{50}$. The maximum must be at an endpoint: $10^2 + 0^2 = \mathbf{100}$.

**56** *First method:* Use the identity $\sin x \sin(10-x) = \frac{1}{2}\cos(2x-10) - \frac{1}{2}\cos 10$. The maximum when $2x = 10$ is $\frac{1}{2} - \frac{1}{2}\cos 10 = .92$. The minimum when $2x - 10 = \pi$ is $-\frac{1}{2} - \frac{1}{2}\cos 10 = -.08$. *Second method:* $\sin x \sin(10-x)$ has derivative $\cos x \sin(10-x) - \sin x \cos(10-x)$ which is $\sin(10-x-x)$. This is zero when $10 - 2x$ equals 0 or $\pi$. Then $\sin x \sin(10-x)$ is $(\sin 5)(\sin 5) = .92$ or $\sin(5 + \frac{\pi}{2})\sin(5 - \frac{\pi}{2}) = -.08$.

**58** Time on $AX$ is $\frac{\sqrt{r^2+x^2}}{v}$. Time on $XB$ is $\frac{\sqrt{q^2+(s-x)^2}}{w}$. Add to find total time. The derivatives give $\mathbf{\frac{x}{v\sqrt{r^2+x^2}} = \frac{s-x}{w\sqrt{q^2+(s-x)^2}}}$ or $\frac{\sin a}{v} = \frac{\sin b}{w}$.

**60** The squared distance $\ell^2 = x^2 + y^2 = x^2 + (5-2x)^2$ has derivative $2x - 4(5-2x) = 0$ at $\mathbf{x = 2}$. At that point $\mathbf{y = 1}$.

**62** The squared distance $x^2 + (y - \frac{1}{3})^2 = x^2 + (x^2 - \frac{1}{3})^2$ has derivative $2x + 4x(x^2 - \frac{1}{3}) = 0$ at $\mathbf{x = 0}$. Don't just cancel the factor $x$! The nearest point is (0,0). Writing the squared distance as $x^2 + (y - \frac{1}{3})^2 = y + (y - \frac{1}{3})^2$ we forget that $y = x^2 \geq 0$. Zero is an **endpoint** and it gives the minimum.

**64** The triangle has one side from $(-1, 1)$ to (3,9) on the line $y = 2x + 3$. Its length is $\sqrt{4^2 + 8^2} = 4\sqrt{5}$. The height of the triangle is the distance from this line to $(x, x^2)$. By the hint the distance is $\frac{|x^2-2x-3|}{\sqrt{1+2^2}}$. This is a minimum at $\mathbf{x = 1}$, where it equals $\frac{4}{\sqrt{5}}$. The minimum area is $\frac{1}{2}(4\sqrt{5})\frac{4}{\sqrt{5}} = 8$.

**66** To find where the graph of $y(x)$ has greatest slope, solve $\mathbf{\frac{d^2y}{dx^2} = 0}$ **maximize** $\frac{dy}{dx}$). For $y = \frac{1}{1+x^2}$ and $\frac{dy}{dx} = \frac{-2x}{(1+x^2)^2}$ and $\frac{d^2y}{dx^2} = \frac{(1+x^2)^2(-2)+2x(4x)(1+x^2)}{(1+x^2)^4} = 0$, the slope is greatest when $(1+x^2)(-2) + 2x(4x) = 0$ or $6x^2 - 2 = 0$ or $\mathbf{x = \frac{1}{\sqrt{3}}}$.

**68** Suppose $y$ is fixed. The minimum of $x^2+xy-y^2$ is where $2x+y=0$ so $m(y)=(-\frac{y}{2})^2-\frac{y}{2}y-y^2=-\frac{5}{4}\mathbf{y^2}$. The maximum of $m(y)$ is **zero**. Now $x$ is fixed. For the maximum of $x^2+xy-y^2$ take the $y$ derivative to get $x-2y=0$ or $y=\frac{1}{2}x$. The maximum is $M(y)=x^2+x(\frac{1}{2}x)-(\frac{1}{2}x)^2=\frac{5}{4}\mathbf{x^2}$. The minimum of $M(y)$ is **zero**. So max of min equals min of max.

**70** When $\frac{1}{x}$ is $2\pi, 4\pi, 6\pi, \cdots$ the slope $1-2\cos(\frac{1}{x})+4x\sin(\frac{1}{x})$ equals $1-2+0=-1$. So the wavy function $x+2x^2\sin(\frac{1}{x})$ is decreasing when $x=\frac{1}{2\pi},\frac{1}{4\pi},\cdots$, nearer and nearer to $x=0$ (but it is increasing at $x=0$).

**72** $y(x)=-|x|$ for $|x|\le 1$ has $y_{\min}=-1$ and $y_{\max}=0$.

## 3.3 Second Derivatives: Bending and Acceleration (page 110)

The direction of bending is given by the sign of $\mathbf{f''(x)}$. If the second derivative is **positive** in an interval, the function is concave up (or convex). The graph bends **upward**. The tangent lines are **below** the graph. If $f''(x)<0$ then the graph is concave **down**, and the slope is **decreasing**.

At a point where $f'(x)=0$ and $f''(x)>0$, the function has a **minimum**. At a point where $\mathbf{f'(x)=0}$ **and** $\mathbf{f''(x)<0}$, the function has a maximum. A point where $f''(x)=0$ is an **inflection** point, provided $f''$ changes sign. The tangent line **crosses** the graph.

The centered approximation to $f'(x)$ is $[\mathbf{f(x+\Delta x)-f(x-\Delta x)}]/2\Delta x$. The 3-point approximation to $f''(x)$ is
$[\mathbf{f(x+\Delta x)-2f(x)+f(x-\Delta x)}]/(\Delta x)^2$. The second-order approximation to $f(x+\Delta x)$ is $f(x)+f'(x)\Delta x+\frac{1}{2}\mathbf{f''(x)(\Delta x)^2}$. Without that extra term this is just the **linear** (or **tangent**) approximation. With that term the error is $\mathbf{O((\Delta x)^3)}$.

**3** $y=-1-x^2$; no $\cdots$ **5** False **7** True **9** True ($f'$ has 8 zeros, $f''$ has 7)
**11** $x=3$ is min: $f''(3)=2$ **13** $x=0$ not max or min; $x=\frac{9}{2}$ is min: $f''(\frac{9}{2})=81$
**15** $x=\frac{3\pi}{4}$ is max: $f''(\frac{3\pi}{4})=-\sqrt{2}$; $x=\frac{7\pi}{4}$ is min: $f''(\frac{7\pi}{4})=\sqrt{2}$
**17** Concave down for $x<-1$ and $x>\frac{1}{3}$ (inflection points)
**19** $x=3$ is max: $f''(3)=-4$; $x=2,4$ are min but $f''=0$ **21** $f(\Delta x)=f(-\Delta x)$ **23** $1+x-\frac{x^2}{2}$
**25** $1-\frac{x^2}{6}$ **27** $1-\frac{1}{2}x-\frac{1}{8}x^2$ **29** Error $\frac{1}{2}f''(x)\Delta x$ **31** Error $0\Delta x+\frac{1}{3}f'''(x)(\Delta x)^2$
**37** $\frac{1}{.99}=1.010\overline{101}$; $\frac{1}{1.1}=.90\overline{909}$ **39** Inflection **41** 18 vs. 17 **43** Concave up; below

**2** We want inflection points $\frac{d^2y}{dx^2}=0$ at $x=0$ and $x=1$. Take $\frac{d^2y}{dx^2}=x-x^2$. This is positive ($y$ is concave up) between 0 and 1. Then $\mathbf{y=\frac{1}{6}x^3-\frac{1}{12}x^4}$. (Intermediate step: the first derivative is $\frac{1}{2}x^2-\frac{1}{3}x^3$). Alternative: $y=-x^2$ for $x<0$, then $y=+x^2$ up to $x=1$, then $y=2-x^2$ for $x>1$.

**4** Set $f''(x)=x-2$. Then $f'(x)=\frac{1}{2}(x-2)^2$. Then $\mathbf{f(x)=\frac{1}{6}(x-2)^3}$.

**6** **True:** If $f'=0$ at the endpoints then $f''=0$ at an in-between point. In Section 3.7 this will be Rolle's theorem (applied to $f'$).

**8** **True:** If $f(x)$ is 9th degree then $f''(x)$ is 7th degree. Any 7th curve degree curve crosses the axis because $cx^7$ has opposite signs as $x\to\infty$ and $x\to-\infty$. Then $f''=0$ gives an inflection point. (False if $f(x)$ is 10th degree)

**10** **False:** Take any $f(x)$ that does have seven inflection points and nine zeros. Add a large constant to raise the graph of $f(x)$. Then $f(x)+C$ has fewer zeros but the same inflection points.

