CHAPTER 14 MULTIPLE INTEGRALS

14.1 Double Integrals (page 526)

The most basic double integral has the form $\int\int_R dA$ or $\int\int_R dy\, dx$ or $\int\int_R dx\, dy$. It is the integral of 1 over the region R in the xy plane. The integral equals the ***area of*** R. When we write dA, we are not committed to xy coordinates. The coordinates could be r and θ (polar) or any other way of chopping R into small pieces. When we write $dy\, dx$, we are planning to chop R into vertical strips (width dx) and then chop each strip into very small pieces (height dy). The y integral assembles the pieces and the x integral assembles the strips.

Suppose R is the rectangle with $1 \leq x \leq 4$ and $2 \leq y \leq 7$. The side lengths are 3 and 5. The area is 15:

$$\int_{x=1}^{4}\int_{y=2}^{7} 1\, dy\, dx = \int_{x=1}^{4}(7-2)dx = [5x]_1^4 = 20 - 5 = 15$$

The inner integral gave $\int_2^7 1\, dy = [y]_2^7 = 7 - 2$. This is the height of the strips.

My first point is that this is nothing new. We have written $\int y\, dx$ for a long time, to give the area between a curve and the x axis. The height of the strips is y. *We have short-circuited the inner integral* of $1\, dy$.

Remember also the area between two curves. That is $\int(y_2 - y_1)dx$. Again we have already done the inner integral, between the lower curve y_1 and the upper curve y_2. The integral of $1\, dy$ was just $y_2 - y_1 =$ height of strip. We went directly to the outer integral – the x-integral that adds up the strips.

So what is new? First, the regions R get more complicated. The limits of integration are not as easy as $\int_2^7 dy$ or $\int_1^4 dx$. Second, we don't always integrate the function "1". In particular, double integrals often give ***volume***:

$$\int_1^4\int_2^7 f(x,y)\, dy\, dx \text{ is the volume between the surface } z = f(x,y) \text{ and the } xy \text{ plane.}$$

To be really truthful, volume starts as a triple integral. It is $\int\int\int 1\, dz\, dy\, dx$. The inner integral $\int 1\, dz$ gives z. The lower limit on z is 0, at the xy plane. The upper limit is $f(x,y)$, at the surface. So the inner integral $\int 1\, dz$ between these limits is $f(x,y)$. When we find volume from a *double* integral, we have short-circuited the z-integral $\int 1\, dz = f(x,y)$.

The second new step is to go beyond areas and volumes. We can compute masses and moments and averages of all kinds. The integration process is still $\int\int_R f(x,y)dy\, dx$, if we choose to do the y-integral first. In reality the main challenge of double integrals is to find the limits. You get better by doing examples. We borrow a few problems from Schaum's Outline and other sources, to display the steps for double integrals – and the difference between $\int\int f(x,y)dy\, dx$ and $\int\int f(x,y)dx\, dy$.

1. Evaluate the integral $\int_{x=0}^{1}\int_{y=0}^{x}(x+y)dy\, dx$. Then reverse the order to $\int\int(x+y)dx\, dy$.

   - The inner integral is $\int_{y=0}^{x}(x+y)dy = [xy + \frac{1}{2}y^2]_0^x = \frac{3}{2}x^2$. This is a function of x. The outer integral is a completely ordinary x-integral $\int_0^1 \frac{3}{2}x^2 dx = [\frac{1}{2}x^3]_0^1 = \frac{1}{2}$.
   - Reversing the order is simple for rectangles. But we don't have a rectangle. The inner integral goes from $y = 0$ on the x axis up to $y = x$. This top point is on the 45° line. *We have a triangle* (see figure). When we do the x-integral first, it starts at the 45° line and ends at $x = 1$. The inner x-limits can depend on y, they can't depend on x. The outer limits are *numbers* 0 and 1.

$$\begin{array}{llll} \textbf{\textit{inner}} & \int_y^1(x+y)dx & = [\frac{1}{2}x^2 + xy]_y^1 & = \frac{1}{2} + y - \frac{3}{2}y^2 \\ \textbf{\textit{outer}} & \int_0^1(\frac{1}{2} + y - \frac{3}{2}y^2)dy & = [\frac{1}{2}y + \frac{1}{2}y^2 - \frac{1}{2}y^3]_0^1 & = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{1}{2}. \end{array}$$

The answer $\frac{1}{2}$ is the same in either order. The work is different.

2. Evaluate $\int\int_R y^2 dA$ in both orders $dA = dy\, dx$ and $dA = dx\, dy$. The region R is bounded by $y = 2x$, $y = 5x$, and $x = 1$. Please draw your own figures – vertical strips in one, horizontal strips in the other.

   - The vertical strips run from $y = 2x$ up to $y = 5x$. Then x goes from 0 to 1:

$$\textbf{\textit{inner}} \quad \int_{2x}^{5x} y^2 dy = [\frac{1}{3}y^3]_{2x}^{5x} = \frac{1}{3}(125x^3 - 8x^3) = 39x^3 \qquad \textbf{\textit{outer}} \quad \int_0^1 39x^3 dx = \frac{39}{4}.$$

For the reverse order, the limits are not so simple. The figure shows why. In the lower part, horizontal strips go between the sloping lines. The inner integral is an x-integral so change $y = 5x$ and $y = 2x$ to $x = \frac{1}{5}y$ and $x = \frac{1}{2}y$. The outer integral in the lower part is from $y = 0$ to 2:

$$\textbf{\textit{inner}} \quad \int_{y/5}^{y/2} y^2 dx = [y^2 x]_{y/5}^{y/2} = (\frac{1}{2} - \frac{1}{5})y^3 \qquad \textbf{\textit{outer}} \quad \int_0^2 (\frac{1}{2} - \frac{1}{5})y^3 dy = (\frac{1}{2} - \frac{1}{5})\frac{2^4}{4}.$$

