

CHAPTER 6 EXPONENTIALS AND LOGARITHMS

6.1 An Overview (page 234)

The laws of logarithms which are highlighted on pages 229 and 230 apply just as well to “natural logs.” Thus $\ln yz = \ln y + \ln z$ and $b = e^{\ln b}$. Also important :

$$b^x = e^{x \ln b} \quad \text{and} \quad \ln x^a = a \ln x \quad \text{and} \quad \ln 1 = 0.$$

Problems 1 – 4 review the rules for logarithms. Don’t use your calculator. Find the exponent or power.

1. $\log_7 \frac{1}{49}$ 2. $\log_{12} 72 + \log_{12} 2$ 3. $\log_{10} 6 \cdot \log_6 x$ 4. $\log_{0.5} 8$

- To find $\log_7 \frac{1}{49}$, ask yourself “seven to *what power* is $\frac{1}{49}$?” Since $\frac{1}{49} = 7^{-2}$, the power is $\log_7 \frac{1}{49} = -2$.
- $\log_{12} 72 + \log_{12} 2 = \log_{12}(72 \cdot 2) = \log_{12} 144$. Since the bases are the same (everything is base 12), the log of the product equals the sum of the logs. To find $\log_{12} 144$, ask 12^{what power} = 144. The power is 2, so $\log_{12} 144 = 2$.
- Follow the change of base formula $\log_a x = (\log_a b)(\log_b x)$. Here $a = 10$ and $b = 6$. The answer is $\log_{10} x$.
- To find $\log_{0.5} 8$, ask $\frac{1}{2}$ ^{what power} = 8. Since $\frac{1}{2} = 2^{-1}$ and $8 = 2^3$, the power is -3 . Therefore $\log_{0.5} 8 = -3$.

5. Solve $\log_x 10 = 2$. (This is Problem 6.1.6c) The unknown is the base x .

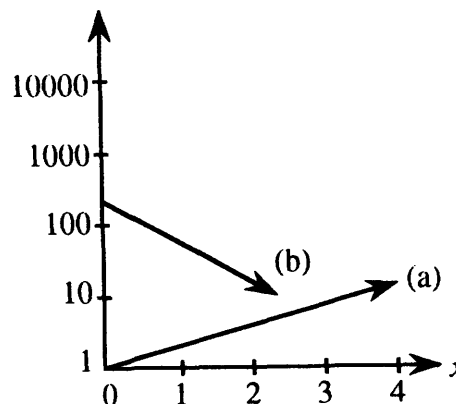
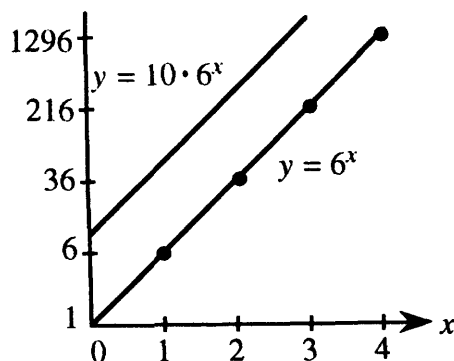
- The statement $\log_x 10 = 2$ means exactly the same as $x^2 = 10$. Therefore $x = \sqrt{10}$. We can’t choose $x = -\sqrt{10}$ since bases must be positive.

6. Draw the graphs for $y = 6^x$ and $y = 5 \cdot 6^x$ on semilog paper (preferably homemade).

- The x axis is scaled normally. The y axis is scaled so that $6^0 = 1$, $6^1 = 6$, $6^2 = 36$, and $6^3 = 216$ are one unit apart. The axes cross at $(0,1)$, not at $(0,0)$ as on regular paper. Both graphs are straight lines. The line $y = 10 \cdot 6^x$ crosses the vertical axis when $x = 0$ and $y = 10$.