**12** $f'(x) = 3x^2 - 12x = 0$ at $x = 0$ and $x = 4$. The second derivative $6x - 12$ is $-12$ then $+12$. So $\mathbf{x = 0}$ **is a maximum point,** $\mathbf{x = 4}$ **is a minimum point.**

**14** $f'(x) = 11x^{10} - 60x^9 = 0$ at $x = 0$ and $x = \frac{60}{11}$. The second derivative $110x^9 - 540x^8$ has $f''(0) = 0$ and and $f''(\frac{60}{11}) = 110(\frac{60}{11})^9 - 540(\frac{60}{11})^8 > 0$. Then $x = 0$ is **not a minimum or maximum** (check function) and $\mathbf{x = \frac{60}{11}}$ **is a minimum point.**

**16** $f'(x) = 1 + 2\cos 2x = 0$ at $\cos 2x = -\frac{1}{2}$ or $2x = 120°$ or $240°(\frac{2\pi}{3}$ or $\frac{4\pi}{3})$. At those two stationary points $f''(x) = -4\sin 2x$ is negative ($f$ is a maximum) and then positive ($f$ is a minimum).

**18** $f'(x) = \cos x + \sec^2 x$ gives $f''(x) = -\sin x + 2\sec^2 x \tan x = \mathbf{\sin x(-1 + \frac{2}{\cos^3 x})}$. The inflection points are $0, \pi, 2\pi, \cdots$ where $\sin x = 0$. (Note $\cos^3 x = 2$ is impossible.) Then $\mathbf{f'' > 0}$ **for** $\mathbf{0 < x < \frac{\pi}{2}}$ **and** $\mathbf{\pi < x < \frac{3\pi}{2}}$ (concave up). In the other intervals $f$ is concave down. Watch blow-up at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ when $\cos x = 0$.

**20** $f'(x) = \cos x + 3(\sin x)^2 \cos x$ gives $f''(x) = -\sin x - 3(\sin x)^2 \sin x + 6 \sin x(\cos x)^2 = \mathbf{5\sin x - 9\sin^3 x}$. Inflection points where $\sin x = 0$ (at $0, \pi, \cdots$) and also where $(\sin x)^2 = \frac{5}{9}$ (an angle $x$ in each quadrant). Concavity is up-down-up from $0$ to $\pi$. Then down-up-down from $\pi$ to $2\pi$.

**22** An odd function has $f(0) = 0$ and $f(-\Delta x) = -f(\Delta x)$. The second difference is zero (so is $f''$ at $x = 0$).

**24** $f(x) = \tan x$ has $f'(x) = \sec^2 x$ and $f''(x) = 2\sec^2 x \tan x$. The quadratic approximation is $0 + 1(x - 0) + \frac{1}{2}0(x - 0)^2$ or just $\mathbf{x}$.

**26** $f(x) = 1 + x + x^2$ has $f'(x) = 1 + 2x$ and $f''(x) = 2$. Substituting $x = 0$ gives $1 + 1(x - 0) + \frac{1}{2}2(x - 0)^2 = 1 + x + x^2$ **(of course)**. Substituting $x = 1$ gives $3 + 3(x - 1) + \frac{1}{2}2(x - 1)^2$, which is again $1 + x + x^2$.

**28** $y = (1 - x)^{-2}$ has $y' = -2(1 - x)^{-3}$ and $y'' = 6(1 - x)^{-4}$. Substituting $x = 0$ gives $1, -2, 6$. The quadratic is $1 - 2x + \frac{1}{2}6x^2 = \mathbf{1 - 2x + 3x^2}$.

**30** $f(-\Delta x) \approx f(0) + f'(0)(-\Delta x) + \frac{1}{2}f''(0)(-\Delta x)^2$

**32** The cubic term is $\frac{1}{6}f'''(0)(\Delta x)^3$. When $f = x^3$ and $f' = 3x^2$ and $f'' = 6x$ and $f''' = 6$, the third degree approximation is $f(\Delta x) = 0 + 0 + 0 + \frac{1}{6}6(\Delta x)^3 = (\Delta x)^3$, exactly right.

**34**

| $f(x) = \frac{1}{1-x}$ | $\frac{f(\Delta x) - f(0)}{\Delta x} - f'(0)$ | $\frac{f(\Delta x) - f(-\Delta x)}{2\Delta x} - f'(0)$ | $\frac{f(\Delta x) - 2f(0) + f(-\Delta x)}{(\Delta x)^2} - f''(0)$ | $1 + x + x^2 - \frac{1}{1-x}$ |
|---|---|---|---|---|
| $\Delta x = 1/4$ | $1/3 = .333$ | $1/15 = .067$ | $2/15 = .133$ | $-1/48 = -.021$ |
| $\Delta x = 1/8$ | $1/7 = .142$ | $1/63 = .016$ | $2/63 = .032$ | $-1/448 = -.002$ |

**36** At $x = 0.1$ the difference $\frac{1}{.9} - (1.11) = .00111\cdots$ comes from the omitted terms $x^3 + x^4 + x^5 + \cdots$. At $x = 2$ the difference is $\frac{1}{1-2} - (1 + 2 + 4) = -8$. This is large because $x = 2$ is far from the basepoint $x = 0$.

**38** For $\cos x$ the approximation around $x = 0$ is $\cos 0 + (\sin 0)x + \frac{1}{2}(-\cos 0)x^2 = 1 - \frac{1}{2}x^2$. At $1° = \frac{\pi}{180}$ radians this is $1 - \frac{1}{2}(\frac{\pi}{180})^2 = .999847691$ : correct to 8 decimals. At 1 radian this is $1 - \frac{1}{2} = .5$. The correct $\cos 1 = .54$ is not so close because $x = 1$ is far from the basepoint $x = 0$.

**40** If $f(x)$ is even then $f'(0) = 0$ and $f(x) \approx f(0) + \frac{1}{2}f''(0)x^2$. The tangent line $y = f(0)$ is horizontal (linear term not present).

**42** $f(1) = 3, f(2) = 2 + 4 + 8 = 14, f(3) = 3 + 9 + 27 = 39$. The second difference is $\frac{39 - 28 + 3}{1^2} = 14$. The true $f'' = 2 + 6x$ is also 14. The error involves $f''''$ which in this example is zero.

**44** $y(x)$ is concave up if **the line segment between any two points on its graph stays above the graph.**

# 3.4 Graphs (page 119)

The position, slope, and bending of $y = f(x)$ are decided by $\mathbf{f(x), f'(x)}$, and $\mathbf{f''(x)}$. If $|f(x)| \to \infty$ as $x \to a$, the line $x = a$ is a vertical **asymptote**. If $f(x) \to b$ for large $x$, then $x = b$ is a **horizontal asymptote**. If

$f(x) - mx \to b$ for large $x$, then $y = mx + b$ is a **sloping asymptote.** The asymptotes of $y = x^2/(x^2-4)$ are $\mathbf{x = 2, x = -2, y = 1}$. This function is even because $y(-x) = \mathbf{y(x)}$. The function $\sin kx$ has period $\mathbf{2\pi/k}$.

Near a point where $dy/dx = 0$, the graph is extremely **flat**. For the model $y = Cx^2, x = .1$ gives $y = \mathbf{.01C}$. A box around the graph looks long and **thin**. We **zoom** in to that box for another digit of $x^*$. But solving $dy/dx = 0$ is more accurate, because its graph **crosses** the $x$ **axis**. The slope of $dy/dx$ is $\mathbf{d^2y/dx^2}$. Each derivative is like an **infinite** zoom.

To move $(a,b)$ to $(0,0)$, shift the variables to $X = \mathbf{x - a}$ and $Y = \mathbf{y - b}$. This **centering** transform changes $y = f(x)$ to $Y = \mathbf{-b + f(X + a)}$. The original slope at $(a,b)$ equals the new slope at $\mathbf{(0,0)}$. To stretch the axes by $c$ and $d$, set $\mathbf{x} = cX$ and $\mathbf{y} = dY$. The **zoom** transform changes $Y = F(X)$ to $\mathbf{y = dF(x/c)}$. Slopes are multiplied by $\mathbf{d/c}$. Second derivatives are multiplied by $\mathbf{d/c^2}$.

