# CHAPTER 7 TECHNIQUES OF INTEGRATION

## 7.1 Integration by Parts (page 287)

Integration by parts is the reverse of the **product rule**. It changes $\int u\,dv$ into **uv** minus $\int$ **v du**. In case $u = x$ and $dv = e^{2x}dx$, it changes $\int xe^{2x}dx$ to $\frac{1}{2}xe^{2x}$ minus $\int \frac{1}{2}e^{2x}dx$. The definite integral $\int_0^2 xe^{2x}dx$ becomes $\frac{3}{4}e^4$ minus $\frac{1}{4}$.

In choosing $u$ and $dv$, the **derivative** of $u$ and the **integral** of $dv/dx$ should be as simple as possible. Normally $\ln x$ goes into **u** and $e^x$ goes into **v**. Prime candidates are $u = x$ or $x^2$ and $v = \sin x$ or $\cos x$ or $e^x$. When $u = x^2$ we need **two** integrations by parts. For $\int \sin^{-1} x\,dx$, the choice $dv = dx$ leads to $x \sin^{-1} x$ minus $\int x\,dx/\sqrt{1-x^2}$.

If $U$ is the unit step function, $dU/dx = \delta$ is the unit **delta** function. The integral from $-A$ to $A$ is $U(A) - U(-A) = 1$. The integral of $v(x)\delta(x)$ equals $v(0)$. The integral $\int_{-1}^{1} \cos x\ \delta(x)dx$ equals **1**. In engineering, the balance of forces $-dv/dx = f$ is multiplied by a displacement $u(x)$ and integrated to give a balance of **work**.

**1** $-x\cos x + \sin x + C$ **3** $-xe^{-x} - x + C$ **5** $x^2 \sin x + 2x\cos x - 2\sin x + C$

**7** $\frac{1}{2}(2x+1)\ln(2x+1) + C$ **9** $\frac{1}{2}e^x(\sin x - \cos x) + C$ **11** $\frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx) + C$

**13** $\frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + C$ **15** $x(\ln x)^2 - 2x\ln x + 2x + C$ **17** $x \sin^{-1} x + \sqrt{1-x^2} + C$

**19** $\frac{1}{2}(x^2+1)\tan^{-1} x - \frac{x}{2} + C$ **21** $x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\cos x + C$

**23** $e^x(x^3 - 3x^2 + 6x - 6) + C$ **25** $x \tan x + \ln(\cos x) + C$ **27** $-1$ **29** $-\frac{3}{4}e^{-2} + \frac{1}{4}$ **31** $-2$

**33** $3\ln 10 - 6 + 2\tan^{-1} 3$ **35** $u = x^n, v = e^x$ **37** $u = x^n, v = \sin x$ **39** $u = (\ln x)^n, v = x$

**41** $u = x \sin x, v = e^x \to \int e^x \sin x\,dx$ in **9** and $-\int x\cos x\, e^x dx$. Then $u = -x\cos x, v = e^x \to \int e^x \cos x\,dx$ in **10** and $-\int x \sin x\, e^x dx$ (move to left side): $\frac{e^x}{2}(x\sin x - x\cos x + \cos x)$. Also try $u = xe^x, v = -\cos x$.

**43** $\int \frac{1}{2}u \sin u\,du = \frac{1}{2}(\sin u - u\cos u) = \frac{1}{2}(\sin x^2 - x^2 \cos x^2)$; odd **45** $3\cdot$ step function; $3e^x\cdot$ step function

**49** $0; x\delta(x)] - \int \delta(x)dx = -1; v(x)\delta(x)] - \int v(x)\delta(x)dx$ **51** $v(x) = \int_x^1 f(x)dx$

**53** $u(x) = \frac{1}{k}\int_0^x v(x)dx; \frac{1}{k}(\frac{x}{2} - \frac{x^3}{6}); \frac{x}{k}$ for $x \le \frac{1}{2}, \frac{1}{k}(2x - x^2 - \frac{1}{4})$ for $x \ge \frac{1}{2}; \frac{x}{k}$ for $x \le \frac{1}{2}, \frac{1}{2k}$ for $x \ge \frac{1}{2}$.

**55** $u = x^2, v = -\cos x \to -x^2 \cos x - (2x)\sin x - \int 2\sin x\,dx$ **57** Compare **23**

**59** $uw']_0^1 - \int_0^1 u'w' - u'w]_0^1 + \int_0^1 u'w' = [uw' - u'w]_0^1$

**61** No mistake: $e^x \cosh x - e^x \sin hx = 1$ is part of the constant $C$

**2** $uv - \int v\,du = x(\frac{1}{4}e^{4x}) - \int \frac{1}{4}e^{4x}dx = e^{4x}(\frac{x}{4} - \frac{1}{16}) + C$

**4** $uv - \int v\,du = x(\frac{1}{3}\sin 3x) - \int \frac{1}{3}\sin 3x\,dx = \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x + C$

**6** $uv - \int v\,du = (\ln x)\frac{x^2}{2} - \int \frac{x^2}{2}\frac{dx}{x} = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

**8** $uv - \int v\,du = x^2(\frac{1}{4}e^{4x}) - \int(\frac{1}{4}e^{4x})2x\,dx =$ (Problem 2) $e^{4x}(\frac{x^2}{4} - \frac{x}{8} + \frac{1}{32}) + C$

**10** $\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$. Another integration by parts produces $e^x(\sin x + \cos x) - \int e^x \cos x\,dx$. Move the last integral to the left side and divide by 2: answer $\frac{1}{2}e^x(\sin x + \cos x) + C$.

**12 Not by parts.** Substitute $u = x^2, du = 2x\,dx : \int \frac{1}{2}e^{-u}du = -\frac{1}{2}e^{-u} = -\frac{1}{2}e^{-x^2} + C$.

**14** $\int \cos(\ln x)dx = uv - \int v du = \cos(\ln x)x + \int x \sin(\ln x)\frac{1}{x}dx =$ **again by parts gives** $\cos(\ln x)x + \sin(\ln x)x - \int x\cos(\ln x)\frac{1}{x}dx$. Move the last integral to the left and divide by 2: answer $\frac{x}{2}(\cos(\ln x) + \sin(\ln x)) + C$.

**16** $uv - \int v\,du = (\ln x)\frac{x^3}{3} - \int \frac{x^3}{3}\frac{dx}{x} = (\ln x)\frac{x^3}{3} - \frac{x^3}{9} + C$.

**18** $uv - \int v\,du = \cos^{-1}(2x)x + \int x\frac{2\,dx}{\sqrt{1-(2x)^2}} = x\cos^{-1}(2x) - \frac{1}{2}(1-4x^2)^{1/2} + C$.

**20** $\int x^2 \sin x\,dx = x^2(-\cos x) + \int \cos x(2x\,dx) =$ **again by parts gives** $-x^2\cos x + (\sin x)2x - \int \sin x(2\,dx) =$ answer: $-x^2\cos x + 2x\sin x + 2\cos x + C$.

**22** $uv - \int v\,du = x^3(-\cos x) + \int(\cos x)3x^2dx =$ (use Problem 5) $= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$.

**24** $uv - \int v\,du = \sec^{-1}x(\frac{x^2}{2}) - \int \frac{x^2}{2}\frac{dx}{|x|\sqrt{1-x^2}} = \frac{x^2}{2}\sec^{-1}x + \frac{1}{2}\sqrt{1-x^2} + C$.

**26** $uv - \int v\,du = x\cosh x - \int \cosh x\,dx = x\cosh x - \sinh x + C$.

**28** $\int_0^1 e^{\sqrt{x}}dx = \int_{u=0}^1 e^u(2u\,du) = 2e^u(u-1)]_0^1 = \mathbf{2}$. **30** $\ln(x^2) = 2\ln x; \int_1^e 2\ln x\,dx = [2(x\ln x - x)]_1^e = \mathbf{2}$.

**32** $\int_{-\pi}^{\pi} x\sin x\,dx = [\sin x - x\cos x]_{-\pi}^{\pi} = \mathbf{2\pi}$.

**34** $\int_0^{\pi/2} x^2\sin x\,dx =$ (Problem 20) $[-x^2\cos x + 2x\sin x + 2\cos x]_0^{\pi/2} = \pi - \mathbf{2}$.

**36** $\int x^n e^{ax}dx = x^n\frac{e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}dx$. **38** $\int x^n \sin x\,dx = -x^n\cos x + n\int x^{n-1}\cos x\,dx$.

