

CHAPTER 6 EXPONENTIALS AND LOGARITHMS

6.1 An Overview (page 234)

In $10^4 = 10,000$, the exponent 4 is the **logarithm** of 10,000. The base is $b = 10$. The logarithm of 10^m times 10^n is $m + n$. The logarithm of $10^m/10^n$ is $m - n$. The logarithm of $10,000^x$ is $4x$. If $y = b^x$ then $x = \log_b y$. Here x is any number, and y is always positive.

A base change gives $b = a^{\log_a b}$ and $b^x = a^{x \log_a b}$. Then 8^5 is 2^{15} . In other words $\log_2 y$ is $\log_2 8$ times $\log_8 y$. When $y = 2$ it follows that $\log_2 8$ times $\log_8 2$ equals 1.

On ordinary paper the graph of $y = mx + b$ is a straight line. Its slope is m . On semilog paper the graph of $y = Ab^x$ is a straight line. Its slope is $\log b$. On log-log paper the graph of $y = Ax^k$ is a straight line. Its slope is k .

The slope of $y = b^x$ is $dy/dx = cb^x$, where c depends on b . The number c is the limit as $h \rightarrow 0$ of $\frac{b^h - 1}{h}$. Since $x = \log_b y$ is the inverse, $(dx/dy)(dy/dx) = 1$. Knowing $dy/dx = cb^x$ yields $dx/dy = 1/cb^x$. Substituting b^x for y , the slope of $\log_b y$ is $1/cy$. With a change of letters, the slope of $\log_b x$ is $1/cx$.

1 5; -5; -1; $\frac{1}{5}$; $\frac{3}{2}$; 2 5 1; -10; 80; 1; 4; -1 7 $n \log_b x$ 9 $\frac{10}{3}$; $\frac{3}{10}$ 13 10^5
 15 0; $I_{SF} = 10^7 I_0$; $8.3 + \log_{10} 4$ 17 $A = 7, b = 2.5$ 19 $A = 4, k = 1.5$
 21 $\frac{1}{cx}$; $\frac{2}{cx}$; $\log 2$ 23 $y - 1 = cx$; $y - 10 = c(x - 1)$ 25 $(.1^{-h} - 1)/(-h) = (10^h - 1)/(-h)$
 27 $y'' = c^2 b^x$; $x'' = -1/cy^2$ 29 Logarithm

- 2 (a) 5 (b) 25 (c) 1 (d) 2 (e) 10^4 (f) 3
 4 The graph of 2^{-x} goes through $(0, 1)$, $(1, \frac{1}{2})$, $(2, \frac{1}{4})$. The mirror image is $x = \log_{\frac{1}{2}} y$ (y is now horizontal):
 $\log_{1/2} 2 = -1$ and $\log_{1/2} 4 = -2$.
 6 (a) 7 (b) 3 (c) $\sqrt{10}$ (d) $\frac{1}{4}$ (e) $\sqrt{8}$ (f) 5
 8 $\log_b a = (\log_b d)(\log_d a)$ and $(\log_b d)(\log_d c) = \log_b c$. Multiply left sides, multiply right sides, cancel $\log_b d$.
 10 Number of decimal digits $\approx$ logarithm to base 10. For 2^{1000} this logarithm is $1000 \log_{10} 2 \approx 1000(.3) = 300$.
 12 $y = \log_{10} x$ is a straight line on "inverse" semilog paper: y axis normal, x axis scaled logarithmically (so $x = 1, 10, 100$ are equally spaced). Any equation $y = \log_b x + C$ will have a straight line graph.
 14 $y = 10^{1-x}$ drops from 10 to 1 to .1 with slope -1 on semilog paper; $y = \frac{1}{2}\sqrt{10^x}$ increases with slope $\frac{1}{2}$ from $y = \frac{1}{2}$ at $x = 0$ to $y = 5$ at $x = 2$.
 16 If 440/second is the frequency of middle A, then the next A is 880/second. The 12 steps from A to A are approximately multiples of $2^{1/12}$. So 7 steps multiplies by $2^{7/12} \approx 1.5$ to give $(1.5)(440) = 660$. The seventh note from A is E.
 18 $\log y = 2 \log x$ is a straight line with slope 2; $\log y = \frac{1}{2} \log x$ has slope $\frac{1}{2}$.
 20 $g(f(y)) = y$ gives $g'(f(y)) \frac{df}{dy} = 1$ or $cg(f(y)) \frac{df}{dy} = 1$ or $cy \frac{df}{dy} = 1$ or $\frac{df}{dy} = \frac{1}{cy}$.
 22 The slope of $y = 10^x$ is $\frac{dy}{dx} = c10^x$ (later we find that $c = \ln 10$). At $x = 0$ and $x = 1$ the slope is c and $10c$. So the tangent lines are $y - 1 = c(x - 0)$ and $y - 10 = 10c(x - 1)$.

24 $h = 1$ gives $c = 9$; $h = .1$ gives $c = 2.6$; $h = .01$ gives $c = 2.339$; $h = .001$ gives $c = 2.305$; the limit is $c = \ln 10 = 2.3026$.

26 (The right base is $b = e$.) With $h = \frac{1}{4}$ we pick the base so that $\frac{b^{1/4}-1}{1/4} = 1$ or $b^{1/4} = (1 + \frac{1}{4})$ or $b = (1 + \frac{1}{4})^4 = \frac{625}{256}$. Generally $b = (1 + h)^{1/h}$ which approaches e as $h \rightarrow 0$.

28 $c = \lim_{h \rightarrow 0} \frac{10^h - 1}{h} = \lim_{h \rightarrow 0} \frac{10^{2h} - 1}{2h} = \frac{1}{2} \lim_{h \rightarrow 0} \frac{100^h - 1}{h} = \frac{1}{2} C$.

6.2 The Exponential e^x (page 241)

The number e is approximately **2.78**. It is the limit of $(1 + h)$ to the power $1/h$. This gives 1.01^{100} when $h = .01$. An equivalent form is $e = \lim (1 + \frac{1}{n})^n$.

When the base is $b = e$, the constant c in Section 6.1 is **1**. Therefore the derivative of $y = e^x$ is $dy/dx = e^x$. The derivative of $x = \log_e y$ is $dx/dy = 1/y$. The slopes at $x = 0$ and $y = 1$ are both **1**. The notation for $\log_e y$ is $\ln y$, which is the natural logarithm of y .

The constant c in the slope of b^x is $c = \ln b$. The function b^x can be rewritten as $e^{x \ln b}$. Its derivative is $(\ln b)e^{x \ln b} = (\ln b)b^x$. The derivative of $e^{u(x)}$ is $e^{u(x)} \frac{du}{dx}$. The derivative of $e^{\sin x}$ is $e^{\sin x} \cos x$. The derivative of e^{cx} brings down a factor c .

The integral of e^x is $e^x + C$. The integral of e^{cx} is $\frac{1}{c}e^{cx} + C$. The integral of $e^{u(x)} du/dx$ is $e^{u(x)} + C$. In general the integral of $e^{u(x)}$ by itself is impossible to find.

1 $49e^{7x}$ **3** $8e^{8x}$ **5** $3^x \ln 3$ **7** $(\frac{2}{3})^x \ln \frac{2}{3}$ **9** $\frac{-e^x}{(1+e^x)^2}$ **11** 2 **13** xe^x **15** $\frac{4}{(e^x + e^{-x})^3}$
17 $e^{\sin x} \cos x + e^x \sin e^x$ **19** .1246, .0135, .0014 are close to $\frac{e}{2n}$ **21** $\frac{1}{e}; \frac{1}{e}$
23 $Y(h) = 1 + \frac{1}{10}$; $Y(1) = (1 + \frac{1}{10})^{10} = 2.59$ **25** $(1 + \frac{1}{x})^x < e < e^x < e^{3x/2} < e^{2x} < 10^x < x^x$
27 $\frac{e^{3x}}{3} + \frac{e^{7x}}{7}$ **29** $x + \frac{2^x}{\ln 2} + \frac{3^x}{\ln 3}$ **31** $\frac{(2e)^x}{\ln(2e)} + 2e^x$ **33** $\frac{e^{x^2}}{2} - \frac{e^{-x^2}}{2}$
35 $2e^{x/2} + \frac{e^{2x}}{2}$ **37** e^{-x} drops faster at $x = 0$ (slope -1); meet at $x = 1$; $e^{-x^2}/e^{-x} < e^{-9}/e^{-3} < \frac{1}{100}$ for $x > 3$
39 $y - e^a = e^a(x - a)$; need $-e^a = -ae^a$ or $a = 1$
41 $y' = x^x(\ln x + 1) = 0$ at $x_{\min} = \frac{1}{e}$; $y'' = x^x[(\ln x + 1)^2 + \frac{1}{x}] > 0$
43 $\frac{d}{dx}(e^{-x}y) = e^{-x} \frac{dy}{dx} - e^{-x}y = 0$ so $e^{-x}y = \text{Constant}$ or $y = Ce^x$
45 $\frac{e^{2x}}{2} \Big|_0^1 = \frac{e^2 - 1}{2}$ **47** $\frac{2^x}{\ln 2} \Big|_{-1}^1 = \frac{2 - \frac{1}{2}}{\ln 2} = \frac{3}{2 \ln 2}$ **49** $-e^{-x} \Big|_0^\infty = 1$ **51** $e^{1+x} \Big|_0^1 = e^2 - e$ **53** $2^{\sin x} \Big|_0^\pi = 0$
55 $\int \frac{du/dx}{e^u} dx = -e^{-u} + C$; $\int (e^u)^2 \frac{du}{dx} dx = \frac{1}{2}e^{2u} + C$ **57** $yy' = 1$ gives $\frac{1}{2}y^2 = x + C$ or $y = \sqrt{2x + 2C}$
59 $\frac{dF}{dx} = (n - x)x^{n-1}/e^x < 0$ for $x > n$; $F(2x) < \frac{\text{constant}}{e^x} \rightarrow 0$ **61** $\frac{6!}{\sqrt{12\pi}} \approx 117$; $(\frac{6}{e})^6 \approx 116$; 7 digits

