

Notes on the Text: Classroom Experience

Chapter 1 makes a start on three topics: functions in general, the sine and cosine in particular, and computing. Here are brief comments so you will know what is needed later.

1. *Functions and graphs* (essential): Section 1.1 is a solid starting point for calculus. You may give it more than a day, especially if you introduce $f(t + 2)$ and $f(t) + 2$ and $f(2t)$ and $2f(t)$.

These notes offer ideas about other graphing activities. One purpose is to maintain the interest of those who have already taken calculus, without giving them an enormous advantage.

My favorite is the forward-back function graphed on page 4. Section 1.2 goes on to other piecewise linear models like income tax – and says explicitly that “I hope you like them but you don’t have to learn them.” Any of these examples, especially the delta function mentioned briefly, can be passed over. It is Section 1.3 that compares *average* to *instantaneous* for $y = x^2$.

2. *Sines and cosines* (these are optional in Chapter 1): My intention is to see trigonometry in use – for *points on a circle* as well as sides of a triangle. Many classes will not spend substantial time on the review, but it must be available. The figure on page 31 leads neatly to $\cos(s - t)$.

Section 2.4 computes derivatives of $\sin x$ and $\cos x$ in the normal way from $\frac{\Delta y}{\Delta x}$. The limits of $\frac{\sin h}{h}$ and $\frac{1 - \cos h}{h}$ are fully developed there. But students may understand these functions better (and also the motion described by $x = \cos t, y = \sin t$) by following a point on a circle. That is the outstanding example of a parameter.

3. *Computing in calculus* (optional): The computing section is placed in a way that allows you to discuss it *or not*. This topic is especially dependent on the local situation. (M.I.T. does not do much computing in the first year, and does nothing with graphing calculators.) But calculators are so convenient that we will see them more and more. They have the advantage of requiring less faculty time, as well as being personal and portable and not too expensive.

The valuable thing is to see graphs (better than numbers). The example of 3^x versus x^3 is quite good – those graphs are surprisingly close for $2.2 < x < 3.2$. It is a challenge to find their intersection. It is a *real* challenge to find the only value of b for which b^x never goes below x^b for positive x . The main point is to see how this happens – the graphs of e^x and x^e are *tangent* at $x = e$. In Section 6.2 we know the derivatives and verify $e^x \geq x^e$.

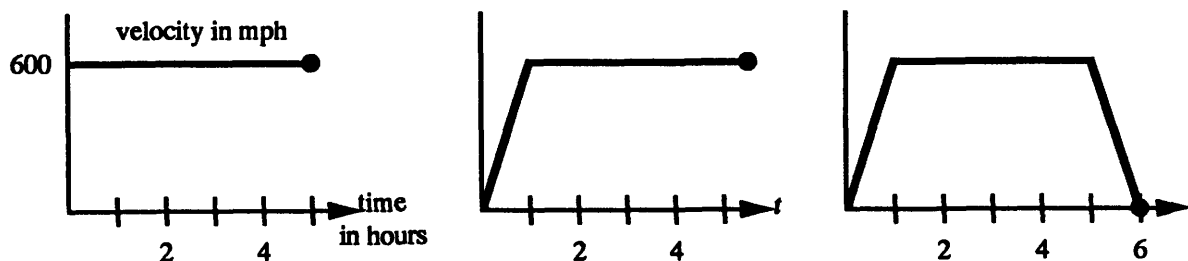
Computing needs to be separate from the stream of ideas that launch calculus in Chapter 2. “1000 Points of Light” is purely for entertainment. See the College Mathematics Journal of November 1990, and a forthcoming American Mathematical Monthly paper by Richert.

Graphing Activities

Ithaca College has developed a course that begins with a study of graphs and includes larger projects throughout the course. They find that “posing questions about graphs results in abundant class participation and student response.” I can confirm this from the Boston Workshop for Mathematics Faculty.