**40** $\int x(\ln x)^n dx = (\ln x)^n\frac{x^2}{2} - \int \frac{x^2}{2}n(\ln x)^{n-1}\frac{dx}{x} = \frac{x^2}{2}(\ln x)^n - \frac{n}{2}\int x(\ln x)^{n-1}dx$.

**42** Try $u = \tan^{-1}x$ and $dv = xe^x\,dx$ so $v = (x-1)e^x$. Then $\int v\,du = \int \frac{x-1}{1+x^2}e^x\,dx$. I believe this cannot be done in closed form; that is true for $\int \frac{e^x}{x}dx$.

**44** (a) $e^0 = \mathbf{1}$; (b) $\mathbf{v(0)}$ (c) $\mathbf{0}$ (limits do not enclose zero).

**46** $\int_{-1}^1 \delta(2x)dx = \int_{u=-2}^2 \delta(u)\frac{du}{2} = \mathbf{\frac{1}{2}}$. Apparently $\delta(2x)$ equals $\frac{1}{2}\delta(x)$; both are zero for $x \neq 0$.

**48** $\int_0^1 \delta(x-\frac{1}{2})dx = \int_{-1/2}^{1/2}\delta(u)du = \mathbf{1}; \int_0^1 e^x\delta(x-\frac{1}{2})dx = \int_{-1/2}^{1/2} e^{u+\frac{1}{2}}\delta(u)du = \mathbf{e^{1/2}}; \delta(x)\delta(x-\frac{1}{2}) = \mathbf{0}$.

**50** $\int_{-1}^1 U(x)\frac{dU}{dx}dx =$ (directly) $[\frac{1}{2}(U(x))^2]]_0^1 = \mathbf{\frac{1}{2}}$.

**52** $-\frac{dv}{dx} = x$ gives $v = -\frac{x^2}{2} + C = -\frac{x^2}{2} + \frac{1}{2}; -\frac{dv}{dx} = U(x-\frac{1}{2})$ gives a change in slope at $x = \frac{1}{2}$ :
$v = C$ for $x \le \frac{1}{2}$ and $v = C - (x-\frac{1}{2})$ for $x \ge \frac{1}{2}$; take $\mathbf{C = \frac{1}{2}}$ to make $v(1) = 0$;
$-\frac{dv}{dx} = \delta(x-\frac{1}{2})$ gives $v = C$ for $x < \frac{1}{2}$ and $v = C - 1$ for $x > \frac{1}{2}$; take $\mathbf{C = 1}$ to make $v(1) = 0$.

**54** $\frac{\Delta U}{\Delta x} = \frac{1}{\Delta x}$ over the interval from $x = -\Delta x$ to $x = 0$. Elsewhere $\Delta U = 0$. The area under the graph is $(\frac{1}{\Delta x})\Delta x = \mathbf{1}$. As $\Delta x \to 0$ the area is tall and thin. In the limit $\int \delta(x)dx = 1$.

**56** $(-1)^n\int \frac{d^nu}{dx^n}v_{(n-1)}dx = (-1)^n\frac{d^nu}{dx^n}v_{(n)} + (-1)^{n+1}\int \frac{d^{n+1}u}{dx^{n+1}}v_{(n)}dx$.

**58** $\int_0^x f'(t)dt = [uv]_0^x - \int_0^x v\,du = [f'(t)(t-x)]_0^x + \int_0^x (x-t)f''(t)dt = xf'(0) + \int_0^x (x-t)f''(t)dt$.

**60** $A = \int_1^e \ln x\,dx = [x\ln x - x]_1^e = \mathbf{1}$ is the area under $y = \ln x$. $B = \int_0^1 e^y dy = \mathbf{e - 1}$ is the area to the left of $y = \ln x$. Together the area of the rectangle is $1 + (e-1) = e$.

**62** The derivative is $C(ae^{ax}\cos bx - be^{ax}\sin bx) + D(ae^{ax}\sin bx + be^{ax}\cos bx)$. This equals $e^{ax}\cos bx$ if $Ca + Db = 1$ and $-Cb + Da = 0$. These two equations give $C = \mathbf{\frac{a}{a^2+b^2}}$ and $D = \mathbf{\frac{b}{a^2+b^2}}$. Knowing the correct form in advance seems easier than integrating.

## 7.2 Trigonometric Integrals (page 293)

To integrate $\sin^4 x\ \cos^3 x$, replace $\cos^2 x$ by $\mathbf{1 - \sin^2 x}$. Then $(\sin^4 x - \sin^6 x)\cos\ x\,dx$ is $(\mathbf{u^4 - u^6})du$. In terms of $u = \sin x$ the integral is $\mathbf{\frac{1}{5}u^5 - \frac{1}{7}u^7}$. This idea works for $\sin^m x\ \cos^n x$ if $m$ or $n$ is **odd**.

If both $m$ and $n$ are **even,** one method is integration by **parts.** For $\int \sin^4 x\,dx$, split off $dv = \sin x\,dx$.

Then $-\int v\,du$ is $\int 3\sin^2 x\cos^2 x$. Replacing $\cos^2 x$ by $1-\sin^2 x$ creates a new $\sin^4 x\,dx$ that combines with the original one. The result is a *reduction* to $\int \sin^2 x\,dx$, which is known to equal $\frac{1}{2}(x-\sin x\cos x)$.

The second method uses the double-angle formula $\sin^2 x = \frac{1}{2}(1-\cos 2x)$. Then $\sin^4 x$ involves $\cos^2 2x$. Another doubling comes from $\cos^2 2x = \frac{1}{2}(1+\cos 4x)$. The integral contains the sine of $4x$.

To integrate $\sin 6x\cos 4x$, rewrite it as $\frac{1}{2}\sin 10x + \frac{1}{2}\sin 2x$. The integral is $-\frac{1}{20}\cos 10x - \frac{1}{4}\cos 2x$. The definite integral from 0 to $2\pi$ is zero. The product $\cos px\ \cos qx$ is written as $\frac{1}{2}\cos(p+q)x + \frac{1}{2}\cos(p-q)x$. Its integral is also zero, except if $p=q$ when the answer is $\pi$.

With $u=\tan x$, the integral of $\tan^9 x\ \sec^2 x$ is $\frac{1}{10}\tan^{10}x$. Similarly $\int \sec^9 x(\sec\ x\ \tan\ x\,dx) = \frac{1}{10}\sec^{10}x$. For the combination $\tan^m x\ \sec^n x$ we apply the identity $\tan^2 x = 1+\sec^2 x$. After reduction we may need $\int \tan\ x\,dx = -\ln\cos x$ and $\int \sec\ x\,dx = \ln(\sec x+\tan x)$.

**1** $\int(1-\cos^2 x)\sin x\,dx = -\cos x + \frac{1}{3}\cos^3 x + C$ **3** $\frac{1}{2}\sin^2 x + C$
**5** $\int(1-u^2)^2u^2(-du) = -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$ **7** $\frac{2}{3}(\sin x)^{3/2} + C$
**9** $\frac{1}{8}\int \sin^3 2x\,dx = \frac{1}{16}(-\cos 2x + \frac{1}{3}\cos^3 2x) + C$ **11** $\frac{\pi}{2}$ **13** $\frac{1}{3}(\frac{3x}{2} + \frac{\sin 6x}{4}) + C$
**15** $x + C$ **17** $\frac{1}{6}\cos^5 x\sin x + \frac{5}{6}\int \cos^4 x\,dx$; use equation (5)
**19** $\int_0^{\pi/2}\cos^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2}\cos^{n-2}x\,dx = \cdots = \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{1}{2}\int_0^{\pi/2}dx$
**21** $I = -\sin^{n-1}x\cos x + (n-1)\int\sin^{n-2}x\cos^2 x\,dx = -\sin^{n-1}x\cos x + (n-1)\int\sin^{n-2}x\,dx - (n-1)I.$
So $nI = -\sin^{n-1}x\ \cos x + (n-1)\int\sin^{n-2}x\,dx.$
**23** $0,+,0,0,0,-$ **25** $-\frac{2}{3}\cos^3 x, 0$ **27** $-\frac{1}{2}(\frac{\cos 2x}{2} + \frac{\cos 200x}{200}), 0$ **29** $\frac{1}{2}(\frac{\sin 200x}{200} + \frac{\sin 2x}{2}), 0$
**31** $-\frac{1}{2}\cos x, 0$ **33** $\int_0^\pi x\sin x\,dx = \int_0^\pi A\sin^2 x\,dx \to A = 2$ **35** Sum = zero = $\frac{1}{2}$ (left + right)
**37** $p$ is even **39** $p-q$ is even **41** $\sec x + C$ **43** $\frac{1}{3}\tan^3 x + C$ **45** $\frac{1}{3}\sec^3 x + C$
**47** $\frac{1}{3}\tan^3 x - \tan x + x + C$ **49** $\ln|\sin x| + C$ **51** $\frac{1}{2\cos^2 x} + C$ **53** $A = \sqrt{2}, -\sqrt{2}\sin(x+\frac{\pi}{4})$
**55** $4\sqrt{2}$ **57** $\frac{1000}{\sqrt{3}}$ **59** $\frac{1-\cos x+\sin x}{1+\cos x+\sin x} + C$ **61** $p$ and $q$ are 10 and 1