7. What are the equations of the functions represented in the right graph?

- This is base 10 semilog paper, so both lines graph functions $y = A \cdot 10^{x \log b}$, where A is the intercept on the vertical axis and $\log b$ is the slope. One graph has $A = 1$ and the slope is $\frac{1}{4}$, so $y = 10^{x/4}$. The intercept on the second graph is 300 and the slope is $\frac{2}{3}$, so $y = 300 \cdot 10^{-2x/3}$.



Read-throughs and selected even-numbered solutions :

In $10^4 = 10,000$, the exponent 4 is the **logarithm** of 10,000. The base is $b = 10$. The logarithm of 10^m times 10^n is $m + n$. The logarithm of $10^m/10^n$ is $m - n$. The logarithm of $10,000^x$ is $4x$. If $y = b^x$ then $x = \log_b y$. Here x is any number, and y is always positive.

A base change gives $b = a^{\log_a b}$ and $b^x = a^{x \log_a b}$. Then 8^5 is 2^{15} . In other words $\log_2 y$ is $\log_2 8$ times $\log_8 y$. When $y = 2$ it follows that $\log_2 8$ times $\log_8 2$ equals 1.

On ordinary paper the graph of $y = mx + b$ is a straight line. Its slope is m . On semilog paper the graph of $y = Ab^x$ is a straight line. Its slope is $\log b$. On log-log paper the graph of $y = Ax^k$ is a straight line. Its slope is k .

The slope of $y = b^x$ is $dy/dx = cb^x$, where c depends on b . The number c is the limit as $h \rightarrow 0$ of $\frac{b^h - 1}{h}$. Since $x = \log_b y$ is the inverse, $(dx/dy)(dy/dx) = 1$. Knowing $dy/dx = cb^x$ yields $dx/dy = 1/cb^x$. Substituting b^x for y , the slope of $\log_b y$ is $1/cy$. With a change of letters, the slope of $\log_b x$ is $1/cx$.

6 (a) 7 (b) 3 (c) $\sqrt{10}$ (d) $\frac{1}{4}$ (e) $\sqrt{8}$ (f) 5

12 $y = \log_{10} x$ is a straight line on "inverse" semilog paper: y axis normal, x axis scaled logarithmically (so $x = 1, 10, 100$ are equally spaced). Any equation $y = \log_b x + C$ will have a straight line graph.

14 $y = 10^{1-x}$ drops from 10 to 1 to .1 with slope -1 on semilog paper; $y = \frac{1}{2}\sqrt{10^x}$ increases with slope $\frac{1}{2}$ from $y = \frac{1}{2}$ at $x = 0$ to $y = 5$ at $x = 2$.

16 If 440/second is the frequency of middle A , then the next A is 880/second. The 12 steps from A to A are approximately multiples of $2^{1/12}$. So 7 steps multiplies by $2^{7/12} \approx 1.5$ to give $(1.5)(440) = 660$. The seventh note from A is E .

22 The slope of $y = 10^x$ is $\frac{dy}{dx} = c10^x$ (later we find that $c = \ln 10$). At $x = 0$ and $x = 1$ the slope is c and $10c$. So the tangent lines are $y - 1 = c(x - 0)$ and $y - 10 = 10c(x - 1)$.

6.2 The Exponential e^x (page 241)

Problems 1 - 8 use the facts that $\frac{d}{dx} e^u = e^u \frac{du}{dx}$ and $\frac{d}{dx} \ln u = \frac{1}{u} \cdot \frac{du}{dx}$. If the base in the problem is not e , convert to base e . Use the change of base formulas $b^u = e^{(\ln b)u}$ and $\log_b u = \frac{\ln u}{\ln b}$. (And remember that $\ln b$ is just a constant.) In each problem, find dy/dx :

1. $y = \ln 3x$ • Take $u = 3x$ to get $\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx} = \frac{1}{3x} \cdot 3 = \frac{1}{x}$. This is the same derivative as for $y = \ln x$.