**1** $120; 150; \frac{60}{x}$ **3** Odd; $x = 0, y = x$ **5** Even; $x = 1, x = -1, y = 0$ **7** Even; $y = 1$ **9** Even
**11** Even; $x = 1, x = -1, y = 0$ **13** $x = 0, x = -1, y = 0$ **15** $x = 1, y = 1$ **17** Odd **19** $\frac{2x}{x-1}$
**21** $x + \frac{1}{x-4}$ **23** $\sqrt{x^2+1}$ **25** Of the same degree **27** Have degree $P <$ degree $Q$; none
**29** $x = 1$ and $y = 3x + C$ if $f$ is a polynomial; but $f(x) = (x-1)^{1/3} + 3x$ has no asymptote $x = 1$
**31** $(x-3)^2$ **39** $x = \sqrt{2}, x = -\sqrt{2}, y = x$ **41** $Y = 100 \sin \frac{2\pi X}{360}$ **45** $c = 3, d = 10; c = 4, d = 20$
**47** $x^* = \sqrt{5} = 2.236$ **49** $y = x - 2; Y = X; y = 2x$ **51** $x_{\max} = .281, x_{\min} = 6.339; x_{\text{infl}} = 4.724$
**53** $x_{\min} = .393, x_{\max} = 1.53, x_{\min} = 3.33; x_{\text{infl}} = .896, 2.604$
**55** $x_{\min} = -.7398, x_{\max} = .8135; x_{\text{infl}} = .04738; x_{\text{blowup}} = \pm 2.38$ **57** 8 digits

**2** (a) 30 dark lines in 6 seconds (b) 8 beats in 6 seconds is 80 beats/minute; (c) Rule: Heart rate = cycles per 6-second interval times 10.
**4** $x^n$ is odd or even according as $n$ is odd or even. Horizontal asymptote $\mathbf{y = 0}$ if $n = 0$, a sloping asymptote $\mathbf{y = x}$ if $n = 1$, and a vertical asymptote $\mathbf{x = 0}$ if $n$ is negative.
**6** $\frac{x^3}{4-x^2}$ is odd: vertical asymptotes $\mathbf{x = 2}$ and $\mathbf{x = -2}$, sloping asymptote $\mathbf{y = -x}$ for large $x$ (ignore 4).
**8** $\frac{x^2+3}{x+1}$ is not odd or even. Vertical asymptote $\mathbf{x = -1}$ and a sloping asymptote $\mathbf{y = x}$ (ignore 3 and 1).
**10** Periodic and even: Maximum $f(0) = 3$, two local maxima and minima before $f(\pi) = -3$.
**12** $\frac{x}{\sin x}$ is even. Vertical asymptotes at all multiples $\mathbf{x = n\pi}$, except at $x = 0$ where $f(0) = 1$.
**14** $\frac{1}{x-1} - 2x$ is not odd or even. Vertical asymptote $x = 1$, sloping asymptote $y = -2x$.
**16** $\frac{\sin x + \cos x}{\sin x - \cos x}$ is periodic, not odd or even, vertical asymptotes when $\sin x = \cos x$ at $x = \frac{\pi}{4} + n\pi$.
**18** $\frac{1}{x} - \sqrt{x}$ is defined for $x \geq 0$ and has vertical asymptote at $x = 0$. **20** $f(x) = \frac{1}{(x-1)(x-2)}$
**22** $f(x) = \frac{1}{x} + 2x + 3$ **24** $f(x) = \frac{x^3}{(x-1)(x-3)}$ **26** For a sloping asymptote, degree of $P = 1 +$ (degree of $Q$).
**28** $\frac{1}{(x-1)^2}$ goes to $+\infty$ on both sides of the asymptote $x = 1$, but $\frac{1}{x-1}$ goes to $-\infty$ on the left side $(x < 1)$.
**30** (a) False: $\frac{x^4}{x^2+1}$ has no asymptotes (b) True: the second difference on page 108 is even (c) False: $f(x) = 1 + x$ is not even but $f'' = 0$ is even (d) False: $\frac{1}{x^2(1-x)^2}$ never touches zero.
**32** $g(x) = x - \pi$ or $(x-\pi)^3$ or $\cdots$ **34** $x^{1/x}$ is near zero for small $x$, equals 1 at $x = 1$ and $\sqrt{2}$ at $x = 2$.
**36** The asymptotes are $x = -2$ and $y = -1$. **38** This is the second difference $\frac{\Delta^2 f}{\Delta x^2} \approx (\sin x)'' = -\sin x$.
**40** (a) The asymptotes are $y = 0$ and $x = -3.48$. (b) The asymptotes are $y = 1$ and $x = 1$ (double root).
**42** The exact solution is $x^* = \sqrt{3} = 1.73205$. The zoom should find those digits.
**46** The slope is zero near $x = 1.5$. This is the first step in Newton's method. The second step uses $y'$ and $y''$ at $x = 1.5$ to come very near $x^* = \sqrt{3}$.
**48** The exact $x = \sqrt{5}$ solves $\frac{x}{30\sqrt{15+x^2}} = \frac{1}{60}$ or $(2x)^2 = 15 + x^2$ or $x^2 = 5$.
**50** The odd functions $f_n(x)$ approximate a square wave of height $1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}$. (Substitute $x = \frac{\pi}{2}$.)

They overshoot the jump from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$ at $x = 0$. The overshoot moves closer and closer to $x = 0$ but does not become smaller (Gibbs phenomenon).

**52** $\sqrt{3x+1}$ is defined only for $x \geq -\frac{1}{3}$ (local maximum); minimum near $x = .95$; no inflection point.

**54** Maxima at $x = .295, 1.72, 4.09$; minima at $x = 1.39, 2.84, 5.34$; inflection points at $.78, 1.6, 2.3, 3.5, 4.7, 5.9$.

**56** $x \sin \frac{1}{x}$ has derivative $\sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$. Minima and maxima alternate at $x = .22, .13, .09, .07, \cdots$ (approaching zero); inflection points $.16, .11, .08, \cdots$.

**58** Inflection points and second derivatives are **harder** to compute than maximum points and first derivatives. (In examples, derivatives seem easier than integrals. For numerical computation **it is the other way:** derivatives are very sensitive, integrals are smooth.)

## 3.5 Parabolas, Ellipses, and Hyperbolas (page 128)

The graph of $y = x^2 + 2x + 5$ is a **parabola.** Its lowest point (the vertex) is $(x, y) = (\mathbf{-1, 4})$. Centering by $X = x + 1$ and $Y = \mathbf{y - 4}$ moves the vertex to (0,0). The equation becomes $Y = \mathbf{X^2}$. The focus of this centered parabola is $(\mathbf{0, \frac{1}{4}})$. All rays coming straight down are **reflected** to the focus.

The graph of $x^2 + 4y^2 = 16$ is an **ellipse.** Dividing by **16** leaves $x^2/a^2 + y^2/b^2 = 1$ with $a = \mathbf{4}$ and $b = \mathbf{2}$. The graph lies in the rectangle whose sides are $\mathbf{x = \pm 4, y = \pm 2}$. The area is $\pi ab = \mathbf{8\pi}$. The foci are at $x = \pm c = \pm\sqrt{\mathbf{12}}$. The sum of distances from the foci to a point on the ellipse is always **8**. If we rescale to $X = x/4$ and $Y = y/2$ the equation becomes $\mathbf{X^2 + Y^2 = 1}$ and the graph becomes a **circle.**

The graph of $y^2 - x^2 = 9$ is a **hyperbola.** Dividing by 9 leaves $y^2/a^2 - x^2/b^2 = 1$ with $a = \mathbf{3}$ and $b = \mathbf{3}$. On the upper branch $y \geq \mathbf{0}$. The asymptotes are the lines $\mathbf{y = \pm x}$. The foci are at $y = \pm c = \pm\sqrt{\mathbf{18}}$. The **difference** of distances from the foci to a point on this hyperbola is **6.**

All these curves are conic sections – the intersection of a **plane** and a **cone.** A steep cutting angle yields a **hyperbola.** At the borderline angle we get a **parabola.** The general equation is $Ax^2 + \mathbf{Bxy + Cy^2 + Dx + Ey + F} = 0$. If $D = E = 0$ the center of the graph is at **(0,0)**. The equation $Ax^2 + Bxy + Cy^2 = 1$ gives an ellipse when $\mathbf{4AC > B^2}$. The graph of $4x^2 + 5xy + 6y^2 = 1$ is an **ellipse.**

**1** $dy/dx = 0$ at $\frac{-b}{2a}$ **3** $V = (1, -4), F = (1, -3.75)$ **5** $V = (0,0), F = (0, -1)$ **7** $F = (1,1)$