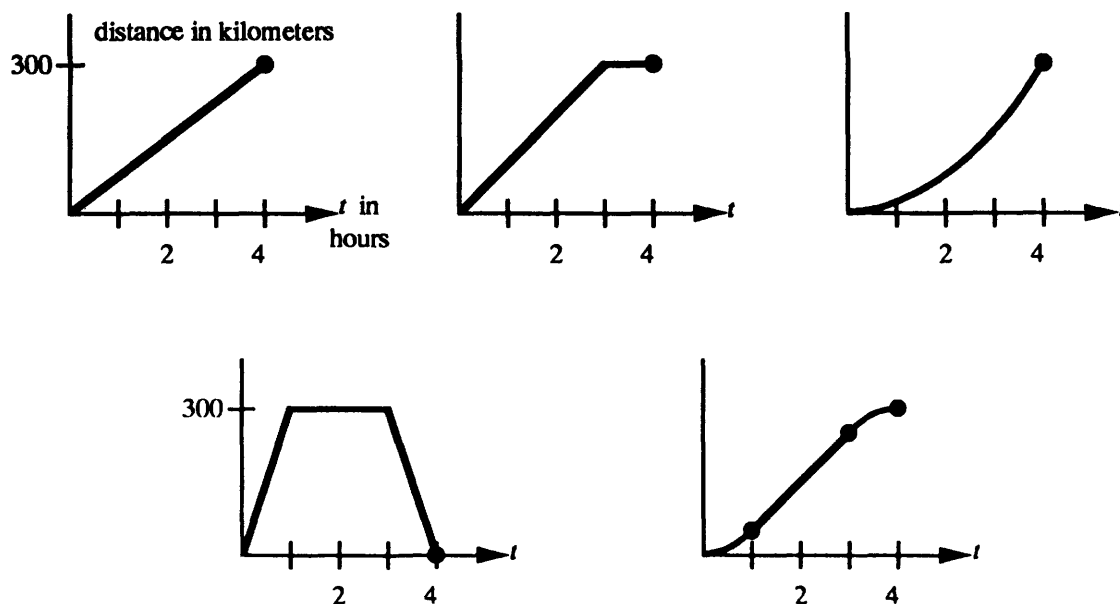
The Ithaca materials will be published in the future – here are three examples suggested by and adapted from their graphing activities. They can all be done *in class*.

EXAMPLE 1 These graphs describe flights from Los Angeles toward Washington:



- Explain each flight in complete sentences. Does the third flight end in Washington or California?
- Try to give a formula for each velocity (in terms of time).
- Sketch a graph of the distance traveled versus time – same horizontal axis, distances on the vertical axis (with numbers).
- Which graph is most realistic? Can you make it more realistic?
- If the first plane climbs slowly over bad weather in Chicago, how does that affect the graphs of velocity and distance?

EXAMPLE 2 These graphs describe the distance traveled from New York toward Boston.



- (a) Describe the five trips. For each graph, when is the distance increasing and when is it decreasing?
- (b) Try to give reasonable rules for the distances in terms of time.
- (c) Sketch the five velocity graphs. Which (if any) are realistic?
- (d) Give rules for the velocities in terms of time.
- * (e) Draw the distance graph for travel from your room to this class.
- * (f) Draw the height graph for raising a flag hand over hand.

- Here is a word problem that uses the connection between $v = 8t$ and $f = 4t^2$, and then finds roots of a quadratic. It could be discussed after Section 1.3 or with tangent lines.

Problem: A car goes 240 feet in 8 seconds. Up to time t_1 it has $v = 8t$ (constant acceleration); after that time it has $v = 8t_1$ (constant velocity). Find t_1 , by computing the distance $f(t)$ and solving $f(8) = 240$. Draw the graphs of $f(t)$ and $v(t)$.

Solution: $4t_1^2 + 8t_1(8 - t_1) = 240$ yields $t_1 = 6$. The graph of $v(t)$ is piecewise linear. The graph of $f(t)$ shows a parabola changing into its tangent line at $t = 6$.

EXAMPLE 3 Students start lining up for registration at 8:00 at the rate of 200 students per hour. From 12 noon the rate decreases steadily until 2:00, when 1,100 students have arrived and the rate stabilizes at 100 per hour. At 9:00, 250 students per hour are allowed to enter. At 1:00 the gym closes for a faculty lunch. After 2:00 the entering rate is 150 students per hour. At 4:00 the last student arrives and enters.