The upper part has horizontal strips from $x = \frac{1}{5}y$ to $x = 1$. The outer limits are $y = 2$ and $y = 5$:

$$\textbf{\textit{inner}} \quad \int_{y/5}^{1} y^2 dx = [y^2 x]_{y/5}^{1} = y^2 - \frac{1}{5}y^3 \qquad \textbf{\textit{outer}} \quad \int_2^5 (y^2 - \frac{1}{5}y^3)dy = [\frac{1}{3}y^3 - \frac{1}{20}y^4]_2^5 = \frac{125-8}{3} - \frac{625-16}{20}$$

Add the two parts, preferably by calculator, to get 9.75 which is $\frac{39}{4}$. *Same answer.*

3. Reverse the order of integration in $\int_0^2 \int_0^{x^2} (x + 2y)dy\, dx$. What *volume* does this equal?

   - The region is bounded by $y = 0$, $y = x^2$, and $x = 2$. When the x-integral goes first it starts at $x = \sqrt{y}$. It ends at $x = 2$, where the horizontal strip ends. Then the outer y-integral ends at $y = 4$:

$$\textit{The reversed order is} \quad \int_0^4 \int_{\sqrt{y}}^2 (x + 2y)dx\, dy. \qquad \textit{Don't reverse } x + 2y \textit{ into } y + 2x!$$

Read-throughs and selected even-numbered solutions :

The double integral $\int\int_R f(x, y)dA$ gives the volume between R and **the surface $\mathbf{z = f(x,y)}$**. The base is first cut into small **squares** of area ΔA. The volume above the ith piece is approximately $\mathbf{f(x_i, y_i)\Delta A}$. The limit of the sum $\sum \mathbf{f(x_i, y_i)\Delta A}$ is the volume integral. Three properties of double integrals are $\int\int \mathbf{(f + g)dA} = \int\int \mathbf{fdA}$ $+ \int\int \mathbf{gdA}$ and $\int\int \mathbf{cfdA} = \mathbf{c} \int\int \mathbf{fdA}$ and $\int\int_{\mathbf{R}} \mathbf{f\ dA} = \int\int_{\mathbf{S}} \mathbf{f\ dA} + \int\int_{\mathbf{T}} \mathbf{f\ dA}$ **if R splits into S and T.**

If R is the rectangle $0 \le x \le 4, 4 \le y \le 6$, the integral $\int\int x\, dA$ can be computed two ways. One is $\int\int x\, dy\, dx$, when the inner integral is $\mathbf{xy]_4^6 = 2x}$. The outer integral gives $\mathbf{x^2]_0^4 = 16}$. When the x integral comes first it equals $\int\ x\,dx = \frac{1}{2}\mathbf{x}^2]_0^4 = 8$. Then the y integral equals $8\mathbf{y}]_4^6 = \mathbf{16}$. This is the volume between **the base rectangle and the plane $\mathbf{z} = \mathbf{x}$.**

The area R is $\iint \mathbf{1}dy\,dx$. When R is the triangle between $x = 0, y = 2x$, and $y = 1$, the inner limits on y are **$\mathbf{2x}$ and $\mathbf{1}$**. This is the length of a **thin vertical** strip. The (outer) limits on x are **$\mathbf{0}$ and $\frac{\mathbf{1}}{\mathbf{2}}$**. The area is $\frac{\mathbf{1}}{\mathbf{4}}$. In the opposite order, the (inner) limits on x are **$\mathbf{0}$ and $\frac{1}{2}\mathbf{y}$**. Now the strip is **horizontal** and the outer integral is $\int_0^1 \frac{1}{2}\mathbf{y}\,\mathbf{dy} = \frac{\mathbf{1}}{\mathbf{4}}$. When the density is $\rho(x,y)$, the total mass in the region R is $\iint \rho\,\mathbf{dx}\,\mathbf{dy}$. The moments are $M_y = \iint \rho\mathbf{x}\,\mathbf{dx}\,\mathbf{dy}$ and $M_x = \iint \rho\mathbf{y}\,\mathbf{dx}\,\mathbf{dy}$. The centroid has $\bar{x} = M_y/M$.

10 The area is all below the axis $y = 0$, where horizontal strips cross from $x = y$ to $x = |y|$ (which is $-y$). Note that the y integral stops at $y = 0$. Area $= \int_{-1}^0 \int_y^{-y} dx\,dy = \int_{-1}^0 -2y\,dy = [-y^2]_{-1}^0 = \mathbf{1}$.

16 The triangle in Problem 10 had sides $x = y, x = -y$, and $y = -1$. Now the vertical strips go from $y = -1$ up to $y = x$ on the left side: area $= \int_{-1}^0 \int_{-1}^x dy\,dx = \int_{-1}^0 (x+1)dx = \frac{1}{2}(x+1)^2]_{-1}^0 = \frac{1}{2}$. The strips go from -1 up to $y = -x$ on the right side: area $= \int_0^1 \int_{-1}^{-x} dy\,dx = \int_0^1 (-x+1)dx = \frac{1}{2}$. Check: $\frac{1}{2} + \frac{1}{2} = 1$.