**9** $V = (0, \pm 3); F = (0, \pm\sqrt{8})$ **11** $V = (0, \pm 1); F = (0, \pm\sqrt{\frac{5}{4}})$ **13** Two lines, $a = b = c = 0; V = F = (0,0)$

**15** $y = 5x^2 - 4x$ **17** $y + p = \sqrt{x^2 + (y-p)^2} \to 4py = x^2; F = (0, \frac{1}{12}), y = -\frac{1}{12}; (\pm\frac{\sqrt{11}}{6}, \frac{11}{12})$

**19** $x = ay^2$ with $a > 0; y = \frac{(x+p)^2}{4p}; y = -ax^2 + ax$ with $a > 0$

**21** $\frac{x^2}{4} + y^2 = 1; \frac{(x-1)^2}{4} + (y-1)^2 = 1$ **23** $\frac{x^2}{25} + \frac{y^2}{9} = 1; \frac{(x-3)^2}{36} + \frac{(y-1)^2}{32} = 1; x^2 + y^2 = 25$

**25** Circle, hyperbola, ellipse, parabola **27** $\frac{dy}{dx} = -\frac{4}{5}; y = -\frac{4}{5}x + 5$ **29** $\frac{5}{4}; \frac{9}{40} = \frac{1}{2}(\frac{5}{4} - \frac{4}{5})$

**31** Circle; $(3,1); 2; X = \frac{x-3}{2}, Y = \frac{y-1}{2}$ **33** $3x'^2 + y'^2 = 2$ **35** $y^2 - \frac{1}{3}x^2 = 1; \frac{y^2}{9} - \frac{4x^2}{9} = 1; y^2 - x^2 = 5$

**37** $\frac{x^2}{25} - \frac{y^2}{39} = 1$ **39** $y^2 - 4y + 4, 2x^2 + 12x + 18; -14, (-3, 2)$, right-left

**41** $F = (\pm\frac{\sqrt{5}}{2}, 0); y = \pm\frac{x}{2}$ **43** $(x + y + 1)^2 = 0$

**45** $(a^2 - 1)x^2 + 2abxy + (b^2 - 1)y^2 + 2acx + 2bcy + c^2 = 0; 4(a^2 + b^2 - 1)$; if $a^2 + b^2 < 1$ then $B^2 - 4AC < 0$

**2** $y = 3x^2 - 12x$ has $\frac{dy}{dx} = 6x - 12$ and $\mathbf{x_{\min} = 2}$. At this minimum, $3x^2$ is **half** as large as $12x$. Then the shift $X = x - 2$ and $Y = y + 12$ centers the equation to $Y = 3X^2$.

**4** $y = (x-1)^2$ has vertex at $(\mathbf{1,0})$ and focus above it at $(1, \frac{1}{4})$. Note $a = 1$.

**6** $4x = y^2$ has vertex at (0,0) and opens to the right. The focus is at $(\frac{1}{16}, 0)$. (Note the coefficient of $y^2$ in $x = \frac{1}{4}y^2$.)

**8** $x^2 + 9y^2 = 9$ is the ellipse $\frac{x^2}{3^2} + \frac{y^2}{1^2} = 1$ with vertices $(\pm 3, 0)$ on the major axis and foci $(\pm\sqrt{8}, 0)$.

**10** $\frac{x^2}{2^2} - \frac{(y-1)^2}{1^2} = 1$ is a hyperbola centered at (0,1). It opens right and left with vertices at $(\pm 2, 1)$ and foci at $(\pm\sqrt{5}, 1)$.

**12** $(y-1)^2 - \frac{x^2}{(1/2)^2} = 1$ is a hyperbola centered at (0,1): $a = 1$ and $b = \frac{1}{2}$. It opens up and down with vertices at (0,2) and (0,0). The foci are $(0, 1 \pm \sqrt{1 + \frac{1}{4}})$.

**14** $xy = 0$ gives the two lines $x = 0$ and $y = 0$, a degenerate hyperbola with vertices and foci all at (0,0).

**16** $y = x^2 - x$ has vertex at $(\frac{1}{2}, -\frac{1}{4})$. To move the vertex to (0,0) set $X = x - \frac{1}{2}$ and $Y = y + \frac{1}{4}$. Then $Y = X^2$.

**18** The parabola $y = 9 - x^2$ opens down with vertex at (0,9).

**20** The path $x = t,\ y = t - t^2$ starts with $\frac{dx}{dt} = \frac{dy}{dt} = 1$ at $t = 0$ (45° angle). Then $y_{\max} = \frac{1}{4}$ at $t = \frac{1}{2}$. The path is the parabola $y = x - x^2$.

**22** When $x^2 = a^2 - b^2$ the equation gives $\frac{a^2-b^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{y^2}{b^2} = \frac{b^2}{a^2}$ or $\mathbf{y} = \frac{\mathbf{b^2}}{\mathbf{a}}$. This is the height above the focus.

**24** Squaring $\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$ yields $x^2 - 2cx + c^2 + y^2 = 4a^2 + x^2 + 2cx + c^2 + y^2$ $-4a\sqrt{(x+c)^2 + y^2}$. This is $-4cx - 4a^2 = -4a\sqrt{(x+c)^2 + y^2}$. Divide by $-4$ and square again: $c^2x^2 + 2a^2cx + a^4 = a^2(x^2 + 2cx + c^2 + y^2)$. Set $c^2 = a^2 - b^2$ and cancel $a^2x^2 + 2a^2cx + a^4$ to leave $-b^2x^2 = -a^2b^2 + a^2y^2$. Divide by $a^2b^2$ to find $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (the ellipse!).

**26** At $z = 0$ the equation becomes $(x - 2y)^2 + (y - 2x)^2 = 1$ or $\mathbf{5x^2 - 8xy + 5y^2 = 1}$. Then $B^2 = \mathbf{64} < 4AC = \mathbf{100}$ (ellipse).

**28** (a) The line touches the circle at $(x_0, y_0)$ because $x_0^2 + y_0^2 = r^2$. It is tangent because the slope $\frac{dy}{dx} = -\frac{\mathbf{x_0}}{\mathbf{y_0}}$ is perpendicular to the slope $\frac{y_0}{x_0}$ of the radius (product of slopes is 1). (b) The derivative of the ellipse equation is $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{\mathbf{x_0\,b^2}}{\mathbf{y_0\,a^2}}$ = slope of the line. (This is **implicit differentiation.**)

**30** $PF_1 = PR$ and $QF_1 = QR$ (because $F_1$ and $R$ are mirror images) so step 2 is the same as step 1. Since step 2 holds for every $Q$ on the tangent line, $P$ must be on the straight line from $F_2$ to $R$. The intersection angles of these lines are $\alpha = \beta$.

**32** The square has side $s$ if the point $(\frac{s}{2}, \frac{s}{2})$ is on the ellipse. This requires $\frac{1}{a^2}(\frac{s}{2})^2 + \frac{1}{b^2}(\frac{s}{2})^2 = 1$ or $s^2 = 4(\frac{1}{a^2} + \frac{1}{b^2})^{-1}$ = area of square.

**34** The Earth has $a = 149{,}597{,}870$ kilometers (Problem 19 on page 469 says $1.5 \cdot 10^8$ km). The eccentricity $e = \frac{c}{a}$ is 0.167 (or .02 on page 356). Then $c = 2.5 \cdot 10^6$ and $b = \sqrt{a^2 - c^2}$. This $b$ is very near $a$; our orbit is nearly a circle. Use $\sqrt{a^2 - c^2} \approx a - \frac{c^2}{2a} \approx a - 2 \cdot 10^4$ km.

**36** The derivative of $y^2 - x^2 = 1$ is $2y\frac{dy}{dx} - 2x = 0$ (again implicit). Then $\frac{dy}{dx} = \frac{\mathbf{x_0}}{\mathbf{y_0}}$ at the point $(x_0, y_0)$ which agrees with the slope of the line. The line goes through the point because $y_0^2 - x_0^2 = 1$.

**38** The cannon was on a **hyperbola** with foci at **Napoleon** and the **Duke of Wellington**. The hyperbola has $2a$ = distance traveled by sound in 1 second.

**40** Complete squares: $y^2 + 2y = (y+1)^2 - 1$ and $x^2 + 10x = (x+5)^2 - 25$. Then $Y = y + 1$ and $X = x + 5$ satisfy $Y^2 - 1 = X^2 - 25$ : the hyperbola is $\mathbf{X^2 - Y^2 = 24}$.