**2** $\int\cos^3 x\,dx = \int(1-\sin^2 x)\cos x\,dx = \sin x - \frac{\sin^3 x}{3} + C$
**4** $\int\cos^5 x\,dx = \int(1-\sin^2 x)^2\cos x\,dx = \int(1-2\sin^2 x+\sin^4 x)\cos x\,dx = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$
**6** $\int\sin^3 x\cos^3 x\,dx = \int\sin^3 x(1-\sin^2 x)\cos x\,dx = \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$
**8** $\int\sqrt{\sin x}\cos^3 x\,dx = \int\sqrt{\sin x}(1-\sin^2 x)\cos x\,dx = \frac{2}{3}(\sin x)^{3/2} - \frac{2}{7}(\sin x)^{7/2} + C$
**10** $\int\sin^2 ax\cos ax\,dx = \frac{\sin^3 ax}{3a} + C$ and $\int\sin ax\cos ax\,dx = \frac{\sin^2 ax}{2a} + C$
**12** $\int_0^\pi\sin^4 x\,dx = \int_0^\pi(\frac{1-\cos 2x}{2})^2dx = \frac{1}{4}\int_0^\pi(1-2\cos 2x + \frac{1+\cos 4x}{2})dx = [\frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32}]_0^\pi = \frac{3\pi}{8}$.
**14** $\int\sin^2 x\cos^2 x\,dx = \int\frac{1-\cos 2x}{2}\frac{1+\cos 2x}{2}dx = \int\frac{1-\cos^2 2x}{4}dx = \int(\frac{1}{4} - \frac{1+\cos 4x}{8})dx = \frac{x}{8} - \frac{\sin 4x}{32} + C$
**16** $\int\sin^2 x\cos^2 2x\,dx = \int\frac{1-\cos 2x}{2}\cos^2 2x\,dx = \int(\frac{1+\cos 4x}{4} - \frac{\cos 2x}{2}(1-\sin^2 2x))dx =$
$\frac{x}{4} + \frac{\sin 4x}{16} - \frac{\sin 2x}{4} + \frac{\sin^3 2x}{12} + C$. This is a hard one.
**18** Equation (7) gives $\int_0^{\pi/2}\cos^n x\,dx = [\frac{\cos^{n-1}x\sin x}{n}]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2}\cos^{n-2}x\,dx$. The integrated term is zero because $\cos\frac{\pi}{2} = 0$ and $\sin 0 = 0$. The exception is $n=1$, when the integral is $[\sin x]_0^{\pi/2} = 1$.
**20** Problem 18 yields $\int_0^{\pi/2}\cos^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2}\cos^{n-2}x\,dx = \frac{n-1}{n}\frac{n-3}{n-2}\int_0^{\pi/2}\cos^{n-4}x\,dx$. For odd $n$ this

continues to $\frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{2}{3}$, times $\int_0^{\pi/2}\cos x\,dx = 1$. Writing from low to high this is $\frac{2}{3}\frac{4}{5}\cdots\frac{n-1}{n}$.

**22** $\int_0^\pi \cos x\,dx = 0$ because the positive area from 0 to $\frac{\pi}{2}$ is balanced by the negative area from $\frac{\pi}{2}$ to $\pi$. This is true for **any odd power** $n = 1, 3, 5, \cdots$ (For even powers $\cos^n x$ is always positive). The substitution $u = \pi - x$ and $du = -dx$ gives $\int_0^\pi \cos^n x\,dx = -\int_\pi^0 \cos^n(\pi - u)du = \int_0^\pi (-1)^n \cos^n u\,du$.

So if $n$ is odd, the integral equals minus the integral and must be zero.

**24** $(\sin x)(\sin x) = -\frac{1}{2}\cos(1+1)x + \frac{1}{2}\cos(1-1)x$ is the double angle formula $\sin^2 x = \frac{1-\cos 2x}{2}$; $(\cos 2x)(\cos x) = \frac{1}{2}\cos(2+1)x + \frac{1}{2}\cos(2-1)x = \frac{\cos 3x + \cos x}{2}$. To derive equation (9), subtract $\cos(s+t) = \cos s \cos t - \sin s \sin t$ from $\cos(s-t) = \cos s \cos t + \sin t \sin t$. Divide by 2. Then set $s = px$ and $t = qx$.

**26** $\int_0^\pi \sin 3x \sin 5x\,dx = \int_0^\pi \frac{-\cos 8x + \cos 2x}{2}dx = [\frac{-\sin 8x}{16} + \frac{\sin 2x}{4}]_0^\pi = 0$.

**28** $\int_{-\pi}^\pi \cos^2 3x\,dx = \int_{-\pi}^\pi \frac{1+\cos 6x}{2}dx = [\frac{x}{2} + \frac{\sin 6x}{12}]_{-\pi}^\pi = \pi$.

**30** $\int_0^{2\pi} \sin x \sin 2x \sin 3x\,dx = \int_0^{2\pi} \sin 2x(\frac{-\cos 4x + \cos 2x}{2})dx = \int_0^{2\pi} \sin 2x(\frac{1 - 2\cos^2 2x + \cos 2x}{2})dx =$
$[-\frac{\cos 2x}{4} + \frac{\cos^3 2x}{6} - \frac{\cos^2 2x}{8}]_0^{2\pi} = 0$. Note: The integral has other forms.

**32** $\int_0^\pi x \cos x\,dx = [x \sin x]_0^\pi - \int_0^\pi \sin x\,dx = [x \sin x + \cos x]_0^\pi = -2$.

**34** $\int_0^\pi 1 \sin 3x\,dx = \int_0^\pi (A \sin x + B \sin 2x + C \sin 3x + \cdots) \sin 3x\,dx$ reduces to $[-\frac{\cos 3x}{3}]_0^\pi = 0 + 0 + C\int_0^\pi \sin^2 3x\,dx$. Then $\frac{2}{3} = C(\frac{\pi}{2})$ and $C = \frac{4}{3\pi}$.

**36** The square wave is $-1$ and 1 periodically. To find $A$, multiply the series by $\sin x$ and integrate from 0 to $\pi$: $\int_0^\pi 1 \sin x\,dx = \int_0^\pi (A \sin x + \cdots) \sin x\,dx$ yields $2 = A(\frac{\pi}{2})$ and $A = \frac{4}{\pi}$. To find $B$, multiply the series by $\sin 2x$ and integrate: $\int_0^\pi 1 \sin 2x\,dx = \int_0^\pi (A \sin x + B \sin 2x + \cdots) \sin 2x\,dx$ yields $0 = B\int_0^\pi \sin^2 2x\,dx$ and $B = 0$.

**38** $\int_0^\pi \cos qx\,dx = [\frac{\sin qx}{q}]_0^\pi = \frac{\sin q\pi}{q}$ which is zero if $q$ is **any nonzero integer.**

**40** "Always zero" means for positive integers $p \neq q$. Then $\int_0^\pi \sin px \sin qx\,dx = \int_0^\pi \frac{-\cos(p+q)x + \cos(p-q)x}{2}dx =$
$[\frac{-\sin(p+q)x}{2(p+q)} + \frac{\sin(p-q)x}{2(p-q)}]_0^\pi = 0$.

**42** $\int \tan 5x\,dx = \int \frac{\sin 5x}{\cos 5x}dx = -\frac{1}{5}\ln|\cos 5x|$ (set $u = \cos 5x$ to find $\int \frac{-du}{5u}$).

**44** First by substituting for $\tan^2 x$ : $\int \tan^2 x \sec x\,dx = \int \sec^3 x\,dx - \int \sec x\,dx$. Use Problem 62 to integrate $\sec^3 x$ : final answer $\frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x|) + C$. Second method from line 1 of Example 11: $\int \tan^2 x \sec x\,dx = \sec x \tan x - \int \sec^3 x\,dx$. Same final answer.