2 $49e^{-7x}$ **4** $8e^{8x}$ **6** $(\ln 3)e^{x \ln 3} = (\ln 3)3^x$ **8** $4(\ln 4)4^{4x}$ **10** $\frac{-1}{(1+x)^2}e^{1/(1+x)}$ **12** $(-\frac{1}{x} + 1)e^{1/x}$ **14** x^2e^x
16 $x^2 + x^2$ has derivative $4x$ **18** $x^{-1/x} = e^{-(\ln x)/x}$ has derivative $(-\frac{1}{x^2} + \frac{\ln x}{x^2})e^{-(\ln x)/x} = (\frac{\ln x - 1}{x^2})x^{-1/x}$
20 $(1 + \frac{1}{n})^{2n} \rightarrow e^2 \approx 7.7$ and $(1 + \frac{1}{n})^{\sqrt{n}} \rightarrow 1$. Note that $(1 + \frac{1}{n})^{\sqrt{n}}$ is squeezed between 1 and $e^{1/\sqrt{n}}$ which approaches 1.
22 $(1.001)^{1000} = 2.717$ and $(1.0001)^{10000} = 2.7181$ have 3 and 4 correct decimals. $(1.00001)^{100000} = 2.71827$ has one more correct decimal. The difference between $(1 + \frac{1}{n})^n$ and e is proportional to $\frac{1}{n}$.

- 24 $y = e^{-x}$ solves $\frac{dy}{dx} = -y$. The difference equation $Y(x + \frac{1}{4}) = Y(x) - \frac{1}{4}Y(x)$ with $Y(0) = 1$ gives $Y(\frac{1}{4}) = \frac{3}{4}$ and $Y(1) = (\frac{3}{4})^4$. (Compare $e^{-1} = .37$ with $(\frac{3}{4})^4 = .32$. See the end of Section 6.6.)
- 26 $\sqrt{e^x}$ is the same as $e^{x/2}$. Its graph at $x = -2, 0, 2$ has the same heights $\frac{1}{e}, 1, e$ as the graph of e^x at $x = -1, 0, 1$.
- 28 $(e^{3x})(e^{7x}) = e^{10x}$ which is the derivative of $\frac{1}{10}e^{10x}$
- 30 $2^{-x} = e^{-x \ln 2}$ which has antiderivative $\frac{-1}{\ln 2}e^{-x \ln 2} = \frac{-1}{\ln 2}2^{-x}$.
- 32 $e^{-x} + x^{-e}$ has antiderivative $-e^{-x} + \frac{x^{1-e}}{1-e}$ 34 $-e^{\cos x} + e^{\sin x}$ 36 $xe^x - e^x$
- 38 e^x meets x^x at $x = e$. Their slopes are e^x and $x^x(1 + \ln x)$ by Example 6. At $x = e$ those slopes are e^e and $2e^e$. The ratio $\frac{x^x}{e^x} = (\frac{x}{e})^x$ approaches infinity.
- 40 At $x = 0$ equality holds: $e^0 = 1 + 0$ and $e^{-0} = 1 - 0$. (a) Beyond $x = 0$ the slope of e^x exceeds the slope of $1 + x$ (this means $e^x > 1 + x$). So e^x increases faster than $1 + x$. (b) Beyond $x = 0$ the slope of e^{-x} is larger than the slope of $1 - x$ (this means $-e^{-x} > -1$). Since they start together, e^{-x} is larger than $1 - x$.
- 42 $x^{1/x} = e^{(\ln x)/x}$ has slope $e^{(\ln x)/x}(\frac{1}{x^2} - \frac{\ln x}{x^2}) = x^{1/x}(\frac{1 - \ln x}{x^2})$. This slope is zero at $x = e$, when $\ln x = 1$.
The second derivative is *negative* so the maximum of $x^{1/x}$ is $e^{1/e}$. Check: $\frac{d}{dx}e^{(\ln x)/x}(\frac{1 - \ln x}{x^2}) = e^{(\ln x)/x}[(\frac{1 - \ln x}{x^3})^2 + \frac{(-2 - 1 + 2 \ln x)}{x^3}] = -\frac{1}{e^3}e^{1/e}$ at $x = e$.
- 44 $x^e = e^x$ at $x = e$. This is the only point where $x^e e^{-x} = 1$ because the derivative is $x^e(-e^{-x}) + ex^{e-1}e^{-x} = (\frac{e}{x} - 1)x^e e^{-x}$. This derivative is positive for $x < e$ and negative for $x > e$. So the function $x^e e^{-x}$ increases to 1 at $x = e$ and then decreases: it never equals 1 again.
- 46 $\int_0^\pi \sin x e^{\cos x} dx = [-e^{\cos x}]_0^\pi = -e^{-1} + e$.
- 48 $\int_{-1}^1 2^{-x} dx = (\text{by Problem 30}) [\frac{-1}{\ln 2}2^{-x}]_{-1}^1 = \frac{-1}{\ln 2}(\frac{1}{2} - 2) = \frac{3}{2 \ln 2}$.
- 50 $\int_0^\infty xe^{-x^2} dx = \int_0^\infty e^{-u} \frac{du}{2} = [-\frac{e^{-u}}{2}]_0^\infty = \frac{1}{2}$. 52 $\int_0^1 e^{1+x^2} x dx = [\frac{1}{2}e^{1+x^2}]_0^1 = \frac{1}{2}(e^2 - e)$
- 54 $\int_0^1 (1 - e^x)^{10} e^x dx = [-\frac{(1 - e^x)^{11}}{11}]_0^1 = -\frac{(1 - e)^{11}}{11}$.
- 56 $y'(x) = 5y(x)$ is solved by $y = Ae^{5x}$ (A is any constant). Choose $A = 2$ so that $y(x) = 2e^{5x}$ has $y(0) = 2$.
- 58 The asymptotes of $(1 + \frac{1}{x})^x = (\frac{x+1}{x})^x = (\frac{x}{x+1})^{-x}$ are $x = -1$ (from the last formula) and $y = e$ (from the first formula).
- 60 The maximum of $x^6 e^{-x}$ occurs when its derivative $(6x^5 - x^6)e^{-x}$ is zero. Then $x = 6$ (note that $x = 0$ is a minimum).
- 62 $\lim_{x \rightarrow \infty} \frac{x^6}{e^x} = \lim_{x \rightarrow \infty} \frac{6x^5}{e^x} = \lim_{x \rightarrow \infty} \frac{30x^4}{e^x} = \lim_{x \rightarrow \infty} \frac{120x^3}{e^x} = \lim_{x \rightarrow \infty} \frac{360x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{720x}{e^x} = \lim_{x \rightarrow \infty} \frac{720}{e^x} = 0$.

6.3 Growth and Decay in Science and Economics (page 250)

If $y' = cy$ then $y(t) = y_0 e^{ct}$. If $dy/dt = 7y$ and $y_0 = 4$ then $y(t) = 4e^{7t}$. This solution reaches 8 at $t = \frac{\ln 2}{7}$. If the doubling time is T then $c = \frac{\ln 2}{T}$. If $y' = 3y$ and $y(1) = 9$ then y_0 was $9e^{-3}$. When c is negative, the solution approaches zero as $t \rightarrow \infty$.

The constant solution to $dy/dt = y + 6$ is $y = -6$. The general solution is $y = Ae^t - 6$. If $y_0 = 4$ then $A = 10$. The solution of $dy/dt = cy + s$ starting from y_0 is $y = Ae^{ct} + B = (y_0 + \frac{s}{c})e^{ct} - \frac{s}{c}$. The output from the source is $\frac{s}{c}(e^{ct} - 1)$. An input at time T grows by the factor $e^{c(t-T)}$ at time t .

At $c = 10\%$, the interest in time dt is $dy = .01 y dt$. This equation yields $y(t) = y_0 e^{.01t}$. With a source term instead of y_0 , a continuous deposit of $s = 4000/\text{year}$ yields $y = 40,000(e - 1)$ after ten years. The deposit

required to produce 10,000 in 10 years is $s = yc/(e^{ct} - 1) = 1000/(e - 1)$. An income of 4000/year forever (!) comes from $y_0 = 40,000$. The deposit to give 4000/year for 20 years is $y_0 = 40,000(1 - e^{-2})$. The payment rate s to clear a loan of 10,000 in 10 years is $1000e/(e - 1)$ per year.

The solution to $y' = -3y + s$ approaches $y_\infty = s/3$.