24 The top of the triangle is (a,b). From $x = 0$ to a the vertical strips lead to $\int_0^a \int_{dx/c}^{bx/a} dy\,dx = [\frac{bx^2}{2a} - \frac{dx^2}{2c}]_0^a = \frac{ba}{2} - \frac{da^2}{2c}$. From $x = a$ to c the strips go up to the third side: $\int_a^c \int_{dx/c}^{b+(x-a)(d-b)/(c-a)} dy\,dx = [bx + \frac{(x-a)^2(d-b)}{2(c-a)} - \frac{dx^2}{2c}]_a^c = b(c-a) + \frac{(c-a)(d-b)}{2} - \frac{dc}{2} + \frac{da^2}{2c}$. The sum is $\frac{ba}{2} + \frac{b(c-a)}{2} + \frac{d(c-a)}{2} - \frac{dc}{2} = \frac{\mathbf{bc-ad}}{\mathbf{2}}$. This is half of a parallelogram.

26 $\int_0^b \int_0^a \frac{\partial f}{\partial x} dx\,dy = \int_0^b [f(a,y) - f(0,y)]dy$.

32 The height is $z = \frac{1-ax-by}{c}$. Integrate over the triangular base ($z = 0$ gives the side $ax + by = 1$) : volume $= \int_{x=0}^{1/a} \int_{y=0}^{(1-ax)/b} \frac{1-ax-by}{c} dy\,dx = \int_0^{1/a} \frac{1}{c}[y - axy - \frac{1}{2}by^2]_0^{(1-ax)/b} dx = \int_0^{1/a} \frac{1}{c}\frac{(1-ax)^2}{2b} dx = -\frac{(1-ax)^3}{6abc}]_0^{1/a} = \frac{\mathbf{1}}{\mathbf{6abc}}$.

36 The area of the quarter-circle is $\frac{\pi}{4}$. The moment is zero around the axis $y = 0$ (by symmetry): $\bar{\mathbf{x}} = \mathbf{0}$. The other moment, with a factor 2 that accounts for symmetry of left and right, is $2\int_0^{\sqrt{2}/2} \int_x^{\sqrt{1-x^2}} y\,dy\,dx = 2\int_0^1 (\frac{1-x^2}{2} - \frac{x^2}{2})dx = 2[\frac{x}{2} - \frac{x^3}{3}]_0^{\sqrt{2}/2} = \frac{\sqrt{2}}{3}$. Then $\bar{y} = \frac{\sqrt{2}/3}{\pi/4} = \frac{\mathbf{4\sqrt{2}}}{\mathbf{3\pi}}$.

14.2 Change to Better Coordinates (page 534)

This title really means "better than xy coordinates." Mostly those are the best, but not always. For regions cut out by circles and rays from the origin, polar coordinates are better. This is the most common second choice, but ellipses and other shapes lead to third and fourth choices. So we concentrate on the change to $r\theta$, but we explain the rules for other changes too.

1. Compute the area between the circles $r = 2$ and $r = 3$ and between the rays $\theta = 0$ and $\theta = \frac{\pi}{2}$.

   - The region is a quarter of a ring. When you draw it, you see that xy coordinates are terrible. Strips start and end in complicated ways. Polar coordinates are extremely easy:

$$\int_0^{\pi/2} \int_{r=2}^3 r\,dr\,d\theta = \int_0^{\pi/2} \frac{1}{2}(3^2 - 2^2)d\theta = \frac{1}{2}(3^2 - 2^2)\frac{\pi}{2} = \frac{5\pi}{4}.$$

Notice first that dA is not $dr\ d\theta$. It is $r\ dr\ d\theta$. The extra factor r gives this the dimension of $(\text{length})^2$. The area of a small polar rectangle is $r\ dr\ d\theta$.

Notice second the result of the inner integral of $r\ dr$. It gives $\frac{1}{2}r^2$. This leaves the outer integral as our old formula $\int \frac{1}{2}r^2 d\theta$ from Chapter 9.

Notice third the result of *limits* on that inner integral. They give $\frac{1}{2}(3^2-2^2)$. This leaves the outer integral as our formula for ring areas and washer areas and areas between two polar curves $r = F_1(\theta)$ and $r = F_2(\theta)$. That area was and still is $\int \frac{1}{2}(F_2^2 - F_1^2)d\theta$. For our ring this is $\int \frac{1}{2}(3^2-2^2)d\theta$.

2. (14.2.4) Find the centroid $(\bar{x},\ \bar{y})$ of the pie-shaped wedge $0 \le r \le 1$, $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$. The average height $\bar{y}$ is $\int\int y\ dA / \int\int dA$. This corresponds to moment around the x axis divided by total mass or area.
   - The area is $\int_{\pi/4}^{3\pi/4}\int_0^1 r\ dr\ d\theta = \int_{\pi/4}^{3\pi/4}\frac{1}{2}d\theta = \frac{1}{2}(\frac{\pi}{2})$. The integral $\iint y\ dA$ has $y = r\sin\theta$:
   $$\int_{\pi/4}^{3\pi/4}\int_0^1 (r\ \sin\theta) r\ dr\ d\theta = \int_{\pi/4}^{3\pi/4}[\frac{1}{3}r^3]_0^1\ \sin\theta\ d\theta = \frac{1}{3}[-\cos\theta]_{\pi/4}^{3\pi/4} = \frac{1}{3}\cdot 2\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{3}.$$
   Now divide to find the average $\bar{y} = (\frac{\sqrt{2}}{3})/(\frac{\pi}{4}) = \frac{4\sqrt{2}}{3\pi}$. This is the height of the centroid.