Why? The answer lies in the laws of logarithms: $\ln 3x = \ln 3 + \ln x$. Since $\ln 3$ is a constant, its derivative is zero. Because $\ln 3x$ and $\ln x$ differ only by a constant, they have the same derivative.

2. $y = \ln \cos 3x$. Assume $\cos 3x$ is positive so $\ln \cos 3x$ is defined.

• Take $u = \cos 3x$. Then $\frac{du}{dx} = -3 \sin 3x$. The answer is $\frac{dy}{dx} = \frac{1}{\cos 3x} (-3 \sin 3x) = -3 \tan 3x$.

When we find a derivative we also find an integral: $-\int 3 \tan 3x = \ln \cos 3x + C$.

3. $y = \ln(\ln x^2)$.

- Take $u = \ln x^2$. Then $\frac{du}{dx} = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$. This means

$$\frac{dy}{dx} = \left(\frac{1}{\ln x^2}\right)\left(\frac{2}{x}\right) = \frac{2}{x \ln x^2} = \frac{1}{x \ln x}.$$

Surprise to the author: This is also the derivative of $\ln(\ln x)$. Why does $\ln(\ln x^2)$ have the same derivative?

3. $y = \log_{10} \sqrt{x^2 + 5}$ • First change the base from 10 to e , by dividing by $\ln 10$. Now you are differentiating $\ln u$ instead of $\log u$: $y = \frac{1}{\ln 10} \ln \sqrt{x^2 + 5}$. For square roots, it is worthwhile to use the law that $\ln u^{1/2} = \frac{1}{2} \ln u$. Then $\ln \sqrt{x^2 + 5} = \frac{1}{2} \ln(x^2 + 5)$. [THIS IS NOT $\frac{1}{2}(\ln x^2 + \ln 5)$] This function is now

$$y = \frac{\ln(x^2 + 5)}{2 \ln 10} \quad \text{and} \quad \frac{dy}{dx} = \left(\frac{1}{2 \ln 10}\right)\left(\frac{1}{x^2 + 5}\right)(2x) = \frac{1}{\ln 10} \frac{x}{x^2 + 5}.$$

4. $y = \ln \frac{(x^4 - 8)^5}{(x^6 + 5x) \cos x}$ • Here again the laws of logarithms allow you to make things easier. Multiplication of numbers is addition of logs. Division is subtraction. Powers of u become multiples of $\ln u$: $y = 5 \ln(x^4 - 8) - \ln(x^6 + 5x) - \ln \cos x$. Now dy/dx is long but easy:

$$\frac{dy}{dx} = \frac{5}{x^4 - 8}(4x^3) - \frac{6x^5 + 5}{x^6 + 5x} - \frac{(-\sin x)}{\cos x} = \frac{20x^3}{x^4 - 8} - \frac{6x^5 + 5}{x^6 + 5x} + \tan x.$$

5. $y = e^{\tan x}$ • dy/dx is $e^u du/dx = e^{\tan x}(\sec^2 x)$.

6. $y = \sin(e^{2x})$ • Set $u = e^{2x}$. Then $du/dx = 2e^{2x}$. Using the chain rule,

$$\frac{d}{dx}(\sin u) = (\cos u)\left(\frac{du}{dx}\right) = (\cos e^{2x})(2e^{2x}).$$

7. $y = 10^{x^2}$ • First change the base from 10 to e : $y = (e^{\ln 10})^{x^2} = e^{x^2 \ln 10}$. Let $u = x^2 \ln 10$. Then $\frac{du}{dx} = 2x \ln 10$. (Remember $\ln 10$ is a constant, you don't need the product rule.) We have

$$\frac{dy}{dx} = e^u \frac{du}{dx} = e^{x^2 \ln 10}(2x \ln 10).$$

8. $y = x^{-1/x}$ (This is Problem 6.2.18)