- 1 $t^2 + y_0$ 3 $y_0 e^{2t}$ 5 $10 e^{4t}; t = \frac{\ln 10}{4}$ 7 $\frac{1}{4} e^{4t} + 9.75; t = \frac{\ln 361}{4}$ 11 $c = \frac{\ln 2}{2}; t = \frac{\ln 10}{c}$
 13 $\frac{5568}{-.7} \ln(\frac{1}{5})$ 15 $c = \frac{\ln 2}{20}; t = \frac{1}{c} \ln(\frac{8}{5})$ 17 $t = \frac{\ln(1/240)}{\ln(.98)}$ 19 $e^c = 3$ so $y_0 = e^{-3c} 1000 = \frac{1000}{27}$
 21 $p = 1013 e^{ch}; 50 = 1013 e^{20c}; c = \frac{1}{20} \ln(\frac{50}{1013}); p(10) = 1013 e^{10c} = 1013 \sqrt{\frac{50}{1013}} = \sqrt{(1013)(50)}$
 23 $c = \frac{\ln 2}{3}; (\frac{1}{2})^3 = \frac{1}{8}$ 25 $y = y_0 - at$ reaches y_1 at $t = \frac{y_0 - y_1}{a}$; then $y = Ae^{-at/y_1}$ 27 F; F; T; T
 29 $A = \frac{1}{3}, B = -\frac{1}{3}$ 31 $e^t - 1$ 33 $1 - e^{-t}$ 35 $6; 6 + Ae^{-2t}; 6 - 6e^{-2t}; 6 + 4e^{-2t}; 6$
 37 $4; 4 - \frac{1}{e}; 4$ 39 $ye^{-t}; y(t) = te^t$ 41 $A = 1, B = -1, C = -1$ 43 $e^{.0725} > .075$ 45 $s(e - 1); \frac{s(e - 1)}{e}$
 47 $(1.02)(1.03) \rightarrow 5.06\%; 5\%$ by Problem 27 49 $20,000 e^{(20 - T)(.5)} = 34,400$ (it grows for $20 - T$ years)
 51 $s = -cy_0 e^{ct}/(e^{ct} - 1) = -(0.1)(1000)e^{.60}/(e^{.60} - 1)$ 53 $y_0 = \frac{100}{.005}(1 - e^{-.005(48)})$
 55 $e^{4c} = 1.20$ so $c = \frac{\ln 1.20}{4}$ 57 $24e^{36.5} = ?$ 59 To $-\infty$; constant; to $+\infty$
 61 $\frac{dY}{dT} = 60cY; \frac{dY}{dT} = 60(-Y + 5)$; still $Y_\infty = 5$
 63 $y = 60e^{ct} + 20, 60 = 60e^{12c} + 20, c = \frac{1}{12} \ln(\frac{40}{60}); 100 = 60e^{ct} + 20$ at $t = \frac{1}{c} \ln(\frac{80}{60})$ 65 0

- 2 $\frac{dy}{dt} = -t$ gives $dy = -t dt$ and $y = -\frac{1}{2}t^2 + C$. Then $y = -\frac{1}{2}t^2 + 1$ and $y = -\frac{1}{2}t^2 - 1$ start from 1 and -1.
 4 $\frac{dy}{dt} = -y$ gives $\frac{dy}{y} = -dt$ and $\ln y = -t + C$ and $y = Ae^{-t}$ (where $A = e^C$). (Question: How does a negative y appear, since e^C is positive? Answer: $\int \frac{dy}{y} = \ln|y|$ leads to $|y| = Ae^{-t}$ and allows $y < 0$.) To start from 1 and -1 choose $y = e^{-t}$ and $y = -e^{-t}$.
 6 $\frac{dy}{dt} = 4t$ gives $dy = 4t dt$ and $y = 2t^2 + C = 2t^2 + 10$. This equals 100 when $2t^2 = 90$ or $t = \sqrt{45}$.
 8 $\frac{dy}{dt} = e^{-4t}$ has $y(t) = \frac{e^{-4t}}{-4} + C = \frac{e^{-4t}}{-4} + 10\frac{1}{4}$. This only increases from 10 to $10\frac{1}{4}$ as $t \rightarrow \infty$. Before $t = 0$ we find $y(t) = 1$ when $\frac{e^{-4t}}{-4} = 9\frac{1}{4}$ or $e^{-4t} = 37$ or $t = \frac{\ln 37}{-4}$.
 10 The solutions of $y' = y - 1$ (which is also $(y - 1)' = y - 1$) are $y - 1 = Ae^x$ or $y = Ae^x + 1$. Figure 6.7b is raised by 1 unit. (The solution that was $y = e^x$ is lifted to $y = e^x + 1$. The solution that was $y = 0$ is lifted to $y = 1$.)
 12 To multiply again by 10 takes ten more hours, a total of 20 hours. If $e^{10c} = 10$ (and $e^{20c} = 100$) then $10c = \ln 10$ and $c = \frac{\ln 10}{10} \approx .23$.
 14 Following Example 2, the ratio e^{ct} would be 90% or .9. Then $t = \frac{\ln .9}{c} = (\frac{\ln .9}{\ln \frac{1}{2}})5568 = (\ln 1.8)5568 = 3273$ years. So the material is dated earlier than the year 0.
 16 $8e^{.01t} = 6e^{.014t}$ gives $\frac{8}{6} = e^{.004t}$ and $t = \frac{1}{.004} \ln \frac{8}{6} = 250 \ln \frac{4}{3} = 72$ years.
 18 At $t = 3$ days, $e^{3c} = 40\% = .4$ and $c = \frac{\ln .4}{3} = -.3$. At T days, 20% remember: $e^{-.3T} = 20\% = .2$ at $T = \frac{\ln .2}{(-.3)} = 5.36$ days. (Check after 6 days: $(.4)^2 = 16\%$ will remember.)
 20 If y is divided by 10 in 4 time units, it will be divided by 10 again in 4 more units. Thus $y = 1$ at $t = 12$. Returning to $t = 0$ multiplies by 10 so $y_0 = 1000$.
 22 Exponential decay is $y = Ae^{ct}$. Then $y(0) = A$ and $y(2t) = Ae^{2ct}$. The square root of $y(0)y(2t) = A^2 e^{2ct}$ is $y(t) = Ae^{ct}$. One way to find $y(3t) = Ae^{3ct}$ is $y(0)(\frac{y(2t)}{y(0)})^{3/2}$. (A better question is to find $y(4t) = Ae^{4ct} = y(0)(\frac{y(2t)}{y(0)})^2 = \frac{(y(2t))^2}{y(0)}$.)

- 24 Go from 4 mg back down to 1 mg in T hours. Then $e^{-.01T} = \frac{1}{4}$ and $-.01T = \ln \frac{1}{4}$ and $T = \frac{\ln \frac{1}{4}}{-.01} = 139$ hours (not so realistic).
- 26 The second-order equation is $(\frac{d}{dt} - c)(\frac{d}{dt} - C)y = \frac{d^2y}{dt^2} - (c+C)\frac{dy}{dt} + cCy = 0$. Check the solution $y = Ae^{ct} + Be^{Ct}$ by substituting into the equation: $c^2Ae^{ct} + C^2Be^{Ct} - (c+C)(cAe^{ct} + CBe^{Ct}) + cC(Ae^{ct} + Be^{Ct})$ does equal zero.
- 28 Given $mv = mv - v\Delta m + m\Delta v - (\Delta m)\Delta v + \Delta m(v - 7)$; cancel terms to leave $m\Delta v - (\Delta m)\Delta v = 7\Delta m$; divide by Δm and approach the limit $m\frac{dv}{dm} = 7$. Then $v = 7 \ln m + C$. At $t = 0$ this is $20 = 7 \ln 4 + C$ so that $v = 7 \ln m + 20 - 7 \ln 4 = 7 \ln \frac{m}{4} + 20$.
- 30 Substitute $y = Ae^{-t} + B$ into $y' = 8 - y$ to find $-Ae^{-t} = 8 - Ae^{-t} - B$. Then $B = 8$. At the start $y_0 = A + B = A + 8$ so $A = y_0 - 8$. Then $y = (y_0 - 8)e^{-t} + 8$ or $y = y_0e^{-t} + 8(1 - e^{-t})$.
- 32 Apply formula (8) to $\frac{dy}{dt} = y - 1$ with $y_0 = 0$. Then $y(t) = \frac{-1}{-1}(e^t - 1) = 1 - e^t$.
- 34 Formula (8) applied to $\frac{dy}{dt} = -y - 1$ with $y_0 = 0$ gives $y = \frac{-1}{-1}(e^{-t} - 1) = e^{-t} - 1$.
- 36 (a) $\frac{dy}{dt} = 3y + 6$ gives $y \rightarrow \infty$ (b) $\frac{dy}{dt} = -3y + 6$ gives $y \rightarrow 2$ (c) $\frac{dy}{dt} = -3y - 6$ gives $y \rightarrow -2$
(d) $\frac{dy}{dt} = 3y - 6$ gives $y \rightarrow -\infty$.
- 38 Solve $y' = y + e^t$ by adding inputs at all times T times growth factors e^{t-T} : $y(t) = \int_0^t e^{t-T} e^T dT = \int_0^t e^t dT = te^t$. Substitute in the equation to check: $(te^t)' = te^t + e^t$.
- 40 Solve $y' + y = 1$ by multiplying to give $e^t y' + e^t y = e^t$. The left side is the derivative of ye^t (by the product rule). Integrate both sides: $ye^t - y_0e^0 = e^t - e^0$ or $ye^t = y_0 + e^t - 1$ or $y = y_0e^{-t} + 1 - e^{-t}$.
- 42 \$1000 changes by (\$1000) $(-.04dt)$, a decrease of $40dt$ dollars in time dt . The printing rate should be $s = 40$.
- 44 First answer: With continuous interest at $c = .09$ the multiplier after a year is $e^{.09} = 1.094$ and the effective rate is 9.4%. Second answer: The continuous rate c that gives an effective annual rate of 9% is $e^c = 1.09$ or $c = \ln 1.09 = .086$ or 8.6%.
- 46 y_0 grows to $y_0e^{(.1)(20)} = 50,000$ so the grandparent gives $y_0 = 50,000e^{-2} \approx \6767 . A continuous deposit s grows to $\frac{s}{.1}(e^{(.1)(20)} - 1) = 50,000$ so the parent deposits $s = \frac{(.1)50,000}{e^{.1} - 1} = \783 per year.
Saving $s = \$1000/\text{yr}$ grows to $\frac{1000}{.1}(e^{.1t} - 1) = 50,000$ when $e^{.1t} = 1 + \frac{5000}{1000}$ or $.1t = \ln 6$ or $t = 17.9$ years.
- 48 The deposit of $4dT$ grows with factor c from time T to time t , and reaches $e^{c(t-T)}4dT$. With $t = 2$ add deposits from $T = 0$ to $T = 1$: $\int_0^1 e^{c(2-T)}4dT = \left[\frac{4e^{c(2-T)}}{-c} \right]_0^1 = \frac{4e^c - 4e^{2c}}{-c}$.
- 50 $y(t) = (5000 - \frac{500}{.08})e^{.08t} + \frac{500}{.08}$ is zero when $e^{.08t} = \frac{-\frac{500}{.08}}{5000 - \frac{500}{.08}} = 5$. Then $.08t = \ln 5$ and $t = \frac{\ln 5}{.08} \approx 20$ years.
(Remember the deposit grows until it is withdrawn.)
- 52 After 365 days the value is $y = e^{(.01)365} = e^{3.65} = \38 .
- 54 (a) Income = expense when $I_0e^{2ct} = E_0e^{ct}$ or $e^t = \frac{E_0}{I_0}$ or $t = \frac{\ln(E_0/I_0)}{c}$. (b) Integrate $E_0e^{ct} - I_0e^{2ct}$ until $e^{ct} = \frac{E_0}{I_0}$. At the upper limit the integral is $\frac{E_0}{c}e^{ct} - \frac{I_0}{2c}e^{2ct} = \frac{1}{c}(\frac{E_0^2}{I_0} - \frac{I_0}{2}\frac{E_0^2}{I_0^2}) = \frac{E_0^2}{2cI_0}$. Lower limit is $t = 0$ so subtract $\frac{E_0}{c} - \frac{I_0}{2c}$: Borrow $\frac{E_0^2}{2cI_0} - \frac{E_0}{c} + \frac{I_0}{2c}$.
- 56 After 10 years (halfway through the mortgage) the variable rate $.09 + .001(10)$ equals the fixed rate $10\% = .1$. Since the variable was lower early, and therefore longer, the variable rate is preferred.
- 58 If $\frac{dy}{dt} = -y + 7$ then $\frac{dy}{dt}$ is zero at $y_\infty = 7$ (this is $-\frac{s}{c} = \frac{7}{1}$). The derivative of $y - y_\infty$ is $\frac{dy}{dt}$, so the derivative of $y - 7$ is $-(y - 7)$. The decay rate is $c = -1$, and $y - 7 = e^{-t}(y_0 - 7)$.
- 60 All solutions to $\frac{dy}{dt} = c(y - 12)$ converge to $y = 12$ provided c is negative.
- 62 (a) False because $(y_1 + y_2)' = cy_1 + s + cy_2 + s$. We have $2s$ not s . (b) True because $(\frac{1}{2}y_1 + \frac{1}{2}y_2)' = \frac{1}{2}cy_1 + \frac{1}{2}s + \frac{1}{2}cy_2 + \frac{1}{2}s$. (c) False because the derivative of $y' = cy + s$ is $(y')' = c(y')$ and s is gone.
- 64 The solution is $y = Ae^{ct} + B$. Substitute $t = 0, 1, 2$ and move B to the left side: $100 - B = A$, $90 - B = Ae^c$, $84 - B = Ae^{2c}$. Then $(100 - B)(84 - B) = (90 - B)(90 - B)$; both sides are A^2e^{2c} . Solve for B : $8400 - 184B + B^2 = 8100 - 180B + B^2$ or $300 = 4B$. The steady state is $B = 75$.
(This problem is a good challenge and was meant to have a star.)