   Symmetry gives $\bar{x} = 0$. The region for negative x is the mirror image of the region for positive x. This answer zero also comes from integrating $x\ dy\ dx$ or $(r\ \cos\theta)(r\ dr\ d\theta)$. Integrating $\cos\theta d\theta$ gives $\sin\theta$. Since $\sin\frac{\pi}{4} = \sin\frac{3\pi}{4}$, the definite integral is zero.

The text explains the "stretching factor" for any coordinates. It is a 2 by 2 determinant J. Write the old coordinates in terms of the new ones, as in $x = r\cos\theta$ and $y = r\sin\theta$. For these polar coordinates the stretching factor is the r in $r\ dr\ d\theta$.

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

3. Explain why $J = 1$ for the coordinte change $x = u\cos\alpha - v\sin\alpha$ and $y = u\sin\alpha + v\cos\alpha$.
   - This is a ***pure rotation***. The xy axes are at a 90° angle and the uv axes are also at a 90° angle (just rotated through the angle α). The area $dA = dx\ dy$ just rotates into $dA = du\ dv$. The factor is
   $$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos^2\alpha + \sin^2\alpha = 1.$$

4. Show that $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$. This is exact even if we can't do $\int_0^1 e^{-x^2}dx$!
   - This is half of Example 4 in the text. There we found the integral $\sqrt{\pi}$ from $-\infty$ to ∞. But e^{-x^2} is an even function – same value for x and $-x$. Therefore the integral from 0 to ∞ is $\frac{1}{2}\sqrt{\pi}$.

Read-throughs and selected even-numbered solutions :

We change variables to improve the **limits** of integration. The disk $x^2 + y^2 \le 9$ becomes the rectangle $0 \le r \le \mathbf{3}, 0 \le \theta \le \mathbf{2\pi}$. The inner limits of $\iint\ dy\ dx$ are $y = \pm\sqrt{\mathbf{9 - x^2}}$. In polar coordinates this area integral becomes $\iint \mathbf{r\ dr\ d\theta} = \mathbf{9\pi}$.

A polar rectangle has sides dr and $\mathbf{r\ d\theta}$. Two sides are not **straight** but the angles are still **90°**. The area between the circles $r = 1$ and $r = 3$ and the rays $\theta = 0$ and $\theta = \pi/4$ is $\mathbf{\frac{1}{8}(3^2 - 1^2) = 1}$. The integral $\iint x\ dy\ dx$

changes to $\iint \mathbf{r^2\cos\theta\, dr\, d\theta}$. This is the **moment** around the **y** axis. Then $\bar{x}$ is the ratio $\mathbf{My/M}$. This is the x coordinate of the **centroid**, and it is the **average** value of x.

In a rotation through α, the point that reaches (u,v) starts at $x = u\cos\alpha - v\sin\alpha, y = \mathbf{u\sin\alpha + v\cos\alpha}$. A rectangle in the uv plane comes from a **rectangle** in xy. The areas are **equal** so the stretching factor is $J = \mathbf{1}$. This is the determinant of the matrix $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$. The moment of inertia $\iint x^2 dx\, dy$ changes to $\iint (\mathbf{u\cos\alpha - v\sin\alpha})^{\mathbf{2}} du\, dv$.

For single integrals dx changes to $(\mathbf{dx/du})du$. For double integrals $dx\, dy$ changes to $J\, du\, dv$ with $J = \mathbf{\partial(x,y)/\partial(u,v)}$. The stretching factor J is the determinant of the 2 by 2 matrix $\begin{bmatrix} \mathbf{\partial x/\partial u} & \mathbf{\partial x/\partial v} \\ \mathbf{\partial y/\partial u} & \mathbf{\partial y/\partial v} \end{bmatrix}$. The functions $x(u,v)$ and $y(u,v)$ connect an xy region R to a uv region S, and $\iint_R dx\, dy = \iint_S \mathbf{J\, du\, dv}$ = area of $\mathbf{R}$. For polar coordinates $x = \mathbf{u\cos v}$ and $y = \mathbf{u\sin v}$ (or $\mathbf{r\sin\theta}$). For $x = u, y = u + 4v$ the 2 by 2 determinant is $J = \mathbf{4}$. A square in the uv plane comes from a **parallelogram** in xy. In the opposite direction the change has $u = x$ and $v = \frac{1}{4}(y - x)$ and a new $J = \mathbf{\frac{1}{4}}$. This J is constant because this change of variables is **linear.**

2 Area $= \int_{-\sqrt{2}/2}^{\sqrt{2}/2}\int_{|x|}^{\sqrt{1-x^2}} dy\, dx$ splits into two equal parts left and right of $x = 0: 2\int_0^{\sqrt{2}/2}\int_x^{\sqrt{1-x^2}} dy\, dx =$ $2\int_0^{\sqrt{2}/2}(\sqrt{1-x^2} - x)dx = [x\sqrt{1-x^2} + \sin^{-1}x - x^2]_0^{\sqrt{2}/2} = \sin^{-1}\frac{\sqrt{2}}{2} = \mathbf{\frac{\pi}{4}}$. The limits on $\iint dx\, dy$ are $\int_0^{\sqrt{2}/2}\int_{-y}^{y} \mathbf{dx\, dy}$ for the lower triangle plus $\int_{\sqrt{2}/2}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \mathbf{dx\, dy}$ for the circular top.