- First change to base e : $y = (e^{\ln x})^{-1/x}$. Since the exponent is $u = -\frac{1}{x} \ln x$, we need the product rule to get $du/dx = -\frac{1}{x}\left(\frac{1}{x}\right) + \frac{1}{x^2} \ln x$. Therefore

$$\frac{dy}{dx} = e^u \frac{du}{dx} = x^{-1/x} \cdot \frac{1}{x^2}(\ln x - 1).$$

Problems 9 - 14 use the definition $e = \lim_{h \rightarrow 0} (1 + h)^{1/h}$. By substituting $h = \frac{1}{n}$ this becomes $e = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$. Evaluate these limits as $n \rightarrow \infty$:

9. $\lim(1 + \frac{1}{n})^{6n}$ 10. $\lim(1 + \frac{1}{6n})^{6n}$ 11. $\lim(1 + \frac{1}{2n})^{3n}$ 12. $\lim(1 + \frac{r}{n})^n$ (r is constant) 13. $\lim(\frac{n+8}{n})^n$

- Rewrite Problem 9 as $\lim((1 + \frac{1}{n})^n)^6$. Since $(1 + \frac{1}{n})^n$ goes to e the answer is $e^6 \approx 403$. The calculator shows $(1 + \frac{1}{1000})^{6000} \approx 402$.
- Problem 10 is different because $6n$ is both inside and outside the parentheses. If you let $k = 6n$, and note $k \rightarrow \infty$ as $n \rightarrow \infty$, this becomes $\lim_{k \rightarrow \infty} (1 + \frac{1}{k})^k = e$. The idea here is: If we have $\lim_{\square \rightarrow \infty} (1 + \frac{1}{\square})^{\square}$ and all the boxes are the same, the limit is e .

- Question 11 can be rewritten as $\lim(1 + \frac{1}{2n})^{2n \cdot \frac{3}{2}} = e^{3/2}$. (The box is $\square = 2n$).
 - In Question 12 write $n = mr$. Then $\frac{r}{n} = \frac{1}{m}$ and we have $\lim(1 + \frac{1}{m})^{mr} = e^r$.
 - In 13 write $(\frac{n+8}{n}) = 1 + \frac{8}{n}$. This is Problem 12 with $r = 8$: $\lim_{n \rightarrow \infty}(1 + \frac{8}{n})^n = e^8$.
14. (This is 6.2.21) Find the limit of $(\frac{11}{10})^{10}$, $(\frac{101}{100})^{100}$, $(\frac{1001}{1000})^{1000}$, \dots . Then find the limit of $(\frac{10}{11})^{10}$, $(\frac{100}{101})^{100}$, $(\frac{1000}{1001})^{1000}$, \dots and the limit of $(\frac{10}{11})^{11}$, $(\frac{100}{101})^{101}$, $(\frac{1000}{1001})^{1001}$, \dots .
- The terms of the first sequence are $(\frac{n+1}{n})^n = (1 + \frac{1}{n})^n$ where $n = 10, 100, 1000, \dots$. The limit is e . The terms of the second sequence are the reciprocals of those of the first. So the second limit is $\frac{1}{e}$. The terms of the third are each $(\frac{n}{n+1})$ times those of the second. Since $\frac{n}{n+1} \rightarrow 1$ as $n \rightarrow \infty$, the third limit is again $\frac{1}{e}$.

The third sequence can also be written $(\frac{n-1}{n})^n$ or $(1 - \frac{1}{n})^n$. Its limit is e^{-1} . See Problem 12 with $r = -1$.

Exercises 6.2.27 and 6.2.45 – 6.2.54 give plenty of practice in integrating exponential functions. Usually the trick is to locate $e^u du$. Problems 15 – 17 are three models.