- 66 (a) The white coffee cools to $y_{\infty} + (y_0 - y_{\infty})e^{ct} = 20 + 40e^{ct}$. (b) The black coffee cools to $20 + 50e^{ct}$. The milk warms to $20 - 10e^{ct}$. The mixture $\frac{5(\text{black coffee}) + 1(\text{milk})}{6}$ has $20 + \frac{250-10}{6}e^{ct} = 20 + 40e^{ct}$. So it doesn't matter when you add the milk!

6.4 Logarithms (page 258)

The natural logarithm of x is $\int_1^x \frac{dt}{t}$ (or $\int_1^x \frac{dx}{x}$). This definition leads to $\ln xy = \ln x + \ln y$ and $\ln x^n = n \ln x$. Then e is the number whose logarithm (area under $1/x$ curve) is 1. Similarly e^x is now defined as the number whose natural logarithm is x . As $x \rightarrow \infty$, $\ln x$ approaches infinity. But the ratio $(\ln x)/\sqrt{x}$ approaches zero. The domain and range of $\ln x$ are $0 < x < \infty$, $-\infty < \ln x < \infty$.

The derivative of $\ln x$ is $\frac{1}{x}$. The derivative of $\ln(1+x)$ is $\frac{1}{1+x}$. The tangent approximation to $\ln(1+x)$ at $x=0$ is x . The quadratic approximation is $x - \frac{1}{2}x^2$. The quadratic approximation to e^x is $1 + x + \frac{1}{2}x^2$.

The derivative of $\ln u(x)$ by the chain rule is $\frac{1}{u(x)} \frac{du}{dx}$. Thus $(\ln \cos x)' = -\frac{\sin x}{\cos x} = -\tan x$. An antiderivative of $\tan x$ is $-\ln \cos x$. The product $p = xe^{5x}$ has $\ln p = 5x + \ln x$. The derivative of this equation is $p'/p = 5 + \frac{1}{x}$. Multiplying by p gives $p' = xe^{5x}(5 + \frac{1}{x}) = 5xe^{5x} + e^{5x}$, which is LD or logarithmic differentiation.

The integral of $u'(x)/u(x)$ is $\ln u(x)$. The integral of $2x/(x^2+4)$ is $\ln(x^2+4)$. The integral of $1/cx$ is $\frac{\ln x}{c}$. The integral of $1/(ct+s)$ is $\frac{\ln(ct+s)}{c}$. The integral of $1/\cos x$, after a trick, is $\ln(\sec x + \tan x)$. We should write $\ln|x|$ for the antiderivative of $1/x$, since this allows $x < 0$. Similarly $\int du/u$ should be written $\ln|u|$.

- 1 $\frac{1}{x}$ 3 $\frac{-1}{x(\ln x)^2}$ 5 $\ln x$ 7 $\frac{\cos x}{\sin x} = \cot x$ 9 $\frac{7}{x}$ 11 $\frac{1}{3} \ln t + C$ 13 $\ln \frac{4}{3}$
 15 $\frac{1}{2} \ln 5$ 17 $-\ln(\ln 2)$ 19 $\ln(\sin x) + C$ 21 $-\frac{1}{3} \ln(\cos 3x) + C$ 23 $\frac{1}{3}(\ln x)^3 + C$
 27 $\ln y = \frac{1}{2} \ln(x^2+1); \frac{dy}{dx} = \frac{x}{\sqrt{x^2+1}}$ 29 $\frac{dy}{dx} = e^{\sin x} \cos x$
 31 $\frac{dy}{dx} = e^x e^x$ 33 $\ln y = e^x \ln x; \frac{dy}{dx} = ye^x(\ln x + \frac{1}{x})$ 35 $\ln y = -1$ so $y = \frac{1}{e}, \frac{dy}{dx} = 0$ 37 0
 39 $-\frac{1}{x}$ 41 $\sec x$ 47 .1; .095; .095310179 49 -.01; -.01005; -.010050335
 51 l'Hôpital: 1 53 $\frac{1}{\ln b}$ 55 $3 - 2 \ln 2$ 57 Rectangular area $\frac{1}{2} + \dots + \frac{1}{n} < \int_1^n \frac{dt}{t} = \ln n$
 59 Maximum at e 61 0 63 $\log_{10} e$ or $\frac{1}{\ln 10}$ 65 $1 - x; 1 + x \ln 2$
 67 Fraction is $y = 1$ when $\ln(T+2) - \ln 2 = 1$ or $T = 2e - 2$ 69 $y' = \frac{2}{(t+2)^2} \rightarrow y = 1 - \frac{2}{t+2}$ never equals 1
 71 $\ln p = x \ln 2$; LD $2^x \ln 2$; ED $p = e^{x \ln 2}, p' = \ln 2 e^{x \ln 2}$
 75 $2^4 = 4^2; y \ln x = x \ln y \rightarrow \frac{\ln x}{x} = \frac{\ln y}{y}; \frac{\ln x}{x}$ decreases after $x = e$, and the only integers before e are 1 and 2.

- 2 $\frac{2}{2x+1}$ 4 $\frac{x(\frac{1}{x}) - (\ln x)}{x^2} = \frac{1 - \ln x}{x^2}$ 6 Use $(\log_e 10)(\log_{10} x) = \log_e x$. Then $\frac{d}{dx}(\log_{10} x) = \frac{1}{\log_e 10} \cdot \frac{1}{x} = \frac{1}{x \ln 10}$.
 8 $y = \ln u$ so $\frac{dy}{dx} = \frac{du/dx}{u} = \frac{1/x}{\ln x} = \frac{1}{x \ln x}$. 10 $y = 7 \ln 4x = 7 \ln 4 + 7 \ln x$ so $\frac{dy}{dx} = \frac{7}{x}$.
 12 $\ln(1+x)$ from $\int \frac{du}{u}$. 14 $\frac{1}{2} \ln(3+2t)|_0^1 = \frac{1}{2}(\ln 5 - \ln 3) = \frac{1}{2} \ln \frac{5}{3}$.
 16 $y = \frac{x^3}{x^2+1}$ equals $x - \frac{x}{x^2+1}$. Its integral is $[\frac{1}{2}x^2 - \frac{1}{2} \ln(x^2+1)]_0^2 = 2 - \frac{1}{2} \ln 5$.
 18 $\int \frac{du}{u^2} = -\frac{1}{u} = [-\frac{1}{\ln x}]_2^e = -1 + \frac{1}{\ln 2}$.