- (a) Draw the graph of the arrival rate. The horizontal axis gives the time (between 8:00 a.m. and 4:00 p.m.).
- (b) Draw the graph of the rate at which students enter the gym vs. time between 8:00 and 4:00.
- (c) Graph the total number of students who have arrived at the gym vs. time between 8:00 and 4:00.
- (d) Graph the number of students who have entered the gym vs. time between 8:00 and 4:00.
- (e) Give a rule for the number of students who have entered as a function of elapsed time since 8:00.
- (f) Graph the length of the waiting line as a function of time.
- (g) Which student has the longest wait?

Comments on Group Work

I am convinced by the experience of others: *It is good for students to work in small groups.* There are so many reports about the success of this idea that it has to be accepted as valuable. It will be implemented in different ways, and the comments from Ithaca about group projects are representative:

“The approach changed the students’ attitudes toward mathematics. The projects engage the curiosity of the good students and challenge them, but this does not come at the expense of average and weaker students. In fact, cooperative work with good, motivated students bolsters the others.”

“A common fear about groups is that one student may do nothing but still get the same grade as the members who did all the work. This has not been a great problem . . . (others say the same). Students experience cooperative learning. *They talk to each other about mathematical ideas* and they form friendships with other mathematics students.”

I personally believe that we too often lose sight of the human part of learning mathematics.

- Section 2.2: My class resists the binomial theorem but likes *Pascal’s triangle*. This inconsistency suggests that theorems are not as attractive as patterns! I made a transparency of Pascal’s triangle down to $(1 + 1)^{15}$, and colored all the even coefficients. They fall into neat triangles which the class can more or less explain.
- Here is a True-False question that looks too good to be true. *The derivative of x^n at $x = n$ is n^n .*

Iteration and Newton’s Method

These topics deserve more attention than in the past. The iteration $x_{n+1} = F(x_n)$ is a simple idea and it is the basis of scientific computing. Its simplicity means that a numerical experiment like $x_{n+1} = \cos x_n$ is instantly available. On the TI-81 no program is necessary: use `.7 ENTER` and on a new line `cos Ans`. Now each press of ENTER gives the next x_{n+1} as the cosine of the previous Answer. Here is the output, six steps at a time:

$x_0 = .7$	
$x_6 = .73$	
$x_{12} = .739$	(keeping only the digits that have settled down: suggested by Rob Corless)
$x_{18} = .7390$	
$x_{24} = .73908$	
$x_{30} = .739085$	
$x_{36} = .7390851$	

The limit of $x_{n+1} = \cos x_n$ is the underlying equation $x = \cos x$. Linear convergence is apparent from this triangle of numbers. At the solution $\frac{dF}{dx} = -\sin x \approx -.68$, so every six steps multiply the error by about $(-.68)^6 \approx \frac{1}{10}$. This explains why one new decimal is correct after six steps. Compare with the *quadratic* convergence of Newton’s method for $x - \cos x = 0$:

$x_0 = .7$ $x_1 = .739$ $x_2 = .739085$ x_3 : 12 correct digits!

Thus two Newton steps match thirty steps of simple iteration. But Newton requires the derivative – in fact 100 partial derivatives for 10 equations in 10 unknowns. This leaves a place for iteration.

What to cover in a calculus course? Certainly we must teach Newton's method, which uses the central idea of linear approximation and the tangent line. I believe that *simple iteration should also be included*, because its convergence is a perfect calculus problem: $x_{n+1} - x = F(x_n) - F(x)$. Approximate the right side by $\frac{dF}{dx}$ times $x_n - x$. The error is multiplied at every step by $\frac{dF}{dx}$. But where is this derivative evaluated? The Mean Value Theorem says *between* x_n and x . Linear approximation says *at the solution* x . Convergence is guaranteed if $|\frac{dF}{dx}| < 1$ everywhere.