15. (This is 6.2.32) Find an antiderivative for $v(x) = \frac{1}{e^x} + \frac{1}{x^e}$.
- The first term is e^{-x} . Its antiderivative is $-e^{-x}$. The second term is just x^n with $n = -e$. Its antiderivative is $\frac{1}{1-e} x^{1-e}$. The answer is $f(x) = -e^{-x} + \frac{1}{1-e} x^{1-e} + C$.
16. Find an antiderivative for $v(x) = 3^{-2x}$. You may change to base e .
- The change produces $e^{-2x \ln 3}$. The coefficient of x in the exponent is $-2 \ln 3$. An antiderivative is $f(x) = \frac{-1}{2 \ln 3} e^{-2x \ln 3}$ or $\frac{-1}{2 \ln 3} 3^{-2x}$. We need -2 and $\ln 3$ in the denominator, the same way that we needed $n + 1$ when integrating x^n .
17. (This is 6.2.52) Find $\int_0^3 e^{(1+x^2)} x dx$. Set $u = 1 + x^2$ and $du = 2x dx$. The integral is $\frac{1}{2} \int_{u(0)}^{u(3)} e^u du$. The new limits of integration are $u(0) = 1 + 0^2 = 1$ and $u(3) = 1 + 3^2 = 10$. Now $\frac{1}{2} \int_1^{10} e^u du = \frac{1}{2} e^u \Big|_1^{10} = \frac{1}{2} (e^{10} - e)$. This is not the same as $\frac{1}{2} e^9$!

Read-throughs and selected even-numbered solutions :

The number e is approximately **2.78**. It is the limit of $(1 + h)$ to the power $1/h$. This gives 1.01^{100} when $h = .01$. An equivalent form is $e = \lim(1 + \frac{1}{n})^n$.

When the base is $b = e$, the constant c in Section 6.1 is **1**. Therefore the derivative of $y = e^x$ is $dy/dx = e^x$. The derivative of $x = \log_e y$ is $dx/dy = 1/y$. The slopes at $x = 0$ and $y = 1$ are both **1**. The notation for $\log_e y$ is $\ln y$, which is the **natural** logarithm of y .

The constant c in the slope of b^x is $c = \ln b$. The function b^x can be rewritten as $e^{x \ln b}$. Its derivative is $(\ln b)e^{x \ln b} = (\ln b)b^x$. The derivative of $e^{u(x)}$ is $e^{u(x)} \frac{du}{dx}$. The derivative of $e^{\sin x}$ is $e^{\sin x} \cos x$. The derivative of e^{cx} brings down a factor c .

The integral of e^x is $e^x + C$. The integral of e^{cx} is $\frac{1}{c} e^{cx} + C$. The integral of $e^{u(x)} du/dx$ is $e^{u(x)} + C$. In general the integral of $e^{u(x)}$ by itself is **impossible** to find.

- 18 $x^{-1/x} = e^{-(\ln x)/x}$ has derivative $(-\frac{1}{x^2} + \frac{\ln x}{x^3})e^{-(\ln x)/x} = (\frac{\ln x - 1}{x^3})x^{-1/x}$
- 20 $(1 + \frac{1}{n})^{2n} \rightarrow e^2 \approx 7.7$ and $(1 + \frac{1}{n})^{\sqrt{n}} \rightarrow 1$. Note that $(1 + \frac{1}{n})^{\sqrt{n}}$ is squeezed between 1 and $e^{1/\sqrt{n}}$ which approaches 1.
- 28 $(e^{3x})(e^{7x}) = e^{10x}$ which is the derivative of $\frac{1}{10}e^{10x}$
- 42 $x^{1/x} = e^{(\ln x)/x}$ has slope $e^{(\ln x)/x}(\frac{1}{x^2} - \frac{\ln x}{x^3}) = x^{1/x}(\frac{1 - \ln x}{x^2})$. This slope is zero at $x = e$, when $\ln x = 1$.
 The second derivative is *negative* so the maximum of $x^{1/x}$ is $e^{1/e}$. Check: $\frac{d}{dx}e^{(\ln x)/x}(\frac{1 - \ln x}{x^2}) = e^{(\ln x)/x}[\frac{1 - \ln x}{x^3} + \frac{-2 - 1 + 2 \ln x}{x^3}] = -\frac{1}{e^3}e^{1/e}$ at $x = e$.
- 44 $x^e = e^x$ at $x = e$. This is the only point where $x^e e^{-x} = 1$ because the derivative is $x^e(-e^{-x}) + ex^{e-1}e^{-x} = (\frac{e}{x} - 1)x^e e^{-x}$. This derivative is positive for $x < e$ and negative for $x > e$. So the function $x^e e^{-x}$ increases to 1 at $x = e$ and then decreases: it never equals 1 again.
- 58 The asymptotes of $(1 + \frac{1}{x})^x = (\frac{x+1}{x})^x = (\frac{x}{x+1})^{-x}$ are $x = -1$ (from the last formula) and $y = e$ (from the first formula).
- 62 $\lim_{x \rightarrow \infty} \frac{x^6}{e^x} = \lim_{x \rightarrow \infty} \frac{6x^5}{e^x} = \lim_{x \rightarrow \infty} \frac{30x^4}{e^x} = \lim_{x \rightarrow \infty} \frac{120x^3}{e^x} = \lim_{x \rightarrow \infty} \frac{360x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{720x}{e^x} = \lim_{x \rightarrow \infty} \frac{720}{e^x} = 0$.