- 20 $\int \frac{\sin x}{\cos x} dx = \int \frac{-du}{u} = -\ln u = -\ln(\cos x)|_0^{\pi/4} = -\ln \frac{1}{\sqrt{2}} + 0 = \frac{1}{2} \ln 2$.
- 22 $\int \frac{\cos 3x}{\sin 3x} dx = \frac{1}{3} \ln(\sin 3x) + C$.
- 24 Set $u = \ln \ln x$. By the chain rule $\frac{du}{dx} = \frac{1}{\ln x} \cdot \frac{1}{x}$. Our integral is $\int \frac{du}{u} = \ln u = \ln(\ln(\ln x)) + C$.
- 26 The graph starts at $-\infty$ when $x = 0$. It reaches zero when $x = \frac{\pi}{2}$ and goes down again. At $x = \pi$ it stops.
- 28 $\ln y = \frac{1}{2} \ln(x^2 + 1) + \frac{1}{2} \ln(x^2 - 1)$. Then $\frac{1}{y} \frac{dy}{dx} = \frac{x}{x^2+1} + \frac{x}{x^2-1} = \frac{2x^2}{x^4-1}$. Then $\frac{dy}{dx} = \frac{2x^3 y}{x^4-1} = \frac{2x^3}{\sqrt{x^4-1}}$.
- 30 $\ln y = -\frac{1}{x} \ln x$ and $\frac{1}{y} \frac{dy}{dx} = \frac{\ln x - 1}{x^2}$ so $\frac{dy}{dx} = \left(\frac{\ln x - 1}{x^2}\right)x^{-1/x}$.
- 32 $\ln y = e \ln x$ and $\frac{1}{y} \frac{dy}{dx} = \frac{e}{x}$ so $\frac{dy}{dx} = \frac{e}{x} x^e = ex^{e-1}$.
- 34 $\ln y = \frac{1}{2} \ln x + \frac{1}{3} \ln x + \frac{1}{6} \ln x = \ln x$ and eventually $\frac{dy}{dx} = 1$.
- 36 $\ln y = -\ln x$ so $\frac{1}{y} \frac{dy}{dx} = -\frac{1}{x}$ and $\frac{dy}{dx} = -\frac{e^{-\ln x}}{x}$. Alternatively we have $y = \frac{1}{x}$ and $\frac{dy}{dx} = -\frac{1}{x^2}$.
- 38 $[\ln x]_1^e + [\ln |x|]_{-2}^{-1} = (\pi - 0) + (0 - \ln |-2|) = \pi - \ln 2$.
- 40 $\frac{d}{dx} \ln x = \frac{1}{x}$. Alternatively use $\frac{1}{x^2} \frac{d}{dx}(x^2) - \frac{1}{x} \frac{d}{dx}(x) = \frac{1}{x}$.
- 42 This is $\int \frac{du}{u}$ with $u = \sec x + \tan x$ so the integral is $\ln(\sec x + \tan x)$. See Problem 41!
- 44 $\frac{d}{dx}(\ln(x-a) - \ln(x+a)) = \frac{1}{x-a} - \frac{1}{x+a} = \frac{(x+a)-(x-a)}{(x-a)(x+a)} = \frac{2a}{x^2-a^2}$.
- 46 **Misprint!** $\frac{1 + \frac{x}{\sqrt{x^2+a^2}}}{x + \sqrt{x^2+a^2}} = \frac{1}{\sqrt{x^2+a^2}} \frac{\sqrt{x^2+a^2} + x}{x + \sqrt{x^2+a^2}} = \frac{1}{\sqrt{x^2+a^2}}$.
- 48 Linear: $e^{-1} \approx 1 + .1 = 1.1$. Quadratic: $e^{-1} \approx 1 + .1 + \frac{1}{2}(.1)^2 = 1.105$. Calculator: $e^{-1} = 1.105170918$.
- 50 Linear: $e^2 \approx 1 + 2 = 3$. Quadratic: $e^2 \approx 1 + 2 + \frac{1}{2}(2^2) = 5$. Calculator: $e^2 = 7.389$.
- 52 Use l'Hôpital's Rule: $\lim_{x \rightarrow 0} \frac{e^x}{1} = 1$.
- 54 Use l'Hôpital's Rule: $\lim_{x \rightarrow 0} \frac{b^x \ln b}{1} = \ln b$. We have redone the derivative of b^x at $x = 0$.
- 56 Upper rectangles $\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \approx .7595$. Lower rectangles: $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \approx .6345$. Exact area $\ln 2 \approx .693$.
- 58 $\frac{1}{t}$ is smaller than $\frac{1}{\sqrt{t}}$ when $1 < t < x$. Therefore $\int_1^x \frac{dt}{t} < \int_1^x \frac{dt}{\sqrt{t}}$ or $\ln x < 2\sqrt{x} - 2$. (In Problem 59 this leads to $\frac{\ln x}{x} \rightarrow 0$. Another approach is from $\frac{x}{e^x} \rightarrow 0$ in Problem 6.2.59. If x is much smaller than e^x then $\ln x$ is much smaller than x .)
- 60 From $\frac{\ln x}{x} \rightarrow 0$ we know $\frac{\ln x^{1/n}}{x^{1/n}} \rightarrow 0$. This is $\frac{1}{n} \frac{\ln x}{x^{1/n}} \rightarrow 0$. Since n is fixed we have $\frac{\ln x}{x^{1/n}} \rightarrow 0$.
- 62 $\frac{1}{x} \ln \frac{1}{x} = -\frac{\ln x}{x} \rightarrow 0$ as $x \rightarrow \infty$. This means $y \ln y \rightarrow 0$ as $y = \frac{1}{x} \rightarrow 0$. (Emphasize: The factor $y \rightarrow 0$ is "stronger" than the factor $\ln y \rightarrow -\infty$.)
- 64 From $\int_1^x t^{h-1} dt = \frac{x^h - 1}{h}$ we find $\int_1^x t^{-1} dt = \lim_{h \rightarrow 0} \frac{x^h - 1}{h}$. The left side is recognized as $\ln x$. (The right side is the "mysterious constant c " when the base is $b = x$. We discovered earlier that $c = \ln b$.)
- 66 $.01 - \frac{1}{2}(.01)^2 + \frac{1}{3}(.01)^3 = .00995033 \dots$ Also $\ln 1.02 \approx .02 - \frac{1}{2}(.02)^2 + \frac{1}{3}(.02)^3 = .01980266 \dots$
- 68 To emphasize: If the ant didn't crawl, the fraction y would be constant (the ant would move as the band stretches). By crawling $v dt$ the fraction y increases by $\frac{v dt}{\text{band length}}$. So $\frac{dy}{dt} = \frac{v}{\ell} = \frac{1}{8t+2}$. Then $y = \frac{1}{8} \ln(8t+2) + C = \frac{1}{8}(\ln(8t+2) - \ln 2)$. This equals 1 when $8 = \ln \frac{8t+2}{2}$ or $4t+1 = e^8$ or $t = \frac{1}{4}(e^8 - 1)$.
- 70 **LD:** $\ln p = x \ln x$ so $\frac{1}{p} \frac{dp}{dx} = 1 + \ln x$ and $\frac{dp}{dx} = p(1 + \ln x) = x^x(1 + \ln x)$. Now find the same answer by **ED:** $\frac{d}{dx}(e^{x \ln x}) = e^{x \ln x} \frac{d}{dx}(x \ln x) = x^x(1 + \ln x)$.
- 72 To compute $\int_1^2 \frac{dx}{x} = \ln 2$ with error $\approx 10^{-5}$ the trapezoidal rule needs $\Delta x \approx 10^{-2}$. *Six Simpson steps:*
 $S_6 = \frac{1}{36}[\frac{1}{1} + \frac{4}{13/12} + \frac{2}{7/6} + \frac{4}{15/12} + \frac{2}{8/6} + \frac{4}{17/12} + \frac{2}{9/6} + \frac{4}{19/12} + \frac{2}{10/6} + \frac{4}{21/12} + \frac{2}{11/6} + \frac{4}{23/12} + \frac{1}{12/6}] = .693149$ compared to $\ln 2 = .693147$. Predicted error $\frac{1}{2880}(\frac{1}{6})^4(6 - \frac{6}{2^4}) = 1.6 \times 10^{-6}$, actual error 1.5×10^{-6} .
- 74 $\frac{1}{\ln 90,000} = .0877$ says that about 877 of the next 10,000 numbers are prime: close to the actual count 879.
- 76 $\frac{\ln x}{x} = \frac{t \ln(\frac{t+1}{t})}{(\frac{t+1}{t})^t}$. This equals $\frac{\ln y}{y} = \frac{(t+1) \ln(\frac{t+1}{t})}{(\frac{t+1}{t})^{t+1}} = \frac{t+1}{t} \frac{\ln(\frac{t+1}{t})}{(\frac{t+1}{t})^t}$. The curve $x^y = y^x$ is asymptotic to $x = 1$, for t near zero. It approaches $x = e, y = e$ as $t \rightarrow \infty$. It is symmetric across the 45°

line (no change by reversing x and y), roughly like the hyperbola $(x-1)(y-1) = (e-1)^2$.

6.5 Separable Equations Including the Logistic Equation (page 266)

The equations $dy/dt = cy$ and $dy/dt = cy + s$ and $dy/dt = u(y)v(t)$ are called **separable** because we can separate y from t . Integration of $\int dy/y = \int c dt$ gives $\ln y = ct + \text{constant}$. Integration of $\int dy/(y + s/c) = \int c dt$ gives $\ln(y + \frac{s}{c}) = ct + C$. The equation $dy/dx = -x/y$ leads to $\int y dy = -\int x dx$. Then $y^2 + x^2 = \text{constant}$ and the solution stays on a circle.

The logistic equation is $dy/dt = cy - by^2$. The new term $-by^2$ represents **competition** when cy represents growth. Separation gives $\int dy/(cy - by^2) = \int dt$, and the y -integral is $1/c$ times $\ln \frac{y}{c-by}$. Substituting y_0 at $t = 0$ and taking exponentials produces $y/(c - by) = e^{ct}y_0/(c - by_0)$. As $t \rightarrow \infty$, y approaches $\frac{c}{b}$. That is the steady state where $cy - by^2 = 0$. The graph of y looks like an S, because it has an inflection point at $\frac{1}{2} \frac{c}{b}$.