6 Area of wedge $= \frac{b}{2\pi}(\pi a^2)$. Divide $\int_0^b\int_0^a (r\cos\theta)r\, dr\, d\theta = \mathbf{\frac{a^3}{3}\sin b}$ by this area $\frac{ba^2}{2}$ to find $\bar{x} = \mathbf{\frac{2a}{3b}}\sin b$. (*Interesting limit:* $\bar{x} \to \frac{2}{3}a$ as the wedge angle b approaches zero: This is like the centroid of a triangle.) For $\bar{y}$ divide $\int_0^b\int_0^a (r\sin\theta)r\, dr\, d\theta = \mathbf{\frac{a^3}{3}(1 - \cos b)}$ by the area $\frac{ba^2}{2}$ to find $\bar{y} = \mathbf{\frac{2a}{3b}(1 - \cos b)}$.

12 $I_x = \int_{\pi/4}^{3\pi/4}\int_0^1 (r\sin\theta + 1)^2 r\, dr\, d\theta = \frac{1}{4}\int \sin^2\theta\, d\theta + \frac{2}{3}\int \sin\theta d\theta + \frac{1}{2}\int d\theta = [\frac{\theta}{8} - \frac{\sin 2\theta}{16} - \frac{2}{3}\cos\theta + \frac{\theta}{2}]_{\pi/4}^{3\pi/4} =$ $\mathbf{\frac{5\pi}{16} + \frac{2}{16} + \frac{4}{3}\frac{\sqrt{2}}{2}}; I_y = \iint (r\cos\theta)^2 r\, dr\, d\theta = \frac{\pi}{16} - \frac{1}{8}$ (as in Problem 11); $I_0 = I_x + I_y = \mathbf{\frac{3\pi}{8} + \frac{4}{3}\frac{\sqrt{2}}{2}}$.

24 Problem 18 has $J = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1$. So the area of R is $1\times$ area of unit square $= \mathbf{1}$.

Problem 20 has $J = \begin{vmatrix} v & u \\ -2u & 2v \end{vmatrix} = 2(u^2 + v^2)$, and integration over the square gives area of $R =$ $\int_0^1\int_0^1 2(u^2 + v^2)du\, dv = \mathbf{\frac{4}{3}}$. Check in x, y coordinates: area of $R = 2\int_0^1 (1 - x^2)dx = \frac{4}{3}$.

26 $\begin{vmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{vmatrix} = \begin{vmatrix} x/r & y/r \\ -y/r^2 & x/r^2 \end{vmatrix} = \frac{x^2+y^2}{r^3} = \frac{1}{r}$. As in equation 12, this new J is $\frac{1}{\text{old} J}$.

34 (a) **False** (forgot the stretching factor J) (b) **False** (x can be larger than x^2) (c) **False** (forgot to divide by the area) (d) **True** (odd function integrated over symmetric interval) (e) **False** (the straight-sided region is a trapezoid: angle from 0 to θ and radius from r_1 to r_2 yields area $\frac{1}{2}(r_2^2 - r_1^2)\sin\theta\cos\theta$).

14.3 Triple Integrals (page 540)

For a triple integral, the plane region R changes to a solid V. The basic integral is $\int\int\int dV = \int\int\int dz\, dy\, dx$.

This integral of "1" equals the volume of V. Similarly $\iiint x\,dV$ gives the moment. Divide by the volume for $\bar{x}$.

As always, the limits are the hardest part. The inner integral of dz is z. The limits depend on x and y (unless the top and bottom of the solid are flat). Then the middle integral is ***not*** $\int dy = y$. We are not integrating "1" any more, when we reach the second integral. We are integrating $z_{top} - z_{\text{bottom}}$ = function of x and y. The limits give y as a function of x. Then the outer integral is an ordinary x-integral (but it is not $\int 1\,dx$!).

1. Compute the triple integral $\int_0^1 \int_0^x \int_0^y dz\,dy\,dx$. What solid volume does this equal?

 - The inner integral is $\int_0^y dz = y$. The middle integral is $\int_0^x y\,dy = \frac{1}{2}x^2$. The outer integral is $\int_0^1 \frac{1}{2}x^2 dx = \frac{1}{6}$. The y-integral goes across to the line $y = x$ and the x-integral goes from 0 to 1.
 - In the xy plane this gives a triangle (between the x axis and the 45° line $y = x$). Then the z-integral goes up to the sloping plane $z = y$. I think we have a ***tetrahedron*** – a pyramid with a triangular base and three triangular sides. *Draw it.*

 Check: Volume of pyramid $= \frac{1}{3}$(base)(height) $= \frac{1}{3}(\frac{1}{2})(1) = \frac{1}{6}$. This is one of the six solids in Problem 14.3.3. It is quickly described by $0 \le z \le y \le x \le 1$. Can you see those limits in our triple integral?

2. Find the limits on $\int\int\int dz\,dy\,dx$ for the volume between the surfaces $x^2 + y^2 = 9$ and $x + z = 4$ and $z = 0$. Describe those surfaces and the region V inside them.

 - The inner integral is from $z = 0$ to $z = 4 - x$. (Key point: We just solved $x + z = 4$ to find z.) The middle integral is from $y = -\sqrt{9 - x^2}$ to $y = +\sqrt{9 - x^2}$. The outer integral is from $x = -3$ to $x = +3$.

 Where did -3 and 3 come from? That is the smallest possible x and the largest possible x, when we are inside the surface $x^2 + y^2 = 9$. This surface is a *circular cylinder*, a pipe around the z axis. It is chopped off by the horizontal plane $z = 0$ and the sloping plane $x + z = 4$. The triple integral turns out to give the volume 36π.