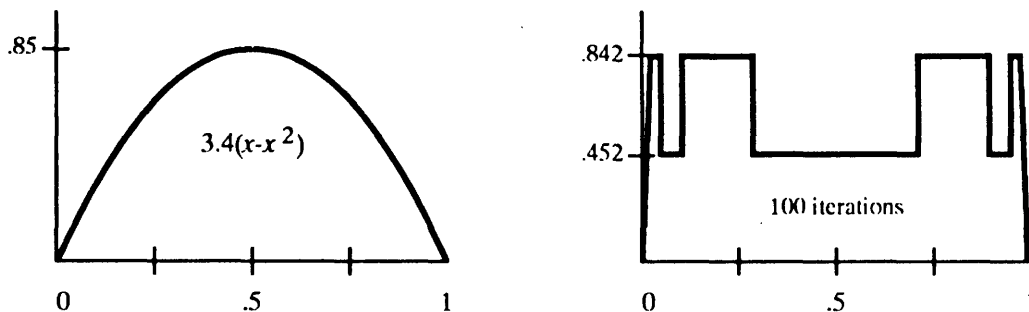
Some new courses compare the iterations

$$1 \quad x_{n+1} = ax_n \qquad 2 \quad x_{n+1} = ax_n + s \qquad 3 \quad x_{n+1} = ax_n - bx_n^2$$

Those discrete systems match the key differential equations in this book: $\frac{dy}{dt} = cy$ and $\frac{dy}{dt} = cy + s$ and $\frac{dy}{dt} = cy - by^2$. The logistic model 3 describes a population in successive years, and no derivative is involved. The solutions can approach a steady state or a cycle or chaos.

Note: The text introduces chaos through Newton's method for $x^2 + 1 = 0$. The iteration is easy to try: $x_{n+1} = \frac{1}{2}(x_n - \frac{1}{x_n})$. Most classes will touch chaos and fractals very lightly – that is absolutely normal, since there is so much calculus to do. But the book must offer the possibility to read something that Newton and Leibniz and even Riemann did not know. I will be grateful for any comments on your experience, and also on the last exercises in Sections 3.6 and 3.7.

• Problem 3.7.48 starts with $y(x) = 3.4(x - x^2)$ and asks for graphs of $y(y(x))$ and $y(y(y(x)))$. Those are polynomials of degree 4 and 8. *The tenth iterate has degree $2^{10} = 1024$* , but this polynomial is easy to graph (by computer). So is the hundredth iterate. Maybe a polynomial of degree 2^{100} has never been seen before! Its graph is very surprising at first:



At each x the graph shows x_{100} , the result of 100 iterations. What do we expect from one more iteration? The graph above will *flip over* for the polynomial of degree 2^{101} , which sends .452 to .842 and vice versa. The function $\cos(\cos(\cos \dots$ would have a horizontal graph at height $x = .7390851$.

The Chain Rule

It is unusual to have a full chapter with this title. The chain rule is generally squeezed in earlier, with a quick explanation of $f(g(x))$. But this is an idea of major importance. It is how new functions are created. The four sections of this chapter (including inverse functions and implicit differentiation and related rates) make a natural development from $f(g(x))$ and the chain rule.

Two special cases we see earlier. One is the power rule $u^{n-1} \frac{du}{dx}$. The other allows examples like $\sin 2x$, when $\frac{du}{dx} = 2$ is a constant. I would call attention to those cases when the general chain rule is presented. And I really feel that $\frac{\Delta x}{\Delta x} = \frac{\Delta x}{\Delta y} \frac{\Delta y}{\Delta x}$ is so clear and simple that a much longer argument could only obscure the key idea.

Inverse functions also deserve serious discussion (and exercises). Draw graphs to see f^{-1} and to explain why f must be one-to-one. This section is preparing for $y = e^x$ and $x = \ln y$. Also for $y = \sin x$ and $x = \sin^{-1} y$ which follow immediately.

To me the correct notation is $x = f^{-1}(y)$. The inverse of $y = x^3$ is $x = y^{1/3}$. We should not switch x and y ! Don't say that "the inverse of the function $y = x^3$ is the function $y = x^{1/3}$."