In biology and chemistry, concentrations y and z react at a rate proportional to y times z . This is the **Law of Mass Action**. In a model equation $dy/dt = c(y)z$, the rate c depends on y . The MM equation is $dy/dt = -cy/(y + K)$. Separating variables yields $\int \frac{y+K}{y} dy = \int -c dt = -ct + C$.

- 1 $7e^t - 5$ 3 $(\frac{3}{2}x^2 + 1)^{1/3}$ 5 x 7 $e^{1-\cos t}$ 9 $(\frac{ct}{2} + \sqrt{y_0})^2$ 11 $y_\infty = 0; t = \frac{1}{by_0}$
- 15 $z = 1 + e^{-t}$, y is in 13 17 $ct = \ln 3, ct = \ln 9$
- 19 $b = 10^{-9}$, $c = 13 \cdot 10^{-3}$; $y_\infty = 13 \cdot 10^6$; at $y = \frac{c}{2b}$ (10) gives $\ln \frac{1}{b} = ct + \ln \frac{10^6}{c-10^6b}$ so $t = 1900 + \frac{\ln 12}{c} = 2091$
- 21 y^2 dips down and up (a valley) 23 $sc = 1 = sbr$ so $s = \frac{1}{c}, r = \frac{c}{b}$
- 25 $y = \frac{N}{1+e^{-Nt/(N-1)}}$; $T = \frac{\ln(N-1)}{N} \rightarrow 0$ 27 Dividing cy by $y + K > 1$ slows down y'
- 29 $\frac{dR}{dy} = \frac{cK}{(y+K)^2} > 0, \frac{cy}{y+K} \rightarrow c$
- 31 $\frac{dY}{dt} = \frac{-Y}{Y+1}$; multiply $e^{y/K} \frac{y}{K} = e^{-ct/K} e^{y_0/K} (\frac{y_0}{K})$ by K and take the K th power to reach (19)
- 33 $y' = (3-y)^2$; $\frac{1}{3-y} = t + \frac{1}{3}$; $y = 2$ at $t = \frac{2}{3}$
- 35 $Ae^t + D = Ae^t + B + Dt + t \rightarrow D = -1, B = -1; y_0 = A + B$ gives $A = 1$
- 37 $y \rightarrow 1$ from $y_0 > 0$, $y \rightarrow -\infty$ from $y_0 < 0$; $y \rightarrow 1$ from $y_0 > 0$, $y \rightarrow -1$ from $y_0 < 0$
- 39 $\int \frac{\cos y dy}{\sin y} = \int dt \rightarrow \ln(\sin y) = t + C = t + \ln \frac{1}{2}$. Then $\sin y = \frac{1}{2}e^t$ stops at 1 when $t = \ln 2$

- 2 $y dy = dt$ gives $\frac{1}{2}y^2 = t + C$. Then $C = \frac{1}{2}$ at $t = 0$. So $y^2 = 2t + 1$ and $y = \sqrt{2t + 1}$.
- 4 $\frac{dy}{y^2+1} = dx$ gives $\tan^{-1} y = x + C$. Then $C = 0$ at $x = 0$. So $y = \tan x$.
- 6 $\frac{dy}{\tan y} = \cos x dx$ gives $\ln(\sin y) = \sin x + C$. Then $C = \ln(\sin 1)$ at $x = 0$. After taking exponentials $\sin y = (\sin 1)e^{\sin x}$. No solution after $\sin y$ reaches 1 (at the point where $(\sin 1)e^{\sin x} = 1$).
- 8 $e^y dy = e^t dt$ so $e^y = e^t + C$. Then $C = e^e - 1$ at $t = 0$. After taking logarithms $y = \ln(e^t + e^e - 1)$.
- 10 $\frac{d(\ln y)}{d(\ln x)} = \frac{dy/y}{dx/x} = n$. Therefore $\ln y = n \ln x + C$. Therefore $y = (x^n)(e^C) = \text{constant times } x^n$.

- 12 $y' = by^2$ gives $y^{-2}dy = b dt$ and $-\frac{1}{y} = bt + C$. Then $C = -\frac{1}{2}$ at $t = 0$. Therefore $y = \frac{-1}{bt - \frac{1}{2}}$ which becomes infinite when $bt = \frac{1}{2}$ or $t = \frac{1}{2b}$.
- 14 (a) Compare $\frac{2}{1+e^{-t}}$ with $\frac{c}{b+de^{-ct}}$. In the exponent $c = 1$. Then $b = d = \frac{1}{2}$. Thus $y' = y - \frac{1}{2}y^2$ with $y_0 = 1$.
 (b) For $\frac{1}{1+e^{-t}}$ the exponent gives $c = 3$. Then also $b = d = 3$. Thus $y' = 3y - 3y^2$ with $y_0 = \frac{1}{2}$.
- 16 Equation (14) is $z = \frac{1}{c}(b + \frac{c-by_0}{y_0}e^{-ct})$. Turned upside down this is $y = \frac{c}{b+de^{-ct}}$ with $d = \frac{c-by_0}{y_0}$.
- 18 Correction: $u = \frac{y}{c-by}$. Then $\frac{du}{dt} = \frac{d}{dt}(\frac{y}{c-by}) = \frac{(c-by)\frac{dy}{dt} - y(-b\frac{dy}{dt})}{(c-by)^2} = \frac{(c-by)\frac{dy}{dt} + by\frac{dy}{dt}}{(c-by)^2} = \frac{c}{(c-by)^2}\frac{dy}{dt}$. Substitute $\frac{dy}{dt} = y(c-by)$ to obtain $\frac{du}{dt} = \frac{cy}{c-by} = cu$. So $u = u_0e^{ct}$.
- 20 $y' = y + y^2$ has $c = 1$ and $b = -1$ with $y_0 = 1$. Then $y(t) = \frac{1}{-1+2e^{-t}}$ by formula (12). The denominator is zero and y blows up when $2e^{-t} = 1$ or $t = \ln 2$.
- 22 If $u = \frac{1}{y^2}$ then $\frac{du}{dt} = \frac{-2y'}{y^3} = \frac{-2(cy-by^3)}{y^3} = -2cu + 2b$. The solution is $u = (u_0 - \frac{2b}{2c})e^{-2ct} + \frac{2b}{2c}$.
 Then $y = [(\frac{1}{y_0^2} - \frac{b}{c})e^{-2ct} + \frac{b}{c}]^{-1/2}$ solves the equation $y' = cy - by^3$ with "cubic competition".
 Another S-curve!
- 24 $y_0 = rY_0$ and $\frac{dY}{dT} = \frac{dy/r}{dt/s}$ so $(\frac{dY}{dT})_0 = \frac{s}{r}y'_0$.
- 26 At the middle of the S-curve $y = \frac{c}{2b}$ and $\frac{dy}{dt} = c(\frac{c}{2b}) - b(\frac{c}{2b})^2 = \frac{c^2}{4b}$. If b and c are multiplied by 10 then so is this slope $\frac{c^2}{4b}$, which becomes steeper.
- 28 If $\frac{cy}{y+K} = d$ then $cy = dy + dK$ and $y = \frac{dK}{c-d}$. At this steady state the maintenance dose replaces the aspirin being eliminated.
- 30 The rate $R = \frac{cy}{y+K}$ is a decreasing function of K because $\frac{dR}{dK} = \frac{-cy}{(y+K)^2}$.
- 34 $\frac{d[A]}{dt} = -r[A][B] = -r[A](b_0 - \frac{n}{m}(a_0 - [A]))$. The changes $a_0 - [A]$ and $b_0 - [B]$ are in the proportion m to n ; we solved for $[B]$.
- 36 To change $cy - by^2$ (with linear term) to $a^2 - x^2$ (no linear term), set $x = \sqrt{b}y - \frac{c}{2\sqrt{b}}$ and $a = \frac{c}{2\sqrt{b}}$.
 (We completed the square in $cy - by^2$.) Now match integrals: The factor $\frac{1}{2a}$ is $\frac{1}{c}$ times $\sqrt{b}$ (from $dx = \sqrt{b} dy$). The ratio $\frac{a+x}{a-x} = \frac{\sqrt{b}y}{\frac{c}{2} - \sqrt{b}y}$ is $\frac{y}{c-by}$.
- 38 The y line shows where y increases (by $y' = f(y)$) and where y decreases. Then the points where $f(y) = 0$ are either approached or left behind.
- 40 $y' = cy(1 - \frac{y}{K})$ agrees with $y' = cy - by^2$ if $K = \frac{c}{b}$. Then $y = K$ is the steady state where $y' = 0$ (this agrees with $y_\infty = \frac{c}{b}$). The inflection point is halfway: $y = \frac{K}{2}$ where $y' = c\frac{K}{2}(1 - \frac{1}{2}) = \frac{c}{4}K$ and $y'' = 0$.

6.6 Powers Instead of Exponentials (page 276)

The infinite series for e^x is $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$. Its derivative is e^x . The denominator $n!$ is called "n factorial" and is equal to $n(n-1)\dots(1)$. At $x = 1$ the series for e is $1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots$.

To match the original definition of e , multiply out $(1 + 1/n)^n = 1 + n(\frac{1}{n}) + \frac{n(n-1)}{2}(\frac{1}{n})^2$ (first three terms). As $n \rightarrow \infty$ those terms approach $1 + 1 + \frac{1}{2}$ in agreement with e . The first three terms of $(1 + x/n)^n$ are $1 + n(\frac{x}{n}) + \frac{n(n-1)}{2}(\frac{x}{n})^2$. As $n \rightarrow \infty$ they approach $1 + x + \frac{1}{2}x^2$ in agreement with e^x . Thus $(1 + x/n)^n$ approaches e^x . A quicker method computes $\ln(1 + x/n)^n \approx nx$ (first term only) and takes the exponential.

Compound interest (n times in one year at annual rate x) multiplies by $(1 + \frac{x}{n})^n$. As $n \rightarrow \infty$, continuous

compounding multiplies by e^x . At $x = 10\%$ with continuous compounding, \$1 grows to $e^{.1} \approx \$1.105$ in a year.