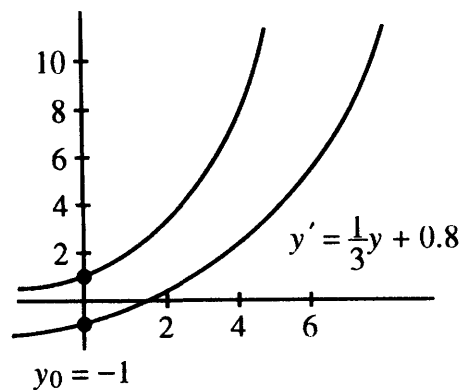
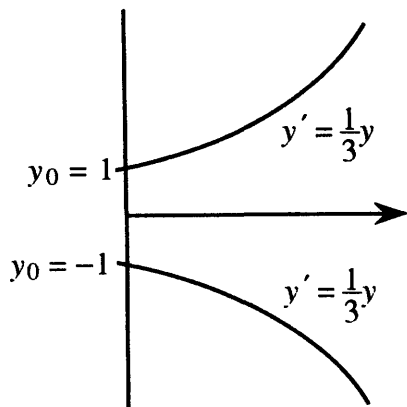
6.3 Growth and Decay in Science and Economics (page 250)

The applications in this section begin to suggest the power of the mathematics you are learning. Concentrate on understanding how to use $y = y_0 e^{ct}$ and $y = y_0 e^{ct} + \frac{s}{c}(e^{ct} - 1)$ as you work the examples.

In Problems 1 and 2, solve the differential equations starting from $y_0 = 1$ and $y_0 = -1$. Draw both solutions on the same graph.

1. $\frac{dy}{dt} = \frac{1}{3}y$ (pure exponential) 2. $\frac{dy}{dt} = \frac{1}{3}y + 0.8$ (exponential with source term)

- Problem 1 says that the rate of change is proportional to y . There are no other complicating terms. Use the exponential law $y_0 e^{ct}$ with $c = \frac{1}{3}$ and $y_0 = \pm 1$. The graphs of $y = \pm e^{t/3}$ are at left below.
- Problem 2 changes Problem 1 into $\frac{dy}{dt} = cy + s$. We have $c = \frac{1}{3}$ and $s = 0.8$. Its solution is $y = y_0 e^{t/3} + \frac{0.8}{\frac{1}{3}}(e^{t/3} - 1) = y_0 e^{t/3} + 2.4(e^{t/3} - 1)$. Study the graphs to see the effect of y_0 and s . With a graphing calculator you can carry these studies further. See what happens if s is very large, or if s is negative. Exercise 6.3.36 is also good for comparing the effects of various c 's and s 's.



3. (This is 6.3.5) Start from $y_0 = 10$. If $\frac{dy}{dt} = 4y$, at what time does y increase to 100?