**46** $\int \sec^4 x\,dx = \int \sec^2 x(1 + \tan^2 x)dx = \tan x + \frac{\tan^3 x}{3} + C$

**48** $\int \tan^5 x\,dx = \int (\sec^2 x - 1)\tan^3 x\,dx = \frac{\tan^4 x}{4} - \int \tan^3 x\,dx = \frac{\tan^4 x}{4} - \int (\sec^2 x - 1)\tan x\,dx =$
$\frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} - \ln|\cos x| + C$

**50** OK to write down $\ln|\csc x - \cot x|$ or $-\ln|\csc x + \cot x|$. For variety set $u = \frac{\pi}{2} - x$ and integrate $-\int \sec u\,du$.

**52** This should have an asterisk! $\int \frac{\sin^6 x}{\cos^3 x}dx = \int \frac{(1-\cos^2 x)^3}{\cos^3 x}dx = \int (\sec^3 x - 3\sec x + 3\cos x - \cos^3 x)dx =$ use Example 11 = Problem 62 for $\int \sec^3 x\,dx$ and change $\int \cos^3 x\,dx$ to $\int (1 - \sin^2 x)\cos x\,dx$.
Final answer $\frac{\sec x \tan x}{2} - \frac{5}{2}\ln|\sec x + \tan x| + 2\sin x + \frac{\sin^3 x}{3} + C$.

**54** $A = 2 : 2\cos(x + \frac{\pi}{3}) = 2\cos x \cos\frac{\pi}{3} - 2\sin x \sin\frac{\pi}{3} = \cos x - \sqrt{3}\sin x$. Therefore $\int \frac{dx}{(\cos x - \sqrt{3}\sin x)^2} =$
$\int \frac{dx}{4\cos^2(x + \frac{\pi}{3})} = \frac{1}{4}\tan(x + \frac{\pi}{3}) + C$.

**56** Expand $\cos(x - \alpha) = \cos x \cos \alpha + \sin x \sin \alpha$, multiply by $\sqrt{a^2 + b^2}$, and match with $a \cos x + b \sin x$. Then $\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}$ and $\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}$ is correct if $\tan \alpha = \frac{b}{a}$ (the right triangle has sides $a$ and $b$).

**58** When lengths are scaled by $\sec x$, area is scaled by $\sec^2 x$. The area from the equator to latitude $x$ is then proportional to $\int \sec^2 x\,dx = \tan x$.

**60** The graphs of $\sin^2 x$ and $\cos^2 x$ obviously give equal areas between 0 and $\frac{\pi}{2}$ and between $\frac{\pi}{2}$ and $\pi$. The areas add to $\int_0^\pi 1dx = \pi$ so each area is $\frac{\pi}{2}$.

**62** Example 11 ends with $2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$. Divide by 2 to find $\int \sec^3 x\,dx =$ $\mathbf{\frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|)} + C$.

## 7.3 Trigonometric Substitutions (page 299)

The function $\sqrt{1-x^2}$ suggests the substitution $x = \mathbf{\sin}\ \theta$. The square root becomes $\mathbf{\cos}\ \theta$ and $dx$ changes to $\cos\theta\ \mathbf{d\theta}$. The integral $\int (1-x^2)^{3/2}dx$ becomes $\int \mathbf{\cos^4}\theta\ d\theta$. The interval $\frac{1}{2} \le x \le 1$ changes to $\mathbf{\frac{\pi}{6}} \le \theta \le \mathbf{\frac{\pi}{2}}$.

For $\sqrt{a^2-x^2}$ the substitution is $x = \mathbf{a \sin}\ \theta$ with $dx = \mathbf{a \cos}\ \theta\ \mathbf{d\theta}$. For $x^2 - a^2$ we use $x = \mathbf{a \sec}\ \theta$ with $dx = \mathbf{a \sec}\ \theta \tan\theta$. (**Insert:** For $x^2 + a^2$ use $x = a \tan\theta$). Then $\int dx/(1+x^2)$ becomes $\int d\theta$, because $1 + \tan^2\theta = \mathbf{\sec^2}\theta$. The answer is $\theta = \tan^{-1} x$. We already knew that $\mathbf{\frac{1}{1+x^2}}$ is the derivative of $\tan^{-1} x$.

The quadratic $x^2 + 2bx + c$ contains a **linear** term $2bx$. To remove it we **complete** the square. This gives $(x+b)^2 + C$ with $C = \mathbf{c - b^2}$. The example $x^2 + 4x + 9$ becomes $\mathbf{(x+2)^2 + 5}$. Then $u = x + 2$. In case $x^2$ enters with a minus sign, $-x^2 + 4x + 9$ becomes $\mathbf{-(x-2)^2 + 13}$. When the quadratic contains $4x^2$, start by factoring out **4**.

**1** $x = 2\sin\theta; \int d\theta = \sin^{-1}\frac{x}{2} + C$ **3** $x = 2\sin\theta; \int 4\cos^2\theta\ d\theta = 2\sin^{-1}\frac{x}{2} + x\sqrt{1-\frac{x^2}{4}} + C$

**5** $x = \sin\theta; \int \sin^2\theta\ d\theta = \frac{1}{2}\sin^{-1}x - \frac{1}{2}x\sqrt{1-x^2} + C$

**7** $x = \tan\theta; \int \cos^2\theta\ d\theta = \frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{1+x^2} + C$

**9** $x = 5\sec\theta; \int 5(\sec^2\theta - 1)d\theta = \sqrt{x^2-25} - 5\sec^{-1}\frac{x}{5} + C$

**11** $x = \sec\theta; \int \cos\theta\ d\theta = \frac{\sqrt{x^2-1}}{x} + C$ **13** $x = \tan\theta; \int \cos\theta\ d\theta = \frac{x}{\sqrt{1+x^2}} + C$

**15** $x = 3\sec\theta; \int \frac{\cos\theta\ d\theta}{9\sin^2\theta} = \frac{-1}{9\sin\theta} + C = \frac{-x}{9\sqrt{x^2-9}} + C$

**17** $x = \sec\theta; \int \sec^3\theta\ d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln(\sec\theta + \tan\theta) + C = \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\ln(x + \sqrt{x^2-1}) + C$

**19** $x = \tan\theta; \int \frac{\cos\theta\ d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = \frac{-\sqrt{x^2+1}}{x} + C$

**21** $\int \frac{-\sin\theta\ d\theta}{\sin\theta} = -\theta + C = -\cos^{-1}x + C$; with $C = \frac{\pi}{2}$ this is $\sin^{-1}x$

**23** $\int \frac{\tan\theta\sec^2\theta\ d\theta}{\sec^2\theta} = -\ln(\cos\theta) + C = \ln\sqrt{x^2+1} + C$ which is $\frac{1}{2}\ln(x^2+1) + C$

**25** $x = a\sin\theta; \int_{-\pi/2}^{\pi/2} a^2\cos^2\theta\ d\theta = \frac{a^2\pi}{2}$ = area of semicircle **27** $\sin^{-1}x]_{.5}^{1} = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

**29** Like Example 6: $x = \sin\theta$ with $\theta = \frac{\pi}{2}$ when $x = \infty, \theta = \frac{\pi}{3}$ when $x = 2$, $\int_{\pi/3}^{\pi/2}\frac{\cos\theta\ d\theta}{\sin^2\theta} = -1 + \frac{2}{\sqrt{3}}$

**31** $x = 3\tan\theta; \int_{-\pi/2}^{\pi/2}\frac{3\sec^2\theta\ d\theta}{9\sec^2\theta} = \frac{\theta}{3}]_{-\pi/2}^{\pi/2} = \frac{\pi}{3}$ **33** $\int \frac{x^{n+1}+x^{n-1}}{x^2+1}dx = \int x^{n-1}dx = \frac{x^n}{n}$

**35** $x = \sec\theta; \frac{1}{2}(e^f + e^{-f}) = \frac{1}{2}(x + \sqrt{x^2-1} + \frac{1}{x+\sqrt{x^2-1}}) = \frac{1}{2}(x + \sqrt{x^2-1} + x - \sqrt{x^2-1}) = x$