The difference equation $y(t+1) = ay(t)$ yields $y(t) = a^t$ times y_0 . The equation $y(t+1) = ay(t) + s$ is solved by $y = a^t y_0 + s[1 + a + \cdots + a^{t-1}]$. The sum in brackets is $\frac{1-a^{t+1}}{1-a}$ or $\frac{a^{t+1}-1}{a-1}$. When $a = 1.08$ and $y_0 = 0$, annual deposits of $s = 1$ produce $y = \frac{1.08^t - 1}{.08}$ after t years. If $a = \frac{1}{2}$ and $y_0 = 0$, annual deposits of $s = 6$ leave $12(1 - \frac{1}{2^t})$ after t years, approaching $y_\infty = 12$. The steady equation $y_\infty = ay_\infty + s$ gives $y_\infty = s/(1-a)$.

When i = interest rate per period, the value of $y_0 = \$1$ after N periods is $y(N) = (1+i)^N$. The deposit to produce $y(N) = 1$ is $y_0 = (1+i)^{-N}$. The value of $s = \$1$ deposited after each period grows to $y(N) = \frac{1}{i}((1+i)^N - 1)$. The deposit to reach $y(N) = 1$ is $s = \frac{1}{i}(1 - (1+i)^{-N})$.

Euler's method replaces $y' = cy$ by $\Delta y = cy\Delta t$. Each step multiplies y by $1 + c\Delta t$. Therefore y at $t = 1$ is $(1 + c\Delta t)^{1/\Delta t} y_0$, which converges to $y_0 e^c$ as $\Delta t \rightarrow 0$. The error is proportional to Δt , which is too large for scientific computing.

- 1 $1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$ 3 $1 \pm x + \frac{x^2}{2} \pm \frac{x^3}{6} + \cdots$ 5 1050.62; 1050.95; 1051.25
 7 $1 + n(\frac{-1}{n}) + \frac{n(n+1)}{2}(\frac{-1}{n})^2 \rightarrow 1 - 1 + \frac{1}{2}$ 9 square of $(1 + \frac{1}{n})^n$; set $N = 2n$
 11 Increases; $\ln(1 + \frac{1}{x}) - \frac{1}{x+1} > 0$ 13 $y(3) = 8$ 15 $y(t) = 4(3^t)$ 17 $y(t) = t$
 19 $y(t) = \frac{1}{2}(3^t - 1)$ 21 $s(\frac{a^t-1}{a-1})$ if $a \neq 1$; st if $a = 1$ 23 $y_0 = 6$ 25 $y_0 = 3$
 27 $-2, -10, -26 \rightarrow -\infty; -5, -\frac{17}{2}, -\frac{41}{4} \rightarrow -12$ 29 $P = \frac{b}{c+d}$ 31 10.38% 33 $100(1.1)^{20} = \$673$
 35 $\frac{100,000(.1/12)}{1-(1+.1/12)^{-240}} = 965$ 37 $\frac{1000}{.1}(1.1^{20} - 1) = 57,275$ 39 $y_\infty = 1500$ 41 $2; (\frac{53}{52})^{52} = 2.69; e$
 43 $1.0142^{12} = 1.184 \rightarrow$ Visa charges 18.4%

- 2 $y = 1 + 2x + \frac{1}{2}(2x)^2 + \frac{1}{6}(2x)^3 + \cdots$ Integrate each term and multiply by 2 to find the next term.
 4 A larger series is $1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 3$. This is greater than $1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots = e$.
 6 $\ln(1 - \frac{1}{n})^n = n \ln(1 - \frac{1}{n}) \approx n(-\frac{1}{n}) = -1$. Take exponentials: $(1 - \frac{1}{n})^n \approx e^{-1}$. Similarly $\ln(1 + \frac{2}{n})^n = n \ln(1 + \frac{2}{n}) \approx n(\frac{2}{n}) = 2$. Take exponentials: $(1 + \frac{2}{n})^n \approx e^2$.
 8 The exact sum is $e^{-1} \approx .37$ (Problem 6). After five terms $1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{9}{24} = .375$.
 10 By the quick method $\ln(1 + \frac{1}{n^2})^n \approx n(\frac{1}{n^2}) \rightarrow 0$. So $(1 + \frac{1}{n^2})^n \rightarrow e^0 = 1$. Similarly $\ln(1 + \frac{1}{n})^{n^2} \approx n^2(\frac{1}{n}) \rightarrow \infty$ so $(1 + \frac{1}{n})^{n^2} \rightarrow \infty$.
 12 Under the graph of $\frac{1}{t}$, the area from 1 to $1 + \frac{1}{x}$ is $\ln(1 + \frac{1}{x})$. The rectangle inside this area has base $\frac{1}{x}$ and height $\frac{1}{1+\frac{1}{x}}$. Its area is $\frac{1}{x+1}$ so this is below $\ln(1 + \frac{1}{x})$.
 14 $y(0) = 0, y(1) = 1, y(2) = 3, y(3) = 7$ (and $y(n) = 2^n - 1$). 16 $y(t) = (\frac{1}{2})^t$.
 18 $y(t) = t$ (Notice that $a = 1$). 20 $y(t) = 3^t + s[\frac{3^t-1}{2}]$. 22 $y(t) = 5a^t + s[\frac{a^t-1}{a-1}]$.
 24 Ask for $\frac{1}{2}y(0) - 6 = y(0)$. Then $y(0) = -12$. 26 Ask for $-\frac{1}{2}y(0) + 6 = y(0)$. Then $y(0) = 4$.
 28 If $-1 < a < 1$ then $\frac{1-a^t}{1-a}$ approaches $\frac{1}{1-a}$.
 30 The equation $-dP(t+1) + b = cP(t)$ becomes $-2P(t+1) + 8 = P(t)$ or $P(t+1) = -\frac{1}{2}P(t) + 4$. Starting from $P(0) = 0$ the solution is $P(t) = 4[\frac{(-\frac{1}{2})^t - 1}{-\frac{1}{2} - 1}] = \frac{8}{3}(1 - (-\frac{1}{2})^t) \rightarrow \frac{8}{3}$.
 32 $(1 + \frac{.10}{365})^{365} = 1.105156 \cdots$ (Compare with $e^{.1} \approx 1 + .1 + \frac{1}{2}(.1)^2 = 1.105$.) The effective rate is 5.156%.
 34 Present value = $\$1,000 (1.1)^{-20} \approx \148.64 .
 36 Correction to formulas 5 and 6 on page 273: Change .05n to .05/n. In this problem $n = 12$ and $N = 6(12) = 72$ months and .05 becomes .1 in the loan formula: $s = \$10,000 (.1)/12[1 - (1 + \frac{1}{12})^{-72}] \approx \185 .

- 38 Solve $\$1000 = \$8000 \left[\frac{1}{1 - (1.1)^{-n}} \right]$ for n . Then $1 - (1.1)^{-n} = .8$ or $(1.1)^{-n} = .2$. Thus $1.1^n = 5$ and $n = \frac{\ln 5}{\ln 1.1} \approx 17$ years.
- 40 The interest is $(.05)1000 = \$50$ in the first month. You pay \$60. So your debt is now $\$1000 - \$10 = \$990$. Suppose you owe $y(t)$ after month t , so $y(0) = \$1000$. The next month's interest is $.05y(t)$. You pay \$60. So $y(t+1) = 1.05y(t) - 60$. After 12 months $y(12) = (1.05)^{12}1000 - 60 \left[\frac{(1.05)^{12} - 1}{1.05 - 1} \right]$. This is also $\frac{60}{.05} + (1000 - \frac{60}{.05})(1.05)^{12} \approx \841 .
- 42 Compounding n times in a year at 100% per year gives $(1 + \frac{1}{n})^n$. Its logarithm is $n \ln(1 + \frac{1}{n}) \approx n[\frac{1}{n} - \frac{1}{2n^2}] = 1 - \frac{1}{2n}$. Therefore $(1 + \frac{1}{n})^n \approx e(e^{-1/2n}) \approx e(1 - \frac{1}{2n})$.
- 44 Use the loan formula with $.09/n$ not $.09n$: payments $s = 80,000 \frac{.09/12}{[1 - (1 + \frac{.09}{12})^{-360}]} \approx \643.70 . Then 360 payments equal \$231,732.

6.7 Hyperbolic Functions (page 280)

$\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh^2 x - \sinh^2 x = 1$. Their derivatives are $\sinh x$ and $\cosh x$ and zero. The point $(x, y) = (\cosh t, \sinh t)$ travels on the hyperbola $x^2 - y^2 = 1$. A cable hangs in the shape of a catenary $y = a \cosh \frac{x}{a}$.

The inverse functions $\sinh^{-1} x$ and $\tanh^{-1} x$ are equal to $\ln[x + \sqrt{x^2 + 1}]$ and $\frac{1}{2} \ln \frac{1+x}{1-x}$. Their derivatives are $1/\sqrt{x^2 + 1}$ and $\frac{1}{1-x^2}$. So we have two ways to write the antiderivative. The parallel to $\cosh x + \sinh x = e^x$ is Euler's formula $\cos x + i \sin x = e^{ix}$. The formula $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ involves imaginary exponents. The parallel formula for $\sin x$ is $\frac{1}{2i}(e^{ix} - e^{-ix})$.