 If we change $x + z = 4$ to $x^2 + z^2 = 4$, we have a harder problem. The limits on z are $\pm\sqrt{4 - x^2}$. But now x can't be as large as 3. The solid is now an *intersection of cylinders*. I don't know its volume.

Read-throughs and selected even-numbered solutions :

Six important solid shapes are **a box, prism, cone, cylinder, tetrahedron, and sphere.** The integral $\iiint dx\,dy\,dz$ adds the volume **dx dy dz** of small **boxes.** For computation it becomes **three** single integrals. The inner integral $\int dx$ is the **length** of a line through the solid. The variables **y** and **z** are constant. The double integral $\iint dx\,dy$ is the **area** of a slice, with **z** held constant. Then the z integral adds up the volumes of **slices.**

If the solid region V is bounded by the planes $x = 0, y = 0, z = 0$, and $x + 2y + 3z = 1$, the limits on the inner x integral are $\mathbf{0}$ **and** $\mathbf{1 - 2y - 3z}$. The limits on y are $\mathbf{0}$ **and** $\mathbf{\frac{1}{2}(1 - 3z)}$. The limits on z are $\mathbf{0}$ **and** $\mathbf{\frac{1}{3}}$. In the new variables $u = x, v = 2y, w = 3z$, the equation of the outer boundary is $\mathbf{u + v + w = 1}$. The volume of the tetrahedron in uvw space is $\mathbf{\frac{1}{6}}$. From $dx = du$ and $dy = dv/2$ and $dz = \mathbf{dw/3}$, the volume of an xyz box is $dx\,dy\,dz = \mathbf{\frac{1}{6}}du\,dv\,dw$. So the volume of V is $\mathbf{\frac{1}{36}}$.

To find the average height $\bar{z}$ in V we compute $\iiint \mathbf{z\,dV}/\iiint \mathbf{dV}$. To find the total mass if the density is $\rho = e^z$ we compute the integral $\iiint \mathbf{e^z\,dx\,dy\,dz}$. To find the average density we compute $\iiint \mathbf{e^z dV}/\iiint \mathbf{dV}$. In the order $\iiint dz\,dx\,dy$ the limits on the inner integral can depend on **x and y**. The limits on the middle integral can depend on **y**. The outer limits for the ellipsoid $x^2 + 2y^2 + 3z^2 \le 8$ are $\mathbf{-2 \le y \le 2}$.

4 $\int_0^1 \int_0^z \int_0^y x\,dx\,dy\,dz = \int_0^1 \int_0^z \frac{y^2}{2} dy\,dz = \int_0^1 \frac{z^3}{6} dz = \frac{1}{24}$. Divide by the volume $\frac{1}{6}$ to find $\bar{\mathbf{x}} = \mathbf{\frac{1}{4}}$;

$\int_0^1 \int_0^z \int_0^y y\,dx\,dy\,dz = \int_0^1 \int_0^z y^2 dy\,dz = \int_0^1 \frac{z^3}{3} dz = \frac{1}{12}$ and $\bar{\mathbf{y}} = \mathbf{\frac{1}{2}}$; by symmetry $\bar{\mathbf{z}} = \mathbf{\frac{3}{4}}$.

14 Put dz last and stop at $z = 1$: $\int_0^1 \int_0^{4-z} \int_0^{(4-y-z)/2} dx\,dy\,dz = \int_0^1 \int_0^{4-z} \frac{4-y-z}{2} dy\,dz =$ $\int_0^1 \frac{(4-z)^2}{4} dz = [-\frac{(4-z)^3}{12}]_0^1 = \frac{4^3-3^3}{12} = \mathbf{\frac{37}{12}}$.

22 Change variables to $X = \frac{x}{a}, Y = \frac{y}{b}, Z = \frac{z}{c}$; then $dX dY dZ = \frac{dx\,dy\,dz}{abc}$. Volume $= \int\int\int abc\,dX dY dZ =$ $\mathbf{\frac{1}{6}abc}$. Centroid $(\bar{x}, \bar{y}, \bar{z}) = (a\bar{X}, b\bar{Y}, c\bar{Z}) = (\mathbf{\frac{a}{4}}, \mathbf{\frac{b}{4}}, \mathbf{\frac{c}{4}})$. (Recall volume $\frac{1}{6}$ and centroid $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ of standard tetrahedron: this is Example 2.)

30 In one variable, the midpoint rule is correct for the functions 1 and x. In three variables it is correct for **1, x, y, z, xy, xz, yz, xyz.**

14.4 Cylindrical and Spherical Coordinates (page 547)

I notice in Schaum's Outline that very few triple integrals use $dx\,dy\,dz$. Most use cylindrical or spherical coordinates. The small pieces have volume $dV = r\,dr\,d\theta\,dz$ when they are wedges from a cylinder – the base is $r\,dr\,d\theta$ and the height is dz. The pieces of spheres have volume $dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$. The reason for these coordinates is that many curved solids in practice have cylinders or spheres as boundary surfaces.

Notice that r is $\sqrt{x^2+y^2}$ and ρ is $\sqrt{x^2+y^2+z^2}$. Thus $r = 1$ is a cylinder and $\rho = 1$ is a sphere. For a cylinder on its side, you would still use $r\,dr\,d\theta$ but dy would replace dz. Just turn the whole system.