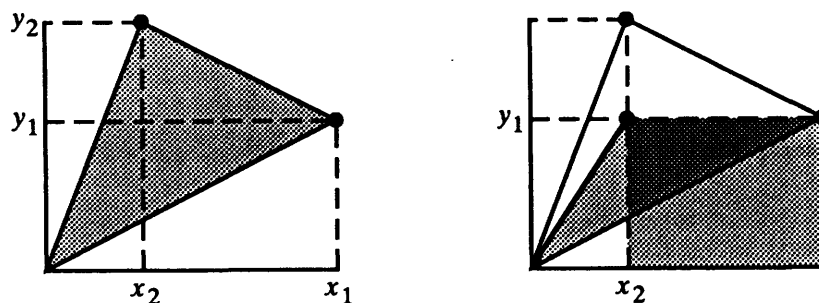
The inverse of $f(g(x))$ is a basic idea, to reverse to $g^{-1}(f^{-1}(z))$. The student can get this.

• Polar Triangles are the Average of Vertical and Horizontal Strips

Most areas are computed in one of three ways: $\int y \, dx$ or $\int x \, dy$ or $\frac{1}{2} \int r^2 \, d\theta$. Those integrals begin with rectangular strips or polar triangles. I was amazed to discover that even for finite strips and triangles (not just in the limit), the three areas are directly related. *The polar area is the average of the two strip areas.*

Any reference to the history of this observation (which cannot be new) will be very gratefully received. I am preparing a brief note on several related ideas.

The figure on the left shows two strips and a triangle, when the function is linear.



The strips are trapezoids. Their areas are $\frac{1}{2}(x_1 - x_2)(y_1 + y_2)$ and $\frac{1}{2}(x_1 + x_2)(y_2 - y_1)$. The average is $\frac{1}{2}(x_1 y_2 - x_2 y_1)$. This formula for the area of a triangle appears in Chapter 11, as half of a vector parallelogram or a cross product or a 2 by 2 determinant. So we have an algebraic proof, when the graph is a straight line. We need a geometrical proof that still applies when the function is not linear and its graph between (x_1, y_1) and (x_2, y_2) is curved.

Geometrical proof: Draw the extra line to (x_2, y_1) in the second figure. The shaded rectangle and triangle have the same height y_1 and the same base. The triangular area is half the rectangular area. The same argument applies above the extra line (rectangle at the top left, triangle down to the origin). That leaves the remaining piece which is above y_1 and beyond x_2 . This piece belongs to both of the strips and also to the triangle, *whether the graph is curved or straight*. Then adding the (curved) trapezoids gives twice the area of the (curved) triangle.

Any class discussion can wait until you have computed area in polar coordinates (Chapter 9).

There is also a “calculus proof” that the triangle area is the average of the strip areas. For infinitesimal pieces the three areas are $x dy$ and $-y dx$ and $\frac{1}{2}(x dy - y dx)$. The minus sign is because a line integral goes counterclockwise from (x_1, y_1) to (x_2, y_2) , so dx is negative. Then the three line integrals, around any region, give the area. This is the first application of Green’s Theorem (page 564), but it is not a surprise – areas are always found from strips. What was a surprise was the realization that $\frac{1}{2}(x dy - y dx)$ is the same as $\frac{1}{2}r^2 d\theta$. By substituting $x = r \cos \theta$ and $y = r \sin \theta$ this is confirmed.

Numerical Integration

This section emphasizes an important fact about numerical integration: *the error is proportional to $(\Delta x)^p$* . The order of accuracy p comes from a simple test: apply the integration rule to the power functions $1, x, x^2, \dots$. *The first power to be wrong is p* . The rule fails to compute $\int_0^1 x^p dx$. The trapezoidal rule has $p = 2$ and Simpson has $p = 4$. New rules are constructed so as to integrate powers correctly.

This error term $(\Delta x)^p$ is what all numerical analysts use – *not* the exact expression which involves the $(p + 1)$ st derivative of $y(x)$. I do not discuss the exact error integrals at all – life is too short.