- 1 $e^x, e^{-x}, \frac{e^{2x} - e^{-2x}}{4} = \frac{1}{2} \sinh 2x$ 7 $\sinh nx$ 9 $3 \sinh(3x + 1)$ 11 $\frac{-\sinh x}{\cosh^2 x} = -\tanh x \operatorname{sech} x$
- 13 $4 \cosh x \sinh x$ 15 $\frac{x}{\sqrt{x^2 + 1}} (\operatorname{sech} \sqrt{x^2 + 1})^2$ 17 $6 \sinh^5 x \cosh x$
- 19 $\cosh(\ln x) = \frac{1}{2}(x + \frac{1}{x}) = 1$ at $x = 1$ 21 $\frac{5}{13}, \frac{13}{5}, -\frac{12}{5}, -\frac{13}{12}, -\frac{5}{12}$ 23 $0, 0, 1, \infty, \infty$
- 25 $\frac{1}{2} \sinh(2x + 1)$ 27 $\frac{1}{3} \cosh^3 x$ 29 $\ln(1 + \cosh x)$ 31 e^x
- 33 $\int y dx = \int \sinh t (\sinh t dt); A = \frac{1}{2} \sinh t \cosh t - \int y dx; A' = \frac{1}{2}; A = 0$ at $t = 0$ so $A = \frac{1}{2}t$.
- 41 $e^y = x + \sqrt{x^2 + 1}, y = \ln[x + \sqrt{x^2 + 1}]$ 47 $\frac{1}{4} \ln \left| \frac{2+x}{2-x} \right|$ 49 $\sinh^{-1} x$ (see 41) 51 $-\operatorname{sech}^{-1} x$
- 53 $\frac{1}{2} \ln 3; \infty$ 55 $y(x) = \frac{1}{c} \cosh cx; \frac{1}{c} \cosh cL - \frac{1}{c}$
- 57 $y'' = y - 3y^2; \frac{1}{2}(y')^2 = \frac{1}{2}y^2 - y^3$ is satisfied by $y = \frac{1}{2} \operatorname{sech}^2 \frac{x}{2}$

- 2 $\frac{d}{dx} \left(\frac{e^x + e^{-x}}{2} \right) = \frac{e^x - e^{-x}}{2} = \sinh x; \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right) = \frac{e^x + e^{-x}}{2} = \cosh x$.
- 4 $\frac{d}{dx} \left(\frac{\sinh x}{\cosh x} \right) = \frac{(\cosh x)^2 - (\sinh x)^2}{(\cosh x)^2} = \frac{1}{(\cosh x)^2} = \operatorname{sech}^2 x$.
- 6 The factor $\frac{1}{2}$ should be removed from Problem 5. Then the derivative of Problem 5 is $2 \cosh x \sinh x + 2 \sinh x \cosh x = 2 \sinh 2x$. Therefore $\sinh 2x = 2 \sinh x \cosh x$ (similar to $\sin 2x$).
- 8 $\left(\frac{e^x - e^{-x}}{2} \right) \left(\frac{e^y + e^{-y}}{2} \right) + \left(\frac{e^x + e^{-x}}{2} \right) \left(\frac{e^y - e^{-y}}{2} \right) = \frac{1}{4} (2e^{x+y} - 2e^{-x-y}) = \sinh(x+y)$. The x derivative gives $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.
- 10 $2x \cosh x^2$ 12 $\sinh(\ln x) = \frac{1}{2}(e^{\ln x} - e^{-\ln x}) = \frac{1}{2}(x - \frac{1}{x})$ with derivative $\frac{1}{2}(1 + \frac{1}{x^2})$.
- 14 $\cosh^2 x - \sinh^2 x = 1$ with derivative zero.
- 16 $\frac{1 + \tanh x}{1 - \tanh x} = e^{2x}$ by the equation following (4). Its derivative is $2e^{2x}$. More directly the quotient rule gives

- $$\frac{(1-\tanh x)\operatorname{sech}^2 x + (1+\tanh x)\operatorname{sech}^2 x}{(1-\tanh x)^2} = \frac{2\operatorname{sech}^2 x}{(1-\tanh x)^2} = \frac{2}{(\cosh x - \sinh x)^2} = \frac{2}{e^{-2x}} = 2e^{2x}.$$
- 18** $\frac{d}{dx} \ln u = \frac{du/dx}{u} = \frac{\operatorname{sech} x \tanh x - \operatorname{sech}^2 x}{\operatorname{sech} x + \tanh x}$. Because of the minus sign we do not get $\operatorname{sech} x$. The integral of $\operatorname{sech} x$ is $\sin^{-1}(\tanh x) + C$.
- 20** $\operatorname{sech} x = \sqrt{1 - (\frac{3}{5})^2} = \frac{4}{5}$, $\cosh x = \frac{5}{4}$, $\sinh x = \sqrt{(\frac{5}{4})^2 - 1} = \frac{3}{4}$, $\coth x = \frac{\sinh x}{\cosh x} = \frac{3}{5}$, $\operatorname{csch} x = \frac{4}{3}$.
- 22** $\cosh x = \sqrt{(2)^2 + 1} = \sqrt{5}$, $\tanh x = \frac{2}{\sqrt{5}}$, $\operatorname{csch} x = \frac{1}{2}$, $\operatorname{sech} x = \frac{1}{\sqrt{5}}$, $\coth x = \frac{\sqrt{5}}{2}$.
- 24** $\sinh(\ln 5) = \frac{e^{\ln 5} - e^{-\ln 5}}{2} = \frac{5 - \frac{1}{5}}{2} = \frac{12}{5}$; $\tanh(2 \ln 4) = \frac{e^{2 \ln 4} - e^{-2 \ln 4}}{e^{2 \ln 4} + e^{-2 \ln 4}} = \frac{16 - \frac{1}{16}}{16 + \frac{1}{16}} = \frac{255}{257}$.
- 26** $\int x \cosh(x^2) dx = \frac{1}{2} \sinh(x^2) + C$. **28** $\frac{1}{3}(\tanh x)^3 + C$.
- 30** $\int \coth x dx = \int \frac{\cosh x}{\sinh x} dx = \ln(\sinh x) + C$. **32** $\sinh x + \cosh x = e^x$ and $\int e^{nx} dx = \frac{1}{n} e^{nx} + C$.
- 34** $y = \tanh x$ is an odd function, with asymptote $y = -1$ as $x \rightarrow -\infty$ and $y = +1$ as $x \rightarrow +\infty$. The inflection point is $(0,0)$.
- 36** $y = \operatorname{sech} x$ looks like a bell-shaped curve with $y_{\max} = 1$ at $x = 0$. The x axis is the asymptote. But note that y decays like $2e^{-x}$ and not like e^{-x^2} .
- 38** To define $y = \cosh^{-1} x$ we require $x \geq 1$. Select the *positive* y (there are two y 's so strictly there is no inverse). For large values, $\cosh y$ is close to $\frac{1}{2}e^y$ so $\cosh^{-1} x$ is close to $\ln 2x$.
- 40** $\frac{1}{2} \ln(\frac{1+x}{1-x})$ approaches $+\infty$ as $x \rightarrow 1$ and $-\infty$ as $x \rightarrow -1$. The function is *odd* (so is the $\tanh$ function). The graph is an S curve rotated by 90° .
- 42** The quadratic equation for e^y has solution $e^y = x \pm \sqrt{x^2 - 1}$. Choose the plus sign so $y \rightarrow \infty$ as $x \rightarrow \infty$. Then $y = \ln(x + \sqrt{x^2 - 1})$ is another form of $y = \cosh^{-1} x$.
- 44** The x derivative of $x = \sinh y$ is $1 = \cosh y \frac{dy}{dx}$. Then $\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}$ = slope of $\sinh^{-1} x$.
- 46** The x derivative of $x = \operatorname{sech} y$ is $1 = -\operatorname{sech} y \tanh y \frac{dy}{dx}$. Then $\frac{dy}{dx} = \frac{-1}{\operatorname{sech} y \tanh y} = \frac{-1}{x\sqrt{1-x^2}}$.
- 48** Set $x = au$ and $dx = a du$ to reach $\int \frac{a du}{a^2(1-u^2)} = \frac{1}{a} \tanh^{-1} u = \frac{1}{a} \tanh^{-1} \frac{x}{a} + C$.
- 50** Not hyperbolic! Just $\int (x^2 + 1)^{-1/2} x dx = (x^2 + 1)^{1/2} + C$.
- 52** Not hyperbolic! $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$.
- 54** (a) $\frac{dv}{dt} = (\sqrt{g})^2 \operatorname{sech}^2 \sqrt{g} t = g(1 - \tanh^2 \sqrt{g} t) = g - v^2$. (b) $\int \frac{dv}{g-v^2} = \int dt$ gives (by Problem 48) $\frac{1}{\sqrt{g}} \tanh^{-1} \frac{v}{\sqrt{g}} = t$ or $\tanh^{-1} \frac{v}{\sqrt{g}} = \sqrt{g} t$ or $\frac{v}{\sqrt{g}} = \tanh \sqrt{g} t$. (c) $f(t) = \int \sqrt{g} \tanh \sqrt{g} t dt = \int \frac{\sinh \sqrt{g} t}{\cosh \sqrt{g} t} \sqrt{g} dt = \ln(\cosh \sqrt{g} t) + C$.
- 56** Change to $dx = \frac{dW}{\frac{1}{2}W^2 - W} = -\frac{dW}{2-W} - \frac{dW}{W}$ and integrate: $x = \ln(2-W) - \ln W = \ln(\frac{2-W}{W})$. Then $\frac{2-W}{W} = e^x$ and $W = \frac{2}{1+e^x}$. (Note: The text suggests $W - 2$ but that is negative. Writing $\frac{2}{1+e^x}$ as $e^{-x/2} \operatorname{sech} \frac{x}{2}$ is not simpler.)
- 58** $\cos ix = \frac{1}{2}(e^{i(ix)} + e^{-i(ix)}) = \frac{1}{2}(e^{-x} + e^x) = \cosh x$. Then $\cos i = \cosh 1 = \frac{e+e^{-1}}{2}$ (real!).
- 60** The derivative of $e^{ix} = \cos x + i \sin x$ is $ie^{ix} = i(\cos x + i \sin x)$ on the left side and $\frac{d}{dx} \cos x + i \frac{d}{dx} \sin x$ on the right side. Comparing we again find $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$.