1. Find the volume inside the cylinder $x^2 + y^2 = 16$ (or $r = 4$), above $z = 0$ and below $z = y$.

   - The solid region is a *wedge*. It goes from $z = 0$ up to $z = y$. The base is half the disk of radius 4. It is not the whole disk, because when y is negative we can't go "up to $z = y$." We can't be above $z = 0$ and below $z = y$, unless y is positive – which puts the polar angle θ between 0 and π. The volume integral seems to be

$$\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^y dz\,dy\,dx \quad \text{or} \quad \int_0^\pi \int_0^4 \int_0^{r\sin\theta} r\,dz\,dr\,d\theta.$$

   The first gives $\int_0^4 \int_0^{\sqrt{16-x^2}} y\,dy\,dx = \int_0^4 \frac{1}{2}(16-x^2)dx = \frac{64}{3}$. The second is $\int_0^\pi \int_0^4 r^2 \sin\theta\,dr\,d\theta =$ $\int_0^\pi \frac{64}{3} \sin\theta\,d\theta = \frac{64}{3} \cdot 2$. ***Which is right?***

2. Find the average distance from the center of the unit ball $\rho \le 1$ to all other points of the ball.

   - We are looking for the average value of ρ, when ρ goes between 0 and 1. But the average is not $\frac{1}{2}$. There is more volume for large ρ than for small ρ. So the average $\bar{\rho}$ over the whole ball will be greater than $\frac{1}{2}$. The integral we want is

$$\bar{\rho} = \frac{1}{\text{volume}} \int\int\int \rho dV = \frac{1}{4\pi/3} \int_0^{2\pi} \int_0^\pi \int_0^1 \rho \cdot \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \frac{3}{4\pi} \cdot \frac{1}{4} \cdot 2 \cdot 2\pi = \frac{3}{4}.$$

   The integration was quick because $\int \rho^3\,d\rho = \frac{1}{4}$ separates from $\int \sin\phi\,d\phi = 2$ and $\int d\theta = 2\pi$.

The same separation gives the volume of the unit sphere as $(\int \rho^2 d\rho = \frac{1}{3}) \times (\int \sin\phi\,d\phi = 2) \times (\int d\theta = 2\pi)$. The volume is $\frac{4\pi}{3}$. ***Notice that the angle ϕ from the North Pole has upper limit*** π (not 2π).

3. Find the centroid of the *upper half* of the unit ball. Symmetry gives $\overline{x} = \overline{y} = 0$. Compute $\overline{z}$.

- The volume is half of $\frac{4\pi}{3}$. The integral of $z\, dV$ (remembering $z = \rho\cos\phi$) is

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho\cos\phi\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^1 \rho^3 d\rho \int_0^{\pi/2}\cos\phi\sin\phi\, d\phi \int_0^{2\pi} d\theta = \frac{1}{4}\cdot\frac{1}{2}\cdot 2\pi = \frac{\pi}{4}.$$

Divide by the volume to find the average $\overline{z} = \frac{\pi}{4}/\frac{2\pi}{3} = \frac{3}{8}$. The ball goes up to $z = 1$, but it is fatter at the bottom so the centroid is below $z = \frac{1}{2}$. This time ϕ stops at $\pi/2$, the Equator.

The text explains Newton's famous result for the gravitational attraction of a sphere. The sphere acts as if all its mass were concentrated at the center. Problem 26 gives the proof.

Read-throughs and selected even-numbered solutions :

The three **cylindrical** coordinates are $r\theta z$. The point at $x = y = z = 1$ has $r = \sqrt{\mathbf{2}}, \theta = \pi/\mathbf{4}, z = \mathbf{1}$. The volume integral is $\iiint \mathbf{r\, dr\, d\theta\, dz}$. The solid region $1 \le r \le 2, 0 \le \theta \le 2\pi, 0 \le z \le 4$ is a **hollow cylinder (a pipe)**. Its volume is $\mathbf{12}\pi$. From the r and θ integrals the area of a **ring (or washer)** equals $\mathbf{3}\pi$. From the z and θ integrals the area of *a* **shell** equals $\mathbf{2\pi rz}$. In $r\theta z$ coordinates **cylinders** are convenient, while **boxes** are not.

The three **spherical** coordinates are $\rho\phi\theta$. The point at $x = y = z = 1$ has $\rho = \sqrt{\mathbf{3}}, \phi = \mathbf{\cos^{-1} 1}/\sqrt{\mathbf{3}}, \theta = \pi/\mathbf{4}$. The angle ϕ is measured from **the z axis.** θ is measured from **the x axis.** ρ is the distance to **the origin,** where r was the distance to **the z axis.** If $\rho\phi\theta$ are known then $\mathbf{x} = \rho\, \mathbf{\sin}\, \phi\, \mathbf{\cos}\, \theta, \mathbf{y} = \rho\, \mathbf{\sin}\, \phi\, \mathbf{\sin}\, \theta, \mathbf{z} = \rho\, \mathbf{\cos}\, \phi$. The stretching factor J is a 3 by 3 **determinant** and volume is $\iiint \mathbf{r^2\, \sin\, \phi\, dr\, d\phi\, d\theta}$.

The solid region $1 \le \rho \le 2, 0 \le \phi \le \pi, 0 \le \theta \le 2\pi$ is a **hollow sphere.** Its volume is $\mathbf{4\pi(2^3 - 1^3)/3}$. From the ϕ and θ integrals the area of a **spherical shell** at radius ρ equals $\mathbf{4\pi\rho^2}$. Newton discovered that the outside gravitational attraction of a **sphere** is the same as for an equal mass located at **the center.**

6 $(x, y, z) = (\frac{3}{2}, \frac{\sqrt{3}}{2}, 1); (r, \theta, z) = (\sqrt{3}, \frac{\pi}{6}, 1)$

14 This is the volume of a half-cylinder (because of $0 \le \theta \le \pi$) : height π, radius π, volume $\mathbf{\frac{1}{2}\pi^4}$.