- *Question: What is the error exponent p for Riemann sums?* The answer depends on the evaluation points x_k^* . We know that $p = 2$ for midpoints and $p = 1$ for left endpoints. Also $p = 2$ for alternating left and right endpoints! To make the question sensible for any Riemann sum, why not pick the points x_k^* *at random*?

This is now a computing problem (and also a probability problem). The student can use the random number generator – an excellent tool for numerical experiments. Try $\int_0^1 x dx$ with intervals $\Delta x = \frac{1}{10}$ and $\frac{1}{20}$; repeat 100 times with random x_k^* . I believe *the expected error is proportional to $(\Delta x)^{1.5}$* . The appearance of a fraction $p = \frac{3}{2}$ is unusual to say the least.

- Jim Cargal suggests more simple integration exercises in Chapter 5.

Differential Equations

A significant part of “calculus reform” is to bring out more clearly the *purpose* of calculus. That leads to an *earlier appearance of differential equations*. In the simplest case (Section 2.2) we are just reversing the derivative, as in $\frac{dy}{dx} = 3x^2$. This is important (it motivates integration). It is also easy to discuss. We cannot let students finish calculus, as many do, without hearing the words “differential equation.”

In the second semester we can do much more. Here is an outline for three specific first-order equations. The applications are in the book along with second-order equations (those use complex numbers). The key equations are

$$1 \quad \frac{dy}{dt} = cy \qquad 2 \quad \frac{dy}{dt} = cy + s \qquad 3 \quad \frac{dy}{dt} = cy - by^2.$$

1 gives exponential growth or decay: $y = y_0 e^{ct}$. Equation 2 has a constant source term. The logistic equation 3 has competition from the nonlinear mass action term – this is basic in the life sciences. With the benefit of hindsight, 2 and 3 can be rewritten to show pure exponential growth:

$$\text{Rewrite 2 as } \frac{d}{dt}\left(y + \frac{s}{c}\right) = c\left(y + \frac{s}{c}\right). \text{ This gives } y + \frac{s}{c} = \left(y_0 + \frac{s}{c}\right)e^{ct}.$$

$$\text{Rewrite 3 as } \frac{d}{dt}\left(\frac{y}{c - by}\right) = c\left(\frac{y}{c - by}\right). \text{ This gives } \frac{y}{c - by} = \left(\frac{y_0}{c - by_0}\right)e^{ct}.$$

Those solutions yield formulas for $y(t)$. For convenience set $d = (c - by_0)/y_0$:

$$2 \quad y(t) = y_0 e^{ct} + \frac{s}{c}(e^{ct} - 1) \qquad 3 \quad y(t) = \frac{c}{b + de^{-ct}} \text{ (the S-curve).}$$

Notice how all these solutions approach *steady states* y_∞ when $c < 0$. The exponential e^{ct} decays to zero. The differential equation approaches $\frac{dy}{dt} = 0$ (no movement):

$$1 \quad y_0 e^{ct} \rightarrow y_\infty = 0 \qquad 2 \quad cy_\infty + s = 0 \text{ gives } y_\infty = -\frac{s}{c} \qquad 3 \quad cy_\infty = by_\infty^2 \text{ gives } y_\infty = \frac{c}{b}.$$

It may be useful to collect together the methods of solution, especially for the equation $\frac{dy}{dt} = cy + s$:

- A. Assume $y = Ae^{ct} + B$ (undetermined coefficients)
- B. Multiply the equation by e^{-ct} (integrating factor)
- C. Add up outputs from each input: $y_0 e^{ct} + \int_0^t e^{c(t-\tau)} s d\tau$ (Duhamel)
- D. Integrate $\int \frac{dy}{cy+s} = \int dt$ (separation of variables)

The logistic equation is also solved by separation of variables and integration of $\frac{1}{cy-by^2}$. This is a good contribution from *partial fractions* (which are quite useful for quadratics – we probably overemphasize the high degree cases). Note that $z = \frac{1}{y}$ converts the logistic equation to $\frac{dz}{dt} = -cz + b$ (linear!).

Notes on Chapter 7

- **7.1 Integration by Parts:** Most likely it will take all your time to practice integration by parts (this is a technique of integration that deserves to survive). If no time is left for the delta function or the application to engineering, that is no problem.