**42** The graph is empty if $A, C, F$ have the **same sign.**

**44** Given any **five** points in the plane, a second-degree curve goes through those points.

**46** The quadratic $ax^2 + bx + c$ has two real roots if $b^2 - 4ac$ is positive and no real roots if $b^2 - 4ac$

is negative. Equal roots if $b^2 = 4ac$.

## 3.6 Iterations $x_{n+1} = F(x_n)$ (page 136)

$x_{n+1} = x_n^3$ describes an **iteration**. After one step $x_1 = \mathbf{x_0^3}$. After two steps $x_2 = F(x_1) = \mathbf{x_1^3} = \mathbf{x_0^9}$. If it happens that input = output, or $x^* = \mathbf{F}(\mathbf{x}^*)$, then $x^*$ is a **fixed** point. $F = x^3$ has **three** fixed points, at $x^* = \mathbf{0}, \mathbf{1}$ **and** $-\mathbf{1}$. Starting near a fixed point, the $x_n$ will converge to it if $|\mathbf{F'(x^*)}| < 1$. That is because $x_{n+1} - x^* = F(x_n) - F(x^*) \approx \mathbf{F'(x^*)(x_n - x^*)}$. The point is called **attracting.** The $x_n$ are repelled if $|\mathbf{F'(x^*)}| > \mathbf{1}$. For $F = x^3$ the fixed points have $F' = \mathbf{0}$ **or** $\mathbf{3}$. The cobweb goes from $(x_0, x_0)$ to $(x_0, x_1)$ to $(\mathbf{x_1}, \mathbf{x_1})$ and converges to $(x^*, x^*) = (\mathbf{0}, \mathbf{0})$. This is an intersection of $y = x^3$ and $y = \mathbf{x}$, and it is super-attracting because $\mathbf{F'} = \mathbf{0}$.

$f(x) = 0$ can be solved iteratively by $x_{n+1} = x_n - cf(x_n)$, in which case $F'(x^*) = \mathbf{1 - cf'(x^*)}$. Subtracting $x^* = x^* - cf(x^*)$, the error equation is $x_{n+1} - x^* \approx m(\mathbf{x_n - x^*})$. The multiplier is $m = \mathbf{1 - cf'(x^*)}$. The errors approach zero if $-\mathbf{1} < \mathbf{m} < \mathbf{1}$. The choice $c_n = \mathbf{1/f'(x^*)}$ produces Newton's method. The choice $c = 1$ is "successive **substitution**" and $c = \mathbf{1/f'(x_0)}$ is modified Newton. Convergence to $x^*$ is **not** certain.

We have three ways to study iterations $x_{n+1} = F(x_n)$ : (1) compute $x_1, x_2, \cdots$ from different $x_0$ (2) find the fixed points $x^*$ and test $|dF/dx| < 1$ (3) draw cobwebs.

**1** $-.366; \infty$ **3** $1; 1$ **5** $\frac{2}{3}; \pm\infty$ **7** $-2; -2$

**9** $\frac{1-\sqrt{3}}{2}$ attracts, $\frac{1+\sqrt{3}}{2}$ repels; $\frac{1}{2}$ attracts, 0 repels; 1 attracts, 0 repels; 1 attracts; $\frac{2}{3}$ attracts, 0 repels; $\pm\sqrt{2}$ repel

**11** Negative **13** .900 **15** .679 **17** $|a| < 1$ **19** Unstable $|F'| > 1$ **21** $x^* = \frac{s}{1-a}; |a| < 1$

**23** \$2000; \$2000 **25** $x_0, b/x_0, x_0, b/x_0, \cdots$ **27** $F' = -\frac{\sqrt{2}}{2}x^{-3/2} = -\frac{1}{2}$ at $x^*$

**29** $F' = 1 - 2cx = 1 - 4c$ at $x^* = 2; 0 < c < \frac{1}{2}$ succeeds

**31** $F' = 1 - 9c(x-2)^8 = 1 - 9c$ at $x^* = 3; 0 < c < \frac{2}{9}$ succeeds

**33** $x_{n+1} = x_n - \frac{x_n^3 - 2}{3x_n^2}; x_{n+1} = x_n - \frac{\sin x_n - \frac{1}{2}}{\cos x_n}$ **35** $x^* = 4$ if $x_0 > 2.5; x^* = 1$ if $x_0 < 2.5$

**37** $m = 1 + c$ at $x^* = 0, m = 1 - c$ at $x^* = 1$ (converges if $0 < c < 2$) **39** 0 **43** $F' = 1$ at $x^* = 0$

**2** $x_{n+1} = 2x_n(1 - x_n) : x_0 = .6, x_1 = .48, x_2 = .4992, \cdots$ approaches $x^* = .5$ and $x_0 = 2, x_1 = -4, x_2 = 8, \cdots$ approaches infinity.

**4** $x_{n+1} = x_n^{-1/2} : x_0 = .6, x_1 = 1.29, x_2 = .88, \cdots$ and $x_0 = 2, x_1 = .707, x_2 = 1.19, \cdots$ both approach $x^* = 1$.

**6** $x_{n+1} = x_n^2 + x_n - 2 : x_0 = .6, x_1 = -1.04, x_2 = -1.9584, \cdots$ and $x_0 = 2, x_1 = 4, x_2 = 18, \cdots$ both approach $+\infty$!

**8** $x_0 = .6, x_1 = .6 \cdots$ approaches $x^* = .6$ and $x_0 = 2, x_1 = 2, \cdots$ approaches $x^* = 2$. Every non-negative number is a fixed point $x^*$.

**10** $x_0 = -1, x_1 = 1, x_2 = -1, x_3 = 1, \cdots$ The double step $x_{n+2} = x_n^9$ has fixed points $x^* = (x^*)^9$, which allows $x^* = 1$ and $x^* = -1$.

**12** $x_{n+1} = x_n^2 - 1 : x_0 = 0, x_1 = -1, x_2 = 0, x_3 = -1, \cdots$. For period 2 look at $x_{n+2} = (x_n^2 - 1)^2 - 1$ and solve $x^* = (x^* - 1)^2 - 1$ to find $x^* = 0, -1, \frac{1}{2} \pm \frac{\sqrt{5}}{2}$ (four period 2 starting points). The sequence $x_0 = .1, x_1 = -.99, -.0199, -.9996, \cdots$ is attracted to $0, -1, 0, -1, \cdots$ (for proof find zero derivative at $x = 0$).

**14** $x = \cos^2 x : x = \mathbf{.6417}$ **16** $x = 2x - 1 : x = 1$ but the iteration $x_{n+1} = 2x_n - 1$ blows up

**18** At $x^* = (a-1)/a$ the derivative $f' = a - 2ax$ equals $f'(x^*) = a - 2(a-1) = 2 - a$. Convergence if $|F'(x^*)| < 1$

or $\mathbf{1} < \mathbf{a} < \mathbf{3}$. (For completeness check $a = 1$ : convergence to zero. Also check $a = 3$ : with $x_0 = .66666$ my calculator gives back $x_2 = .66666$. Apparently period 2.)

**20** $x^* = (x^*)^2 - \frac{1}{2}$ gives $x^*_+ = \frac{1+\sqrt{3}}{2}$ and $x^*_- = \frac{1-\sqrt{3}}{2}$. At these fixed points $F' = 2x^*$ equals $1+\sqrt{3}$ (greater than 1 so $x^*_+$ repels) and $F'(x^*_-) = 1-\sqrt{3}(x^*_-$ attracts). Cobwebs show convergence to $x^*_-$ if $|x_0| < |x^*_+|$, convergence to $x^*_+$ if $|x_0| = |x^*_+|$, divergence to $\infty$ if $|x_0| > |x^*_+|$.

**22** $x_{n+1} = x_n + 4$ adds 4 each time and diverges; $x_{n+1} = -x_n + 4$ oscillates around 2. (Example: $x_0 = 1$, $x_1 = 3, x_2 = 1, \cdots$.) The linear term doesn't pull it in to 2 because $|F'| = 1$ exactly.

**24** The debt $x_n$ at year $n$ leads to debt $x_{n+1} = .95x_n + \$100$ billion. As $n \to \infty$ the steady state is $x^* = .95x^* + \$100$ billion or $.05x^* = \$100$ billion or $\mathbf{x^* = \$2}$ **trillion.** If $x_0 = \$1$ trillion then every $x_n$ equals \$2 trillion $-(.95)^n$ (\$1 trillion).