**37** $x = \cosh\theta; \int d\theta = \cosh^{-1}x + C$

**39** $x = \cosh\theta; \int \sinh^2\theta\ d\theta = \frac{1}{2}(\sinh\theta\cosh\theta - \theta) + C = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln(x + \sqrt{x^2-1}) + C$

**41** $x = \tanh\theta; \int d\theta = \tanh^{-1}x + C$ **43** $(x-2)^2 + 4$ **45** $(x-3)^2 - 9$ **47** $(x+1)^2$

**49** $u = x-2, \int \frac{du}{u^2+4} = \frac{1}{2}\tan^{-1}\frac{u}{2} = \frac{1}{2}\tan^{-1}(\frac{x-2}{2}) + C; u = x-3, \int \frac{du}{u^2-9} = \frac{1}{6}\ln\frac{u-3}{u+3} = \frac{1}{6}\ln\frac{x-6}{x} + C;$
$u = x+1, \int \frac{du}{u^2} = \frac{-1}{u} = \frac{-1}{x+1} + C$

**51** $u = x+b; \int \frac{du}{u^2-b^2+c}$ uses $u = a\sec\theta$ if $b^2 > c, u = a\tan\theta$ if $b^2 < c$, equals $-\frac{1}{u} = \frac{-1}{x+b}$ if $b^2 = c$

**53** $\cos\theta$ is negative $(-\sqrt{1-x^2})$ from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$; then $\int_0^1 - \int_1^{-1} + \int_{-1}^0 \sqrt{1-x^2}dx = \pi$ = area of unit circle

**55** Divide $y$ by 4, multiply $dx$ by 4, same $\int y\,dx$

**57** No $\sin^{-1} x$ for $x > 1$; the square root is imaginary. All correct with complex numbers.

**2** $x = a\sec\theta, x^2 - a^2 = a^2\tan^2\theta, \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta d\theta}{a\tan\theta} = \ln|\sec\theta + \tan\theta| = \mathbf{\ln|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}|} + C$

**4** $x = \frac{1}{3}\tan\theta, 1 + 9x^2 = \sec^2\theta, \int \frac{dx}{1+9x^2} = \int \frac{\frac{1}{3}\sec^2\theta d\theta}{\sec^2\theta} = \frac{\theta}{3} = \mathbf{\frac{1}{3}\tan^{-1}3x} + C.$

**6** $x = \sin\theta, \int \frac{dx}{x^2\sqrt{1-x^2}} = \int \frac{\cos\theta d\theta}{\sin^2\theta\cos\theta} = -\cot\theta = -\mathbf{\frac{\sqrt{1-x^2}}{x}} + C$

**8** $x = a\tan\theta, x^2 + a^2 = a^2\sec^2\theta, \int \sqrt{x^2+a^2}dx = \int a^2\sec^3\theta d\theta$ = use Problem 62 above:
$\frac{a^2}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) = \mathbf{\frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\ln|\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}|} + C$

**10** $x = 3\sin\theta, 9 - x^2 = 9\cos^2\theta, \int \frac{x^3dx}{\sqrt{9-x^2}} = \int \frac{27\sin^3\theta(3\cos\theta\,d\theta)}{3\cos\theta} = \int 27(1-\cos^2\theta)\sin\theta d\theta = -27\cos\theta + 9\cos^3\theta =$
$\mathbf{-27(1-\frac{x^2}{9})^{1/2} + 9(1-\frac{x^2}{9})^{3/2}} + C$

**12** Write $\sqrt{x^6 - x^8} = x^3\sqrt{1-x^2}$ and set $x = \sin\theta : \int\sqrt{x^6-x^8}dx = \int \sin^3\theta\cos\theta(\cos\theta d\theta) =$
$\int \sin\theta(\cos^2\theta - \cos^4\theta)d\theta = -\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5} = \mathbf{-\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2}} + C$

**14** $x = \sin\theta, \int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos\theta d\theta}{\cos^3\theta} = \tan\theta + C = \mathbf{\frac{x}{\sqrt{1-x^2}}} + C.$

**16** $x = \tan\theta, \int \frac{\sqrt{1+x^2}dx}{x} = \int \frac{\sec\theta\sec^2\theta d\theta}{\tan\theta} = \int \frac{\sec\theta(1+\tan^2\theta)d\theta}{\tan\theta} = \int(\csc\theta + \sec\theta\tan\theta)d\theta = \ln|\csc\theta - \cot\theta| + \sec\theta =$
$\mathbf{\ln|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}| + \sqrt{1+x^2}} + C.$

**18** $x = 2\tan\theta, x^2 + 4 = 4\sec^2\theta, \int \frac{x^2dx}{x^2+4} = \int \frac{4\tan^2\theta}{4\sec^2\theta}2\sec^2\theta\,d\theta = \int 2\tan^2\theta d\theta = \int 2(\sec^2\theta - 1)d\theta = 2\tan\theta - 2\theta =$
$\mathbf{x - 2\tan^{-1}\frac{x}{2}} + C.$

**20** $x = \tan\theta, 1 + x^2 = \sec^2\theta, \int \frac{x^2dx}{\sqrt{1+x^2}} = \int \frac{\tan^2\theta}{\sec\theta}\sec^2\theta\,d\theta = \int\tan^2\theta\sec\theta d\theta$ = (use Problem 44 above)
$\frac{1}{2}(\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|) = \mathbf{\frac{1}{2}(x\sqrt{1+x^2} - \ln|\sqrt{1+x^2} + x|)} + C.$

**22** $x = \sec\theta : \int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta d\theta}{\sec\theta\tan\theta} = \theta + C = \mathbf{\sec^{-1}x + C}$. For $x = \csc\theta$ the integral is $\int \frac{-\csc\theta\cot\theta d\theta}{\csc\theta\cot\theta} =$
$-\theta + C = \mathbf{-\csc^{-1}x + C^*}$. Both answers are right: $\sec^{-1}x + \csc^{-1}x$ = sum of complementary angles in Section 4.4 $= \frac{\pi}{2}$ so the arbitrary constant $C^*$ is $C - \frac{\pi}{2}$.

**24** Set $x^2 = \sec\theta$ and $x^4 - 1 = \tan^2\theta$ and $2x\,dx = \sec\theta\tan\theta d\theta$. Then $\int \frac{2x\,dx}{2x^2\sqrt{x^4-1}} = \int \frac{\sec\theta\tan\theta d\theta}{2\sec\theta\tan\theta} = \frac{\theta}{2} =$
$\frac{1}{2}\sec^{-1}(x^2)$.

**26** $x = \sin\theta : \int_{-1}^1 (1-x^2)^{3/2}dx = \int_{-\pi/2}^{\pi/2}\cos^3\theta(\cos\theta d\theta) = 2\int_0^{\pi/2}\cos^4\theta d\theta$ = (Problem 19 of Section 7.2)
$2(\frac{1}{2})(\frac{3}{4})(\frac{\pi}{2}) = \mathbf{\frac{3\pi}{8}}$.

**28** $x = \sec\theta : \int_1^4 \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta d\theta}{\tan\theta} = \ln|\sec\theta + \tan\theta| = [\ln|x + \sqrt{x^2-1}|]_1^4 = \mathbf{\ln(4 + \sqrt{15})}$.

**30** $\int_{-1}^1 \frac{x\,dx}{x^2+1} = [\frac{1}{2}\ln(x^2+1)]_{-1}^1 = 0$ (odd function integrated from $-1$ to 1).

**32** First use geometry: $\int_{1/2}^1 \sqrt{1-x^2}dx$ = half the area of the unit circle beyond $x = \frac{1}{2}$ which breaks into
$\frac{1}{2}$(120° wedge minus 120° triangle) $= \frac{1}{2}(\frac{\pi}{3} - \frac{1}{2}\cdot\frac{1}{2}\cdot 2\sqrt{1-(\frac{1}{2})^2}) = \mathbf{\frac{\pi}{6} - \frac{\sqrt{3}}{8}}$.
Check by integration: $\int_{1/2}^1 \sqrt{1-x^2}dx = [\frac{1}{2}(x\sqrt{1-x^2} + \sin^{-1}x)]_{1/2}^1 = \frac{1}{2}(\frac{\pi}{2} - \frac{1}{2}\frac{\sqrt{3}}{2} - \frac{\pi}{6}) = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$.