The exercises include n integrations by parts, leading to Taylor's formula with integral remainder term. Here is Problem 19 with a clever choice $v = \frac{x^2+1}{2}$: $\int x \tan^{-1} x dx = \frac{x^2+1}{2} \tan^{-1} x - \frac{1}{2} \int 1 dx$.

- **7.2 Trigonometric Integrals:** Example 3 is the integral of $\cos^2 x$. Show why the substitution $u = \cos x$ does not succeed. Then double angle formulas make it easy (also $\cos^4 x$ and $\cos^2 x \sin^2 x$). One integral that is needed later is $\int \sec^3 x dx$.

Maybe the substitution $u = \tan \frac{1}{2} x$ should not come "out of the blue". Draw a unit circle with a typical point $(\cos x, \sin x)$ in the first quadrant. Then draw the chord to that point from the left-hand point $(-1, 0)$. By geometry, the chord makes an angle of $\frac{x}{2}$ with the horizontal. The figure shows that $\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}$. The chord crosses the vertical axis at $(0, u)$. To be truthful, I don't teach this substitution.

- **7.3 Trigonometric Substitutions:** Usually the integral of $\sqrt{1-x^2}$ is pure manipulation. The student must see this as area under a circle. Figure 7.5a shows why the answer has two parts $\frac{1}{2}x\sqrt{1-x^2}$ and $\frac{1}{2}\sin^{-1}x$. Notice the paradox for a complete circle in Problem 53, when the θ integral goes from zero to 2π .
- **7.4 Partial Fractions:** The application in the text is to the integration of rational functions. But partial fractions can also improve the *derivative* (Hendel, College Math Journal 11/91). The fourth derivative of $\frac{1}{x^2+3x+2}$ looks unpleasant – no pattern is visible. Computer algebra can give a formula for any derivative, but it won't be as neat as this:

$$\frac{1}{x^2+3x+2} = \frac{1}{x+1} - \frac{1}{x+2} \text{ so the } n\text{th derivative is } \frac{(-1)^n n!}{(x+1)^{n+1}} - \frac{(-1)^n n!}{(x+2)^{n+1}}.$$

- **7.5 Improper Integrals:** Here is a True-False question to produce discussion:
If $\int_1^\infty y(x)dx$ converges (the area is finite), then $y(x) \rightarrow 0$ as $x \rightarrow \infty$.

Infinite Series

The *geometric series* is absolutely essential. It is easy to get involved with convergence tests and lose that explicit formula. I think some numbers are good in Section 10.1. It might be interesting to take a few steps of the Gauss algorithm on page 372 that computes π .

Apologies for Problem 68 on page 385, which asks for a series whose convergence you cannot decide. Maybe the student gets a chance to be creative. I wouldn't go overboard with the remainder terms in the Taylor series (page 392). But the Taylor series itself is the best connection between finite series and the mainstream of calculus.

Vectors and Matrices

This chapter is an interesting challenge – to present just enough of linear algebra, even as far as A^{-1} and least squares, but not too much. I try to give explanations in \mathbf{R}^2 and \mathbf{R}^3 that remain useful in \mathbf{R}^n . The row picture and column picture of a linear system are both important. Do not forget that a matrix-vector multiplication Av is a *combination of the columns* of A .

Before that come dot products and cross products. Dot products are for everybody – and the formula for $\cos\theta$ is generally well received. Cross products are less fun, but formula (4) for A^{-1} in Section 11.5 is neat. So is the area of a triangle – neater than I realized:

- **Area of a triangle T** The 3 by 3 determinant is in Problem 11.3.44:

$$\text{Area of } T = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} (x_1 y_2 - x_2 y_1) + \frac{1}{2} (x_2 y_3 - x_3 y_2) + \frac{1}{2} (x_3 y_1 - x_1 y_3)$$

This is half of three cross products. Therefore three triangular areas. One triangle connects $(0,0)$ to (x_1, y_1) and (x_2, y_2) . Two more triangles go from $(0,0)$ to the other two edges of T . If $(0,0)$ is inside T , the three small triangles add to the complete triangle. But the sum gives the correct area even when $(0,0)$ is outside T – get the class to see why.