**26** The fixed points satisfy $x^* = (x^*)^2 + x^* - 3$ or $(x^*)^2 = 3$; thus $x^* = \sqrt{3}$ or $x^* = -\sqrt{3}$. The derivative $2x^* + 1$ equals $2\sqrt{3}+1$ or $-2\sqrt{3}+1$; both have $|F'| > 1$. The iterations blow up.

**28** (a) Start with $x_0 > 0$. Then $x_1 = \sin x_0$ is less than $x_0$. The sequence $x_0, \sin x_0, \sin(\sin x_0)\cdots$ decreases to zero (**convergence**: also if $x_0 < 0$.) On the other hand $x_1 = \tan x_0$ is larger than $x_0$. The sequence $x_0, \tan x_0, \tan(\tan x_0), \cdots$ is increasing (slowly repelled from 0). Since $(\tan x)' = \sec^2 x \geq 1$ there is no attractor (**divergence**). (b) $F''$ is $(\sin x)'' = -\sin x$ and $(\tan x)'' = 2\sec^2 x \tan x$.
Theory: When $F''$ changes from + to − as $x$ passes $x_0$, the curve stays closer to the axis than the 45° line (convergence). Otherwise divergence. See Problem 22 for $F'' = 0$.

**30** $f(x) = x^2 - 4x + 3$ equals zero at $x^* = 1$ where $f' = -2$; also $f(x) = 0$ at $x^* = 3$ where $f' = 2$. The iteration $x_{n+1} = x_n - cf(x_n)$ has $F' = 1 + 2c$ at $x^* = 1$ and $F' = 1 - 2c$ at $x^* = 3$. For $-1 < c < 0$ it converges to $x^* = 1$; for $0 < c < 1$ it converges to $x^* = 3$; if $|c| > 1$ it diverges because $|F'| > 1$ at both fixed points.

**32** $f(x) = \frac{1}{1-x} - 3$ equals zero when $1 - x = \frac{1}{3}$ or $x^* = \frac{2}{3}$; at that point $f' = \frac{1}{(1-x)^2} = 9$. The iteration $x_{n+1} = x_n - cf(x_n)$ has $F' = 1 - 9c$ at $x^*$. For $0 < c < \frac{2}{9}$ it converges because then $|F'(x^*)| < 1$.

**34** Newton's method for $f(x) = x^3 - 2 = 0$ is $x_{n+1} = x_n - \frac{1}{3x_n^2}(x_n^3 - 2)$; convergence to $x^* = 2^{1/3} = 1.259921$. Newton's method for $f(x) = \sin x - \frac{1}{2}$ is $x_{n+1} = x_n - \frac{1}{\cos x_n}(\sin x_n - \frac{1}{2})$; convergence (from nearby $x_0$) to $x^* = \frac{\pi}{6} = .523598$.

**36** Newton's method for $f(x) = x^2 - 1 = 0$ is $x_{n+1} = x_n - \frac{1}{2x_n}(x_n^2 - 1) = \mathbf{\frac{x_n}{2} + \frac{1}{2x_n}}$. If $x_0 = 10^6$ then $x_1 = \frac{1}{2}10^6 + \frac{1}{2}10^{-6}$. The distance from $x^* = 1$ was $10^6 - 1$; it is cut approximately in half. But if $x_0$ is close to 1 the multiplier is near zero: $x_0 = 1.05$ gives $x_1 = \frac{1.05}{2} + \frac{1}{2.1} \approx 1.001$.

**38** The iterations from $x_0 = 1$ are $x_{n+1} = x_n - (x_n^2 - \frac{1}{2})$ and $x_{n+1} = x_n - \frac{1}{2}(x_n^2 - \frac{1}{2})$ and $x_{n+1} = x_n - \frac{1}{2x_n}(x_n^2 - \frac{1}{2})$. After one step they give $x_1 = \frac{1}{2}$ and $x_1 = \frac{3}{4}$ and $x_1 = \frac{3}{4}$. After two steps $x_1 = \frac{3}{4} = \mathbf{.750}$ and $x_1 = \frac{23}{32} = \mathbf{.719}$ and $x_1 = \frac{17}{24} = \mathbf{.708}$ (with $x^* = .707$).

**40** The roots of $x^2 + 2 = 0$ are imaginary, and Newton's method $x_{n+1} = x_n - \frac{1}{2x_n}(x_n^2 + 2)$ stays real: convergence is impossible. However the $x_n$ do not approach infinity (if $x_n$ is very large, then $x_{n+1}$ is only half as large). Section 3.7 shows how the $x_n$ jump around chaotically.

**42** The graphs of $\cos x, \cos(\cos x), \cos(\cos(\cos x))$ are approaching the **horizontal line** $\mathbf{y = .7391}\cdots$ (where $x^* = \cos x^*$). For every $x$ this number is the limit.

## 3.7 Newton's Method and Chaos (page 145)

When $f(x) = 0$ is linearized to $f(x_n) + f'(x_n)(x - x_n) = 0$, the solution $x = \mathbf{x_n - f(x_n)/f'(x_n)}$ is Newton's $x_{n+1}$. The **tangent line** to the curve crosses the axis at $x_{n+1}$, while the **curve** crosses at $x^*$. The errors at $x_n$ and $x_{n+1}$ are normally related by $(\text{error})_{n+1} \approx M(\textbf{error})^{\mathbf{2}}_{\mathbf{n}}$. This is **quadratic** convergence. The number of correct decimals **doubles** at every step.

For $f(x) = x^2 - b$, Newton's iteration is $x_{n+1} = \mathbf{\frac{1}{2}(b + \frac{x_n}{b})}$. The $x_n$ converge to $\sqrt{\mathbf{b}}$ if $x_0 > 0$ and to $-\sqrt{\mathbf{b}}$ if $x_0 < 0$. For $f(x) = x^2 + 1$, the iteration becomes $x_{n+1} = \mathbf{\frac{1}{2}(x_n - x_n^{-1})}$. This cannot converge to $\mathbf{i} = \sqrt{-1}$. Instead it leads to chaos. Changing to $z = 1/(x^2+1)$ yields the parabolic iteration $z_{n+1} = \mathbf{4z_n - 4z_n^2}$.

For $a \le 3, z_{n+1} = az_n - az_n^2$ converges to a single **fixed point.** After $a = 3$ the limit is a 2-cycle, which means **that the z's alternate between two values.** Later the limit is a Cantor set, which is a one-dimensional example of a **fractal.** The Cantor set is self-**similar.**

**1** $x_{n+1} = x_n - \frac{x_n^3 - b}{3x_n^2} = \frac{2x_n}{3} + \frac{b}{3x_n^2}$ **5** $x_1 = x_0; x_1$ is not defined $(\infty)$ **7** $x^* = 2$; blows up; $x^* = 2$ if $x_0 < 3$
**11** $x_0 < \frac{1}{2}$ to $x^* = 0; x_0 > \frac{1}{2}$ to $x^* = 1$ **21** $x_{n+1} = x_n - \frac{x_n^k - 7}{kx_n^{k-1}}$ **23** $x_4 = \cot\pi = \infty; x_3 = \cot\frac{8\pi}{7} = \cot\frac{\pi}{7}$
**25** $\pi$ is not a fraction **27** $= \frac{1}{4}x_n^2 + \frac{1}{2} + \frac{1}{4x_n^2} = \frac{(x_n^2+1)^2}{4x_n^2} = \frac{y_n^2}{4(y_n - 1)}$ **29** $16z - 80z^2 + 128z^3 - 64z^4; 4; 2$
**31** $|x_0| < 1$ **33** $\Delta x = 1$, one-step convergence for quadratics **35** $\frac{\Delta f}{\Delta x} = \frac{5.25}{1.5}; x_2 = 1.86$
**37** $1.75 < x^* < 2.5; 1.75 < x^* < 2.125$ **39** $8; 3 < x^* < 4$ **41** Increases by 1; doubles for Newton
**45** $x_1 = x_0 + \cot x_0 = x_0 + \pi$ gives $x_2 = x_1 + \cot x_1 = x_1 + \pi$ **49** $a = 2, Y$'s approach $\frac{1}{2}$

**2** $f(x) = \frac{x-1}{x+1}$ has $f'(x) = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$ so Newton's formula is $x_{n+1} = x_n - \frac{(x_n+1)^2}{2}\frac{x_n-1}{x_n+1} = x_n - \frac{x_n^2-1}{2}$. The fixed points of this $F$ satisfy $x^* = x^* - \frac{(x^*)^2-1}{2}$ which gives $x^* = 1$ and $x^* = -1$. The derivatives $F' = 1 - x^*$ are 0 and $-2$. So the sequence approaches $x^* = 1$, the correct zero of $f(x)$.