**34** $\int \frac{dx}{\cos x} = \int \sec x\,dx = \mathbf{\ln|\sec x + \tan x|} + C; \int \frac{dx}{1+\cos x}(\frac{1-\cos x}{1-\cos x}) = \int \frac{dx}{\sin^2 x} - \int \frac{\cos x\,dx}{\sin^2 x} = \int \csc^2 x\,dx - \int \frac{du}{u^2} =$
$-\cot x + \frac{1}{\sin x} = \mathbf{\frac{1-\cos x}{\sin x}} + C; \int \frac{dx}{\sqrt{1+\cos x}} = \int \frac{dx}{\sqrt{2}\cos\frac{x}{2}} = \mathbf{\sqrt{2}\ln|\sec\frac{x}{2} + \tan\frac{x}{2}|} + C$

**36** $x = \tan\theta$ gives $\int \frac{dx}{\sqrt{x^2+1}} = \int \frac{\sec^2\theta d\theta}{\sec\theta} = \ln(\sec\theta + \tan\theta) = \ln(x + \sqrt{x^2+1}) = \mathbf{g}$. (b) Check $g' = \frac{1 + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}} =$
$\frac{\sqrt{x^2+1}+x}{x+\sqrt{x^2+1}}\frac{1}{\sqrt{x^2+1}}$. (c) Thus $\sinh g = \frac{1}{2}(e^g - e^{-g}) = \frac{1}{2}(x + \sqrt{x^2+1} - \frac{1}{x+\sqrt{x^2+1}}) = \frac{1}{2}(\frac{x^2+2x\sqrt{x^2+1}+x^2+1-1}{x+\sqrt{x^2+1}}) = \mathbf{x}$.
(d) Now go directly to $\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1}x$ by substituting $x = \sinh g$ to reach $\int \frac{\cosh g\,dg}{\cosh g} = g + C$.

**38** $x = \tanh\theta: \int \frac{dx}{x\sqrt{1-x^2}} = \int \frac{\operatorname{sech}^2\theta d\theta}{\tanh\theta \operatorname{sech}\theta} = \int \operatorname{csch}\theta d\theta = -\ln|\operatorname{csch}\theta + \coth\theta| = -\ln(\frac{\sqrt{1-x^2}+1}{x}) + C$

**40** $x = \cosh\theta: \int \frac{\sqrt{x^2-1}}{x^2}dx = \int \frac{\sinh\theta}{\cosh^2\theta}\sinh\theta d\theta = \int \tanh^2\theta d\theta = \int(1-\operatorname{sech}^2\theta)d\theta = \theta - \tanh\theta = \cosh^{-1}x - \frac{\sqrt{x^2-1}}{x} + C$

**42** $x = \sinh\theta: \int \frac{\sqrt{1+x^2}dx}{x^2} = \int \frac{\cosh\theta}{\sinh^2\theta}\cosh\theta d\theta = \int \coth^2\theta d\theta = \int(1+\operatorname{csch}^2\theta)d\theta = \theta - \coth\theta = \sinh^{-1}x - \frac{\sqrt{x^2+1}}{x} + C$

**44** $-x^2 + 2x + 8 = -(x-1)^2 + 9$ **46** $-x^2 + 10$ : no linear term, square already completed

**48** $x^2 + 4x - 12 = (x+2)^2 - 16$

**50** $\int \frac{dx}{\sqrt{9-(x-1)^2}} = \int \frac{du}{\sqrt{9-u^2}}$. Set $u = 3\sin\theta : \int \frac{\cos\theta d\theta}{\cos\theta} = \theta = \sin^{-1}\frac{u}{3} = \sin^{-1}\frac{x-1}{3} + C$;

$\int \frac{dx}{10-x^2} = \frac{1}{2\sqrt{10}} \ln \frac{x-\sqrt{10}}{x+\sqrt{10}} + C; \int \frac{dx}{(x+2)^2-16} = \int \frac{du}{u^2-16} = \frac{1}{8}\ln\frac{2u-8}{2u+8} = \frac{1}{8}\ln\frac{x-2}{x+6} + C$

**52** (a) $u = x - 2$ (b) $u = x + 1$ (c) $u = x - 5$ (d) $u = x - \frac{1}{4}$

**54** (a) If $x = \tan\theta$ then $\int \sqrt{1+x^2}dx = \int \sec^3\theta d\theta$. (b) The integral $\frac{1}{2}[\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)]$ equals $\frac{1}{2}[x\sqrt{x^2+1} + \ln|x + \sqrt{x^2+1}|$. (c) If $x = \sinh\theta$ then $\int \sqrt{1+x^2}dx = \int \cosh^2\theta d\theta$ (d) The integral $\frac{1}{2}[\sinh\theta\cosh\theta + \theta]$ equals $\frac{1}{2}[x\sqrt{1+x^2} + \sinh^{-1}x]$.

**56** The two curves cover the same area! Proof by calculus: $\int_0^4 \frac{dx}{\sqrt{16-x^2}} =$ (with $x = 4u$) $\int_0^1 \frac{4du}{4\sqrt{1-u^2}}$. Proof by geometry: The $x$ scale has factor $\frac{1}{4}$ and the $y$ scale has factor 4, so $dA = dxdy$ is unchanged.

## 7.4 Partial Fractions (page 304)

The idea of partial fractions is to express $P(x)/Q(x)$ as a sum of simpler terms, each one easy to integrate. To begin, the degree of $P$ should be less than the degree of $Q$. Then $Q$ is split into linear factors like $x - 5$ (possibly repeated) and quadratic factors like $x^2 + x + 1$ (possibly repeated). The quadratic factors have two complex roots, and do not allow real linear factors.

A factor like $x-5$ contributes a fraction $A/(x-5)$. Its integral is $A\ln(x-5)$. To compute $A$, cover up $x-5$ in the denominator of $P/Q$. Then set $x = 5$, and the rest of $P/Q$ becomes $A$. An equivalent method puts all fractions over a common denominator (which is Q). Then match the **numerators**. At the same point ($x = 5$) this matching gives $A$.

A repeated linear factor $(x-5)^2$ contributes not only $A/(x-5)$ but also $B/(x-5)^2$. A quadratic factor like $x^2 + x + 1$ contributes a fraction $(Cx + D)/(x^2 + x + 1)$ involving $C$ and $D$. A repeated quadratic factor or a triple linear factor would bring in $(Ex + F)/(x^2 + x + 1)^2$ or $G/(x-5)^3$. The conclusion is that any $P/Q$ can be split into partial fractions, which can always be integrated.

**1** $A = -1, B = 1, -\ln x + \ln(x-1) + C$ **3** $\frac{1}{x-3} - \frac{1}{x-2}$ **5** $\frac{1}{2x} - \frac{2}{x+1} + \frac{5/2}{x+2}$

**7** $\frac{3}{x} + \frac{1}{x^2}$ **9** $3 - \frac{3}{x^2+1}$ **11** $-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}$ **13** $-\frac{1/6}{x} + \frac{1/2}{x-1} - \frac{1/2}{x-2} + \frac{1/6}{x-3}$

**15** $\frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}; A = -\frac{1}{4}, B = \frac{1}{4}, C = 0, D = -\frac{1}{2}$

**17** Coefficients of $y : 0 = -Ab + B$; match constants $1 = Ac; A = \frac{1}{c}, B = \frac{b}{c}$

**19** $A = 1$, then $B = 2$ and $C = 1; \int \frac{dx}{x-1} + \int \frac{(2x+1)dx}{x^2+x+1} =$

$\ln(x-1) + \ln(x^2+x+1) = \ln(x-1)(x^2+x+1) = \ln(x^3-1)$

**21** $u = e^x; \int \frac{du}{u^2-u} = \int \frac{du}{u-1} - \int \frac{du}{u} = \ln(\frac{u-1}{u}) + C = \ln(\frac{e^x-1}{e^x}) + C$

**23** $u = \cos\theta; \int \frac{-du}{1-u^2} = -\frac{1}{2}\int \frac{du}{1-u} - \frac{1}{2}\int \frac{du}{1+u} = \frac{1}{2}\ln(1-u) - \frac{1}{2}\ln(1+u) = \frac{1}{2}\ln\frac{1-\cos\theta}{1+\cos\theta} + C$. We can reach $\frac{1}{2}\ln\frac{(1-\cos\theta)^2}{1-\cos^2\theta} = \ln\frac{1-\cos\theta}{\sin\theta} = \ln(\csc\theta - \cot\theta)$ or a different way $\frac{1}{2}\ln\frac{1-\cos^2\theta}{(1+\cos\theta)^2} = \ln\frac{\sin\theta}{1+\cos\theta} = -\ln\frac{1+\cos\theta}{\sin\theta} = -\ln(\csc\theta + \cot\theta)$

**25** $u = e^x; du = e^x dx = u\,dx; \int \frac{1+u}{(1-u)u}du = \int \frac{2du}{1-u} + \int \frac{du}{u} = -2\ln(1-e^x) + \ln e^x + C = -2\ln(1-e^x) + x + C$

**27** $x + 1 = u^2, dx = 2u\,du; \int \frac{2u\,du}{1+u} = \int[2 - \frac{2}{1+u}]du = 2u - 2\ln(1+u) + C =$
$2\sqrt{x+1} - 2\ln(1+\sqrt{x+1}) + C$

**29** Note $Q(a) = 0$. Then $\frac{x-a}{Q(x)} = \frac{x-a}{Q(x)-Q(a)} \to \frac{1}{Q'(a)}$ by definition of derivative. At a double root $Q'(a) = 0$.