22 The curve $\rho = 1 - \cos\phi$ is a **cardioid** in the xz plane (like $r = 1 - \cos\theta$ in the xy plane). So we have a **cardioid of revolution**. Its volume is $\mathbf{\frac{8\pi}{3}}$ as in Problem 9.3.35.

26 *Newton's achievement* The cosine law (see hint) gives $\cos\alpha = \frac{D^2+q^2-\rho^2}{2qD}$. Then integrate $\frac{\cos\alpha}{q^2}$: $\iiint(\frac{D^2-\rho^2}{2q^3D} + \frac{1}{2qD})dV$. The second integral is $\frac{1}{2D}\iiint\frac{dV}{q} = \frac{4\pi R^3/3}{2D^2}$. The first integral over ϕ uses the same $u = D^2 - 2\rho D\cos\phi + \rho^2 = q^2$ as in the text: $\int_0^\pi \frac{\sin\phi d\phi}{q^3} = \int \frac{du/2\rho D}{u^{3/2}} = [\frac{-1}{\rho D u^{1/2}}]_{\phi=0}^{\phi=\pi} = \frac{1}{\rho D}(\frac{1}{D-\rho} - \frac{1}{D+\rho}) = \frac{2}{D(D^2-\rho^2)}$. The θ integral gives 2π and then the ρ integral is $\int_0^R 2\pi\frac{2}{D(D^2-\rho^2)}\frac{D^2-\rho^2}{2D}\rho^2 d\rho = \frac{4\pi R^3/3}{2D^2}$. The two integrals give $\frac{4\pi R^3/3}{D^2}$ as Newton hoped and expected.

30 $\iint q\, dA = 4\pi\rho^2 D + \frac{4\pi}{3}\frac{\rho^4}{D}$. Divide by $4\pi\rho^2$ to find $\overline{q} = \mathbf{D} + \mathbf{\frac{\rho^2}{3D}}$ for the shell. Then the integral over ρ gives $\iiint q\, dV = \frac{4\pi}{3}R^3 D + \frac{4\pi}{15}\frac{R^5}{D}$. Divide by the volume $\frac{4\pi}{3}R^3$ to find $\overline{q} = \mathbf{D} + \mathbf{\frac{R^2}{5D}}$ for the solid ball.

42 The ball comes to a stop at Australia and **returns** to its starting point. It continues to oscillate in harmonic motion $y = R\cos(\sqrt{c/m}\, t)$.

14 Chapter Review Problems

Review Problems

R1 Integrate $1 + x + y$ over the triangle R with corners (0,0) and (2,0) and (0,2).

R2 Integrate $\frac{x^2}{x^2+y^2}$ over the unit circle using polar coordinates.

R3 Show that $\int_1^2 \int_0^y x\sqrt{y^2 - x^2}\, dx\, dy = \frac{5}{4}$.

R4 Find the area A_n between the curves $y = x^{n+1}$ and $y = x^n$. The limits on x are 0 and 1. Draw A_1 and A_2 on the same graph. Explain why $A_1 + A_2 + A_3 + \cdots$ equals $\frac{1}{2}$.

R5 Convert $y = \sqrt{2x - x^2}$ to $r = 2\cos\theta$. Show that $\int_0^2 \int_0^{\sqrt{2x-x^2}} x\, dy\, dx = \frac{\pi}{2}$ using polar coordinates.

R6 The polar curve $r = 2\cos\theta$ is a unit circle. Find the average $\bar{r}$ for points inside. This is the average distance from points inside to the point (0,0) on the circle. (Answer: $\frac{1}{\text{area}} \iint r dA = \frac{32}{9\pi}$.)

R7 Sketch the region whose area is $\int_0^2 \int_{x^2}^{2x} dy\, dx$. Reverse the order of integration to $\iint dx\, dy$.

R8 Write six different triple integrals starting with $\iiint dx\, dy\, dz$ for the volume of the solid with $0 \le x \le 2y \le 4z \le 8$.

R9 Write six different triple integrals beginning with $\iiint r\, dr\, d\theta\, dz$ for the volume limited by $0 \le r \le z \le 1$. Describe this solid.

Drill Problems

D1 The point with cylindrical coordinates $(2\pi, 2\pi, 2\pi)$ has $x =$ ___, $y =$ ___, $z =$ ___.

D2 The point with spherical coordinates $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ has $x =$ ___, $y =$ ___, $z =$ ___.

D3 Compute $\int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx$ by substituting u for $\sqrt{x}$.

D4 Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by substituting $u = \frac{x}{a}$ and $v = \frac{y}{b}$.

D5 Find $(\bar{x}, \bar{y})$ for the infinite region under $y = e^{-x^2}$. Use page 535 and integration by parts.

D6 What integral gives the area between $x + y = 1$ and $r = 1$?

D7 Show that $\iint_R e^{x^2+y^2} dy\, dx = \frac{\pi}{2}(e - 1)$ when R is the upper half of the unit circle.

D8 If the xy axes are rotated by 30°, the point $(x, y) = (2, 4)$ has new coordinates $(u, v) =$ ___.

D9 In Problem D8 explain why $x^2 + y^2 = u^2 + v^2$. Also explain $dx\, dy = du\, dv$.

D10 ***True or false*** : The centroid of a region is inside that region.