**4** $f(x) = x^{1/3}$ has $f'(x) = \frac{1}{3x^{2/3}}$ so Newton's formula is $x_{n+1} = x_n - 3x_n^{2/3}(x_n^{1/3}) = -2x_n$. The graph of $x^{1/3}$ is vertical at $x = 0$; the tangent line at any $x$ hits the axis at $-2x$.

**6** $f(x) = x^3 - 3x - 1 = 0$: roots near $1.9, -.5, -1.6$

**8** For any $x_0$, the new $x_1$ is on the right side of the root. Then $x_2, x_3, \cdots$ approach steadily from the right.

**10** Newton's method for $f(x) = x^4 - 100$ approaches $x^* = \sqrt{10}$ if $x_0 > 0$ and $x^* = -\sqrt{10}$ if $x_0 < 0$. In this case the error at step $n+1$ equals $\frac{3}{2x^*}$ times (error at step $n$)$^2$. In Problem 9 the multiplier is $\frac{1}{2x^*}$ and convergence is quicker. Note to instructors: The multiplier is $\frac{f''(x^*)}{2f'(x^*)}$ (this is $\frac{1}{2}F''(x^*)$ : see Problem 31 of Section 3.8).

**12** $x^3 - x = 0$ gives $\mathbf{x^* = 1, 0,}$ **and** $\mathbf{-1}$. Newton's method has $x_1 = x_0 - \frac{x_0^3 - x_0}{3x_0^2 - 1} = \frac{2x_0^3}{3x_0^2-1}$. This equals $-x_0$ (producing a cycle) if $x_0 = \pm\sqrt{.2}$. Between these limits we have $|x_1| < |x_0|$ and Newton converges to $x^* = 0$. Between $|x_0| = \sqrt{.2}$ and $|x_0| = \sqrt{1/3}$ the convergence to 1 or $-1$ looks complicated. For $x_0 > \sqrt{1/3}$ there is convergence to $x^* = 1$.

**14** Between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ the graph of $-\tan x$ decreases from $\infty$ to $-\infty$. It crosses the 45° line once (at $x = 0$). In each successive interval of length $\pi$, the same is true: one solution to $x = -\tan x$ in each interval. Roots $\mathbf{x^* = 0}$ and $x^* = 2.03$.

**16** Roots at $x^* = -1.3$ and $x^* = .526$.

**18** (a) From $1 - 2x_{n+1} = (1 - 2x_n)^2 = 1 - 4x_n + 4x_n^2$, cancel the 1's and divide by $-2$. Then $x_{n+1} = 2x_n - 2x_n^2$.
(b) Every step squares $1 - 2x_n$ to find the next $1 - 2x_{n+1}$. So if $|1 - 2x_0| > 1$, repeated squaring blows up. If $|1 - 2x_0| < 1$, then repeated squaring gives $1 - 2x_n \to 0$. Here $|1 - 2x_0| < 1$ puts $2x_0$ between 0 and 2.

**20** Multiply $x_{n+1} = 2x_n - ax_n^2$ by $a$ and subtract from 1 to find $1 - ax_{n+1} = 1 - 2ax_n + a^2x_n^2 = (1 - ax_n)^2$. At each step $1 - ax_n$ is *squared* to find the next $1 - ax_{n+1}$. Then $1 - ax_n \to 0$ (or $x_n \to \frac{1}{a}$) if $|1 - ax_0| < 1$. This puts $1 - ax_0$ between $-1$ and 1: then $0 < x_0 < 2/a$.

**22** Roots at $x^* = -2.11485$ and $x^* = .25410$ and $x^* = 1.86081$.

**24** $\theta = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{16\pi}{9}, \frac{32\pi}{9}, \frac{64\pi}{9} = 7\pi + \frac{\pi}{9}$; this happened at step 6 so $x_6 = x_0$.

**26** If $z_0 = \sin^2\theta$ then $z_1 = 4z_0 - 4z_0^2 = 4\sin^2\theta - 4\sin^4\theta = 4\sin^2\theta(1-\sin^2\theta) = 4\sin^2\theta\cos^2\theta = (2\sin\theta\cos\theta)^2 = \sin^2 2\theta$.

**28** $\frac{1}{y_{n+1}} = \frac{4(y_n-1)}{y_n^2} = \frac{4}{y_n} - \frac{4}{y_n^2}$. With $z = \frac{1}{y}$ this is $z_{n+1} = 4z_n - 4z_n^2$.

**30** Newton's method is $x_{n+1} = x_n - \frac{3x_n^2 - .64}{x_n^3 - .64x_n - .36}$. TO DO

**32** The function $f(x) = x^2$ has a *double root* at $x^* = 0$. Newton's iteration is $x_{n+1} = x_n - \frac{x_n^2}{2x_n} = \frac{x_n}{2}$. Each step multiplies by $\frac{1}{2}$ : there is convergence to $x^* = 0$ but it is only *linear*. The error is not squared.

**34** Halley's method to solve $f(x) = x^2 - 1 = 0$ is $(x_n^2 - 1) + \Delta x(2x_n) + \frac{\Delta x}{2}\frac{1-x_n^2}{2x_n}(2) = 0$ or $\Delta x = \frac{1-x_n^2}{2x_n + \frac{1-x_n^2}{2x_n}}$. (This is Newton's method with an extra term in the denominator.) Substitute $x_0 = 1+\epsilon$ to find $\Delta x = \frac{-2\epsilon-\epsilon^2}{2+2\epsilon+\frac{-2\epsilon-\epsilon^2}{2+2\epsilon}}$. After some calculation $x_1 = x_0 + \Delta x = 1 + \epsilon + \Delta x$ is $1 + O(\epsilon^3)$.

**36** The secant line connecting $x_0 = 1, f(x_0) = -3$ to the next point $x_2 = 2.5, f(x_2) = 2.25$ has slope $\frac{\Delta f}{\Delta x} = \frac{5.25}{1.5}$. The line with this slope is $y + 3 = \frac{5.25}{1.5}(x-1)$. It crosses $y = 0$ at the point $x_2 = 1 + 3\frac{1.5}{5.25} = 1.857$.

**40** Root at $x^* = .29$ (and very flat nearby)

**42** $\frac{3}{4} = \frac{2}{3} + \frac{2}{27} + \frac{2}{9(27)} + \cdots$ (This is $.2020\cdots$ not in decimals but when the base is changed to 3.) The Cantor set removes the interval $(\frac{1}{3}, \frac{2}{3})$ where the first digit is 1; then $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ where the second digit is 1; eventually all numbers containing a "1" in the expansion to base 3 are removed. The number $\frac{3}{4} = .202020\cdots$ is not removed – it remains in the Cantor set.

**44** A Newton step goes from $x_0 = .308$ to $x_1 = x_0 + \frac{\cos x_0}{\sin x_0} = 3.45143$. Then $\frac{\Delta f}{\Delta x} = \frac{\cos x_1 - \cos x_0}{x_1 - x_0} = -.606129$ and a secant step leads to $x_2 = x_1 + \frac{\cos x_1}{.606129} = 1.88$.

**48** The graphs of $Y_1(Y_1(Y_1\cdots(x)))$ become squarer and squarer, going between heights .842 and .452. $Y_9$ is like $Y_8$ but "flipped" – because $Y_1(.842) = .452$ and $Y_1(.452) = .842$. These are fixed points of $Y_1(Y_1(x))$– draw its intersection with the 45° line $y = x$. Note that $Y_9$ is a polynomial of degree $2^9$. *Unusual graphs!*

## 3.8 The Mean Value Theorem and l'Hôpital's Rule (page 152)

The Mean Value Theorem equates the average slope $\Delta f/\Delta x$ over an **interval** $[a,b]$ to the slope $df/dx$ at an unknown **point**. The statement is $\mathbf{\Delta f/\Delta x = f'(x)}$**for some point a < c < b**. It requires $f(x)$ to be **continuous** on the **closed** interval $[a,b]$, with a **derivative** on the open interval $(a,b)$. Rolle's theorem is the special case when $f(a) = f(b) = 0$, and the point $c$ satisfies $\mathbf{f'(c) = 0}$. The proof chooses $c$ as the point where $f$ reaches its **maximum or minimum**.