Notice that the sum does not depend on the position of the origin. Adding a constant to x_1, x_2, x_3 leaves the sum unchanged. So does adding a constant to y_1, y_2, y_3 . Drawing a picture is better, with $(0,0)$ outside and areas canceling to leave the area of T .

- **Area of a polygon P** Barry Cipra actually met two carpet-layers in a Minnesota coffee shop, who wanted a formula for the area of a carpet. Inspired by caffeine he proposed

$$\text{Area of } P = \frac{1}{2} [(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \cdots + (x_n y_1 - x_1 y_n)].$$

It's right, and a good student can see why. *Divide the polygon into triangles*, by connecting each edge to the origin. My experience in this direction was with a lawyer, who wanted the area of a property and not a carpet. "If I count steps walking around the boundary, what is the area?" He was very disappointed when I needed to know the shape. Now I would tell him the carpet-layer formula.

I hope you don't mind saving the equation of a line for Section 12.1, which follows immediately. If you don't want projectiles and curvature and Kepler's Laws, you can stop after 12.1. I like the last law with the $\frac{3}{2}$ power.

Partial Derivatives

Students seem to get the knack of computing $\frac{\partial f}{\partial x}$ and treating y as constant. The idea of the gradient is not so easy. I keep drawing surfaces to teach this chapter, with tangent planes and normal vectors. In explaining the analysis, the summary on page 471 (multivariable vs. single variable) goes a long way. The study of continuity for $f(x, y)$ can get technical quickly – perhaps it is more important to understand the geometry.

One high point of the chapter is to maximize by solving $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. Again, draw the tangent plane. The Steiner problem gives an example with some substance, but it could be omitted. There is a lot to do in Section 13.6. Then 13.7 with constraints and Lagrange multipliers is certainly optional (but important – few calculus texts try to explain what λ represents!)

Caspar Curjel has a book of multivariable exercises (McGraw-Hill) based on figures like those on page 496. In one variable we easily connect $f(x)$ and $\frac{df}{dx}$ to graphs and slopes. It is not so simple for two variables (or three). Should the n -dimensional case be developed in this chapter?

Vector Fields and Green's Theorem

These topics generally come late in multivariable calculus. Many times they don't come at all – for an engineering or physics major they are important, for a biologist or an economist less so. This note is to call attention to the treatment of one specific question: Which pairs of functions $M(x, y)$ and $N(x, y)$ are the partial derivatives of a single function, $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$?

In this case the vector field $M\mathbf{i} + N\mathbf{j}$ is the *gradient* of $f(x, y)$. The field is *conservative*. The integral $\int Mdx + Ndy$ is $\int df = \text{change in } f$. The differential is *exact*. This is a special and highly important situation (not only in physics), and the test is simple: $\frac{\partial M}{\partial y}$ will be $\frac{\partial^2 f}{\partial y \partial x}$ which must equal $\frac{\partial N}{\partial x}$. The function f can be computed based only on Chapter 13, without reaching vector fields.

I feel good about page 560, which brings together the properties of a conservative field. Many texts fail to do this. If gradients are in one place, and path independence is separate, and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is separate again, how can students see the picture? I hope the pattern on pages 570-571 is helpful.

The best fields also satisfy $\frac{\partial M}{\partial x} = -\frac{\partial N}{\partial y}$. The divergence is zero – there is no source. Then $f(x, y)$ satisfies Laplace's equation. This is the high point of the whole topic.

Added in proof (College Mathematics Journal, May 1991): Find the sloping asymptote of

$$y = \frac{x^2 + 3x + 7}{x + 2}.$$

Solution 1. Division gives $y = x + 1 + \frac{5}{x+2}$ so the asymptote is the line $y = x + 1$.

Solution 2. Following a procedure frequently used in calculating limits at infinity,

$$\frac{x^2 + 3x + 7}{x + 2} = \frac{x + 3 + 7/x}{1 + 2/x}.$$

For large x the value is approximately $y = x + 3$, so this line is the asymptote.