**2** $\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$. Cover up $x-1$ and set $x = 1$ to find $A = \frac{1}{2}$. Cover up $x+1$ and set $x = -1$ to find $B = -\frac{1}{2}$. Then $\int \frac{dx}{x^2-1} = \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1) = \frac{1}{2}\ln\frac{x-1}{x+1} + C$. Method 1: $\frac{1}{(x-1)(x+1)} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$ and by matching numerators $A + B = 0$ and $A - B = 1$ so again $A = \frac{1}{2}$ and $B = -\frac{1}{2}$.

**4** $\frac{x}{(x-3)(x-2)} = \frac{3}{x-3} - \frac{2}{x-2}$ **6** $\frac{1}{x(x-1)(x+1)} = -\frac{1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1}$

**8** $\frac{3x+1}{(x-1)^2} = \frac{4}{(x-1)^2} + \frac{3}{x-1}$ (first multiply by $(x-1)^2$ and set $x = 1$ to find the coefficient 4).

**10** $\frac{1}{(x-1)(x^2+1)} = \frac{1/2}{x-1} - \frac{\frac{1}{2}x+\frac{1}{2}}{x^2+1}$ **12** $\frac{x}{x^2-4} = \frac{1/2}{x-2} + \frac{1/2}{x+2}$

$x - 1 + \frac{2}{x-1}$

**14** $x + 1\sqrt{x^2 + 0x + 1}$ so $\frac{x^2+1}{x+1} = x - 1 + \frac{2}{x+1}$ **16** $\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}$

**18** $\frac{x^2}{(x-3)(x+3)} = \frac{A(x+3)+B(x-3)}{(x-3)(x+3)}$ is impossible (no $x^2$ in the numerator on the right side). Divide first to rewrite $\frac{x^2}{(x-3)(x+3)} = 1 + \frac{9}{(x-3)(x+3)} =$ (now use partial fractions) $1 + \frac{3/2}{x-3} - \frac{3/2}{x+3}$.

**20** Integrate $\frac{1/2}{1-y} + \frac{1/2}{1+y}$ to find $-\frac{1}{2}\ln(1-y) + \frac{1}{2}\ln(1+y) = \frac{1}{2}\ln\mathbf{\frac{1+y}{1-y}} = t + C$. At $t = 0$ this is $\frac{1}{2}\ln 1 = 0 + C$ so $C = \mathbf{0}$. Taking exponentials gives $\mathbf{\frac{1+y}{1-y} = e^{2t}}$. Then $1 + y = e^{2t}(1-y)$ and $y = \frac{e^{2t}-1}{e^{2t}+1} = \frac{e^t-e^{-t}}{e^t+e^{-t}} = \mathbf{\tanh t}$. This is the **S-curve**.

**22** Set $u = \sqrt{x}$ so $u^2 = x$ and $2u\,du = dx$. Then $\int \frac{1-\sqrt{x}}{1+\sqrt{x}}dx = \int \frac{1-u}{1+u}2u\,du =$ (divide $u+1$ into $-2u^2+2u$) $\int(-2u + 4 - \frac{4}{u+1})du = -u^2 + 4u - 4\ln(u+1) + C = \mathbf{-x + 4\sqrt{x} - 4\ln(\sqrt{x}+1)} + C$.

**24** Set $u = e^t$ so $du = e^t dt$ or $dt = \frac{du}{u}$. Then $\int \frac{dt}{(e^t-e^{-t})^2} = \int \frac{du/u}{(u-\frac{1}{u})^2} = \int \frac{u\,du}{(u^2-1)^2} = \int(\frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{C}{u+1} + \frac{D}{(u+1)^2})du$. Cover up $(u-1)^2$ and set $u = 1$ to find $B = \frac{1}{4}$; cover up $(u+1)^2$ and set $u = -1$ to find $D = -\frac{1}{4}$; match left and right to find $A = C = 0$. The integral is $-\frac{1}{4}\frac{1}{u-1} + \frac{1}{4}\frac{1}{u+1} = -\frac{1}{2}\frac{1}{u^2-1} = \mathbf{-\frac{1}{2}\frac{1}{e^{2t}-1}}$. Check derivative: $\frac{1}{2}\frac{1}{(e^{2t}-1)^2}(2e^{2t}) = \frac{1}{(e^t-e^{-t})^2}$. Quicker integration: $\int \frac{u\,du}{(u^2-1)^2} = -\frac{1}{2}(u^2-1)^{-1} = \mathbf{-\frac{1}{2}\frac{1}{e^{2t}-1}}$.

**26** Set $u^3 = x - 8$ so $3u^2du = dx$. Then $\int \frac{(x-8)^{1/3}dx}{x} = \int \frac{u(3u^2du)}{u^3+8} =$ (divide first) $\int(3 - \frac{24}{u^3+8})du = 3u - \int \frac{24\,du}{(u+2)(u^2-2u+4)} = 3u - \int(\frac{2}{u+2} + \frac{-2u+8}{u^2-2u+4})du = 3u - 2\ln(u+2) + \int \frac{2(u-1)-6}{(u-1)^2+3}du = 3u - 2\ln(u+2) + \ln((u-1)^2+3) - \frac{6}{\sqrt{3}}\tan^{-1}(\frac{u-1}{\sqrt{3}}) + C$. Finally set $u = (x-8)^{1/3}$.

**28** Set $u^4 = x$ so that $4u^3du = dx$. Then $\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}} = \int \frac{4u^3du}{u^2+u} =$ (divide first) $\int(4u - 4 + \frac{4u}{u^2+u})du = 2u^2 - 4u + 4\ln(u+1) + C = 2\sqrt{x} - 4\sqrt[4]{x} + 4\ln(\sqrt[4]{x}+1) + C$.

**30** Multiply $\frac{1}{x^8-1} = \frac{A}{x-1} + \cdots$ by $x-1$ and let $x$ approach 1 to find $A = \lim\frac{x-1}{x^8-1} = \lim\frac{1}{8x^7} = \mathbf{\frac{1}{8}}$.

## 7.5 Improper Integrals (page 309)

An improper integral $\int_a^b y(x)dx$ has lower limit $a = -\infty$ or upper limit $b = \infty$ or $y$ becomes **infinite** in the interval $a \le x \le b$. The example $\int_1^\infty dx/x^3$ is improper because $\mathbf{b} = \infty$. We should study the limit of $\int_1^b dx/x^3$ as $\mathbf{b} \to \infty$. In practice we work directly with $-\frac{1}{2}x^{-2}]_1^\infty = \mathbf{\frac{1}{2}}$. For $p > 1$ the improper integral $\int_0^{\mathbf{1}} \mathbf{x^{-p}dx}$ is finite. For $p < 1$ the improper integral $\int_0^{\mathbf{1}} \mathbf{x^{-p}dx}$ is finite. For $y = e^{-x}$ the integral from 0 to $\infty$ is **1**.

Suppose $0 \le u(x) \le v(x)$ for all $x$. The convergence of $\int \mathbf{v(x)\,dx}$ implies the convergence of $\int \mathbf{u(x)dx}$. The divergence of $\int u(x)dx$ **implies** the divergence of $\int v(x)dx$. From $-\infty$ to $\infty$, the integral of $1/(e^x + e^{-x})$ converges by comparison with $\mathbf{1/e^{|x|}}$. Strictly speaking we split $(-\infty, \infty)$ into $(-\infty, 0)$ and $(0, \infty)$. Changing to $1/(e^x - e^{-x})$ gives divergence, because $\mathbf{e^x = e^{-x}}$ **at** $\mathbf{x = 0}$. Also $\int_{-\pi}^{\pi} dx/\sin x$ diverges by comparison with $\int \mathbf{dx/x}$. The regions left and right of zero don't cancel because $\infty - \infty$ is **not zero**.