Consequences of the Mean Value Theorem include: If $f'(x) = 0$ everywhere in an interval then $f(x) =$ **constant**. The prediction $f(x) = f(a) + \mathbf{f'(c)}(x-a)$ is exact for some $c$ between $a$ and $x$. The quadratic prediction $f(x) = f(a) + f'(x)(x-a) + \frac{1}{2}\mathbf{f''(c)}(x-a)^2$ is exact for another $c$. The error in $f(a) + f'(a)(x-a)$ is less than $\frac{1}{2}M(x-a)^2$ where $M$ is the maximum of $|\mathbf{f''}|$.

A chief consequence is l'Hôpital's Rule, which applies when $f(x)$ and $g(x) \to \mathbf{0}$ as $x \to a$. In that case the limit of $f(x)/g(x)$ equals the limit of $\mathbf{f'(x)/g'(x)}$, provided this limit exists. Normally this limit is $f'(a)/g'(a)$. If this is also 0/0, go on to the limit of $\mathbf{f''(x)/g''(x)}$.

**1** $c = \sqrt{\frac{4}{3}}$ **3** No $c$ **5** $c = 1$ **7** Corner at $\frac{1}{2}$ **9** Cusp at 0

**11** $\sec^2 x - \tan^2 x =$ constant **13** 6 **15** $-2$ **17** $-1$ **19** $n$ **21** $-\frac{1}{2}$ **23** Not $\frac{0}{0}$

**25** $-1$ **27** $1$; $\frac{1-\sin x}{1+\cos x}$ has no limit **29** $f'(c) = \frac{4^3-1^3}{4-1}; c = \sqrt{7}$

**31** $0 = x^* - x_{n+1} + \frac{f''(c)}{2f'(x_n)}(x^* - x_n)^2$ gives $M \approx \frac{f''(x^*)}{2f'(x^*)}$ **33** $f'(0)$; $\frac{f'(x)}{1}$; singularity **35** $\frac{f(x)}{g(x)} \to \frac{3}{4}$ **37** 1

**2** $\sin 2\pi - \sin 0 = (\pi \cos \pi c)(2-0)$ when $\cos \pi c = 0$ : then $\mathbf{c = \frac{1}{2}}$ or $\mathbf{c = \frac{3}{2}}$.

**4** $(1+2+4)-(1+0+0) = (1+2c)(2-0)$ when $6 = 2(1+2c) =$ or $\mathbf{c = 1}$. (For parabolas $c$ is always halfway between $a$ and $b$).

**6** $(2-1)^9 - (0-1)^9 = 9(c-1)^8(2-0)$ gives $9(c-1)^8 = 1$ : then $\mathbf{c = 1 + (\frac{1}{9})^{1/8}}$ or $\mathbf{c = 1 - (\frac{1}{9})^{1/8}}$.

**8** $f(x) =$ step function has $f(1) = 1$ and $f(-1) = 0$. Then $\frac{f(1)-f(-1)}{1-(-1)} = \frac{1}{2}$ but no point $c$ has $f'(c) = \frac{1}{2}$. MVT does not apply because $f$ is **not continuous** in this interval.

**10** $f(x) = \frac{1}{x^2}$ has $f(1) = 1$ and $f(-1) = 1$, but no point $c$ has $f'(c) = 0$. MVT does not apply because $f(x)$ is **not continuous** in this interval.

**12** $\frac{d}{dx}\csc^2 x = 2\csc x(-\csc x \cot x)$ is equal to $\frac{d}{dx}\cot^2 x = 2\cot x(-\csc^2 x)$. Then $f(x) = \csc^2 x - \cot^2 x$ has $f' = 0$ at every point $c$. By the MVT $f(x)$ must have the same value at every pair of points $a$ and $b$. By trigonometry $\csc^2 x - \cos^2 x = \frac{1}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x} = \frac{\sin^2 x}{\sin^2 x} = 1$ at all points.

**14** $\lim_{x\to 3} \frac{x^2-9}{x+3} = \frac{0}{6}$. (This is not a case for l'Hôpital's Rule! It is just limit of $f(x)$ divided by limit of $g(x)$.)

**16** $L = \lim_{x\to 0} \frac{\sqrt{1-\cos x}}{x} =$ (by l'Hôpital's Rule) $\lim_{x\to 0} \frac{\frac{1}{2}(1-\cos x)^{-1/2}\sin x}{1}$ or $L = \lim \frac{\sin x}{2\sqrt{1-\cos x}}$. This is again $\frac{0}{0}$. But we can multiply by $\frac{x}{\sin x} \to 1$ to reach $\lim_{x\to 0} \frac{x}{2\sqrt{1-\cos x}}$ which is $\frac{1}{2L}$. Thus $L = \frac{1}{2L}$ and $\mathbf{L = \frac{1}{\sqrt{2}}}$.

(Note: The knowledge that $1 - \cos x \approx \frac{x^2}{2}$ also gives $\lim \frac{\sqrt{1-\cos x}}{x} = \lim \frac{x/\sqrt{2}}{x} = \frac{1}{\sqrt{2}}$.)

**18** $\lim_{x\to 1} \frac{x-1}{\sin x} = \frac{0}{\sin 1} = \mathbf{0}$ (not an application of l'Hôpital's Rule).

**20** $\lim_{x\to 0} \frac{(1+x)^n - 1 - nx}{x^2} = \lim_{x\to 0} \frac{n(1+x)^{n-1}-n}{2x} =$ (l'Hôpital again) $\lim_{x\to 0} \frac{n(n-1)(1+x)^{n-2}}{2} = \mathbf{\frac{n(n-1)}{2}}$.

**22** $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}} + \frac{1}{2\sqrt{1-x}}}{1} = \frac{1}{2} + \frac{1}{2} = \mathbf{1}$.

**24** The steps when $f \to \infty$ and $g \to \infty$ are $L = \lim \frac{f}{g} = \lim \frac{1/g}{1/f} =$ **(now comes l'Hôpital for $\frac{0}{0}$)** $\lim \frac{g'/g^2}{f'/f^2} =$ **(here is the limit of a product or quotient)** $(\lim \frac{f^2}{g^2})/\lim \frac{f'}{g'} = L^2/\lim \frac{f'}{g'}$. Cancel $L$ to find $\lim \frac{f'}{g'} = L$.

**26** $\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$ approaches $\frac{\infty}{\infty}$. ok to use l'Hôpital: find $\lim_{x\to 0} \frac{-1/x^2}{1/x^2} = \mathbf{-1}$. Also ok to rewrite the original ratio: $\frac{1+\frac{1}{x}}{1-\frac{1}{x}} = \frac{x+1}{x-1}$ which approaches $\frac{+1}{-1} = \mathbf{-1}$.

**28** $\frac{\csc x}{\cot x} = \frac{1}{\sin x}\frac{\sin x}{\cos x} = \frac{1}{\cos x}$. The limit as $x \to 0$ is $\frac{1}{1} = \mathbf{1}$. ok to use l'Hôpital's Rule for $\frac{\infty}{\infty}$ : $L = \lim_{x\to 0} \frac{\csc x}{\cot x} = \lim_{x\to 0} \frac{-\csc x \cot x}{-\csc^2 x} = \lim \frac{\cot x}{\csc x} = \frac{1}{L}$. This gives $L^2 = 1$ but does not eliminate $L = -1$; add the fact that $\csc x$ and $\cot x$ have the same sign near $x = 0$.

**30** Mean Value Theorem: $f(x) - f(y) = f'(c)(x-y)$. Therefore $|f(x) - f(y)| = |f'(c)||x-y| \le |x-y|$ since we are given that $|f'| \le 1$ at all points. Geometric interpretation: If the tangent slope stays between $-1$ and 1, so does the slope of any secant line.

**32** **No:** The converse of Rolle's theorem is false. The function $f(x) = x^3$ has $f' = 0$ at $x = 0$ (horizontal tangent). But there are no two points where $f(a) = f(b)$ (no horizontal secant line).

**34** $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x^2 \cos \frac{1}{x}}{x} = \lim_{x\to 0}(x \cos \frac{1}{x}) = 0$ because always $|x \cos \frac{1}{x}| \le |x|$. However $\frac{f'(x)}{g'(x)} = \frac{\sin \frac{1}{x} + 2x \cos \frac{1}{x}}{1}$ has no limit because $\sin \frac{1}{x}$ oscillates as $x \to 0$ (its graph is in Section 2.7).

**36** If you travel 3000 miles in 100 hours then at some moment your speed is **30 miles per hour.**

**38** Mean Value Theorem: $f(b) - f(a) = f'(c)(b-a)$ is positive if $f'(c) > 0$ and $b > a$. Therefore $f(b) - f(a) > 0$ and $f(b) > f(a)$. A function with positive slope is increasing (as stated without proof in Section 3.2).