**1** $\frac{x^{1-e}}{1-e}]_1^\infty = \frac{1}{e-1}$ **3** $-2(1-x)^{1/2}]_0^1 = 2$ **5** $\tan^{-1} x]_{-\pi/2}^0 = \frac{\pi}{2}$ **7** $\frac{1}{2}(\ln x)^2]_0^1 = -\infty$
**9** $x \ln x - x]_0^e = -\infty$ **11** $\ln(\ln x)]_{100}^\infty = \infty$ **13** $\frac{1}{2}(x + \sin x \cos x)]_0^\infty = \infty$

**15** $\frac{x^{1-p}}{1-p}]_0^\infty$ diverges for every $p$! **17** Less than $\int_1^\infty \frac{dx}{x^6} = \frac{1}{5}$

**19** Less than $\int_0^1 \frac{dx}{x^2+1} + \int_1^\infty \frac{\sqrt{x}\,dx}{x^2} = \tan^{-1} x]_0^1 - \frac{2}{\sqrt{x}}]_1^\infty = \frac{\pi}{4} + 2$

**21** Less than $\int_1^\infty e^{-x}dx = \frac{1}{e}$, greater than $-\frac{1}{e}$

**23** Less than $\int_0^1 e^2 dx + e\int_1^\infty e^{-(x-1)^2}dx = e^2 + e\int_1^\infty e^{-u^2}du = e^2 + \frac{e}{\sqrt{\pi}}$

**25** $\int_0^1 \frac{\sin^2 x\,dx}{x^2} + \int_1^\infty \frac{\sin^2 x\,dx}{x^2}$ less than $1 + \int_1^\infty \frac{dx}{x^2} = 2$ **27** $p! = p$ times $(p-1)!$; $1 = 1$ times $0!$

**29** $u = x, dv = xe^{-x^2}dx : -x\frac{e^{-x^2}}{2}]_0^\infty + \int_0^\infty \frac{e^{-x^2}}{2}dx = \frac{1}{2}\sqrt{\pi}$ **31** $\int_0^\infty 1000e^{-.1t}dt = -10{,}000e^{-.1t}]_0^\infty = \$10{,}000$

**33** $W = \frac{-GMm}{x}]_R^\infty = \frac{GMm}{R} = \frac{1}{2}mv_0^2$ if $v_0 = \sqrt{\frac{2GM}{R}}$

**35** $\int_0^\infty \frac{dx}{2^x} = \int_0^\infty e^{-x\ln 2}dx = \frac{e^{-x\ln 2}}{-\ln 2}]_0^\infty = \frac{1}{\ln 2}$

**37** $\int_0^{\pi/2}(\sec x - \tan x)dx = [\ln(\sec x + \tan x) + \ln(\cos x)]_0^{\pi/2} = [\ln(1 + \sin x)]_0^{\pi/2} = \ln 2$.
The areas under $\sec x$ and $\tan x$ separately are infinite **39** Only $p = 0$

**2** $\int_0^1 \frac{dx}{x^\pi} = [\frac{x^{1-\pi}}{1-\pi}]_0^1$ diverges at $x = 0$ : infinite area **4** $\int_0^1 \frac{dx}{1-x} = [-\ln(1-x)]_0^1$ diverges at $x = 1$ : infinite area
**6** $\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = [\sin^{-1} x]_{-1}^1 = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi$
**8** $\int_{-\infty}^\infty \sin x\,dx$ is not defined because $\int_a^b \sin x\,dx = \cos a - \cos b$ does not approach a limit as $b \to \infty$ and $a \to -\infty$

**10** $\int_0^\infty xe^{-x}dx = [-xe^{-x}]_0^\infty + \int_0^\infty e^{-x}dx = 0 + 1$

**12** $\int_{-\infty}^\infty \frac{x\,dx}{(x^2-1)^2}$ is not defined because the area around $x = -1$ and $x = 1$ is infinite.

**14** $\int_0^{\pi/2} \tan x\,dx$ is not defined: it is $\int_0^1 \frac{du}{u}$ with $u = \cos x$ and the area is infinite.

**16** $\int_0^\infty \frac{e^x\,dx}{(e^x-1)^p} =$ (set $u = e^x - 1$) $\int_0^\infty \frac{du}{u^p}$ which is infinite: diverges at $u = 0$ if $p \ge 1$, diverges at $u = \infty$ if $p \le 1$.

**18** $\int_0^1 \frac{dx}{x^6+1} < \int_0^1 \frac{dx}{1} = 1$ : convergence **20** $\int_0^1 \frac{e^{-x}dx}{1-x} > \int_0^1 \frac{e^{-1}dx}{1-x} = \infty$ : divergence

**22** $\int_1^\infty x^{-x}dx < \int_1^\infty e^{-x}dx = \frac{1}{e}$ : convergence

**24** $\int_0^1 \sqrt{-\ln x}dx < \int_0^{1/e}(-\ln x)dx + \int_{1/e}^1 1dx = [-x\ln x + x]_0^{1/e} + [x]_{1/e}^1 = \frac{1}{e} + 1$ : convergence (note $x \ln x \to 0$ as $x \to 0$)

**26** $\int_0^\infty (\frac{1}{x} - \frac{1}{1+x})dx$ : the separate integrals would give $\infty - \infty$ which is indeterminate, so combine $\frac{1}{x} - \frac{1}{1+x} = \frac{1+x-x}{x(1+x)} < \frac{1}{x^2}$. The integral is less than $\int_1^\infty \frac{dx}{x^2} = 1$. Convergence.

**28** $\int_0^\infty x^{-1/2}e^{-x}dx$ (set $x = u^2$) $= \int_0^\infty u^{-1}e^{-u^2}2u\,du = 2\int_0^\infty e^{-u^2}du = \sqrt{\pi}$, so this is $(-\frac{1}{2})!$ Then $(p+1)! =$

$(p+1)$ times $p!$ with $p = -\frac{1}{2}$ gives $(\frac{1}{2})! = \frac{1}{2}\sqrt{\pi}$.

**30** $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ is like $\int x^{m-1}dx$ near $x=0$ and $\int (1-x)^{n-1}$ near $x=1$. These are finite if $m-1>-1$ and $n-1>-1$, or $\mathbf{m}>\mathbf{0}$ and $\mathbf{n}>\mathbf{0}$. Then the front inside cover gives $B = \frac{(m-1)!(n-1)!}{(m+n-1)!}$.

**32** To pay $s$ at the end of year $n$, the present deposit must be $\frac{s}{(1+i)^n} = \frac{s}{a^n}$. To pay $s$ at the end of every year (perpetual annuity), the deposit must be $\frac{s}{a} + \frac{s}{a^2} + \cdots = s\frac{1/a}{1-1/a} = \frac{s}{a-1} = \frac{s}{i}$. To receive $s = \$1000$/year with $i = 10\%$ you deposit \$10,000.

**34** Note: $GM = 4 \cdot 10^{14}$ m$^3$/sec$^2$: the lost factor of $10^{10}$ would have a large effect on our universe! The escape velocity is $v_0 = \sqrt{2GM/R}$, so that $R = 2GM/v_0^2 = 2 \cdot 4 \cdot 10^{14}/9 \cdot 10^{16} = \frac{8}{9}10^{-2}$ meters $= .9$ cm.

**36** $\int_a^b \frac{x\,dx}{1+x^2} = [\frac{1}{2}\ln(1+x^2)]_a^b = \frac{1}{2}\ln(1+b^2) - \frac{1}{2}\ln(1+a^2)$. As $b \to \infty$ or as $a \to -\infty$ (separately!) there is no limiting value. If $a = -b$ then the answer is zero – but we are not allowed to connect $a$ and $b$.

**38** $\int_0^\infty \frac{x^{-1/2}dx}{1+x} =$ (set $x = u^2$) $\int_0^\infty \frac{(\frac{1}{u})2u\,du}{1+u^2} = [2\tan^{-1}u]_0^\infty = 2(\frac{\pi}{2}) = \pi$; $\int_0^\infty xe^{-x}\cos x\,dx =$ (by parts) $[\frac{xe^{-x}}{2}(\sin x - \cos x) + \frac{e^{-x}}{2}\sin x]_0^\infty = 0$.

**40** The red area in the right figure has an extra unit square (area 1) compared to the red area on the left.