- The solution is $y = y_0 e^{4t} = 10e^{4t}$. Set $y = 100$ and solve for t :

$$100 = 10e^{4t} \text{ gives } 10 = e^{4t} \quad \text{and} \quad \ln 10 = 4t. \quad \text{Then } t = \frac{1}{4} \ln 10.$$

4. Problem 6.3.6 looks the same as the last question, but the right side is $4t$ instead of $4y$. Note that $\frac{dy}{dt} = 4t$ is *not* exponential growth. The slope $\frac{dy}{dt}$ is proportional to t and the solution is simply $y = 2t^2 + C$. Start at $y_0 = C = 10$. Setting $y = 100$ gives $100 = 2t^2 + 10$ and $t = \sqrt{45}$.

Problems 5 – 10 involve $y = y_0 e^{ct}$.

5. Write the equation describing a bacterial colony growing exponentially. Start with 100 bacteria and end with 10^6 after 30 hours.

- Right away we know $y_0 = 100$ and $y = 100e^{ct}$. We don't yet know c , but at $t = 30$ we have $y = 10^6 = 100e^{30c}$. Taking logarithms of $10^4 = e^{30c}$ gives $4 \ln 10 = 30c$ or $\frac{4 \ln 10}{30} = c$. The equation is $y = 100e^{(\frac{4 \ln 10}{30}t)}$. More concisely, since $e^{\ln 10} = 10$ this is $y = 100 \cdot 10^{4t/30}$.

6. The number of cases of a disease increases by 2% a year. If there were 10,000 cases in 1992, how many will there be in 1995?

- The direct approach is to multiply by 1.02 after every year. After three years $(1.02)^3 10,000 \approx 10,612$.
- We can also use $y = 10^4 e^{ct}$. The 2% increase means $c = \ln(1.02)$. After three years (1992 to 1995) we set $t = 3$: $y = 10^4 e^{3(\ln 1.02)} = 10,612$.

This is *not the same* as $y = 10^4 e^{.02t}$. That is *continuous* growth at 2%. It is continuous compounding, and $e^{.02} = 1.0202 \dots$ is a little different from 1.02.

7. How would Problem 6 change if the number of cases *decreases* by 2%?

- A 2% decrease changes the multiplier to .98. Then $c = \ln(.98)$. In 3 years there would be 9,411 cases.

8. (This is 6.3.15) The population of Cairo grew exponentially from 5 million to 10 million in 20 years. Find the equation for Cairo's population. When was $y = 8$ million?

- Starting from $y_0 = 5$ million = $5 \cdot 10^6$ the population is $y = 5 \cdot 10^6 e^{ct}$. The doubling time $\frac{\ln 2}{c}$ is 20 years. We deduce that $c = \frac{\ln 2}{20} = .035$ and $y = 5 \cdot 10^6 e^{.035t}$. This reaches 8 million = $8 \cdot 10^6$ when $\frac{8}{5} = e^{.035t}$. Then $t = \frac{\ln \frac{8}{5}}{.035} \approx 13.6$ years.

9. If $y = 4500$ at $t = 4$ and $y = 90$ at $t = 10$, what was y at $t = 0$? (We are assuming exponential decay.)

- The first part says that $y = 4500e^{c(t-4)}$. The t in the basic formula is replaced by $(t - 4)$. [The "shifted" formula is $y = y_T e^{c(t-T)}$.] Note that $y = 4500$ when $t = 4$, as required. Since $y = 90$ when $t = 10$, we have $90 = 4500e^{6c}$ and $e^{6c} = \frac{90}{4500} = .02$. This means $6c = \ln .02$ and $c = \frac{1}{6} \ln .02$. Finally, set $t = 0$ to get the amount at that time: $y = 4500e^{(\frac{1}{6} \ln .02)(0-4)} \approx 61074$.

10. (Problem 6.3.13) How old is a skull containing $\frac{1}{5}$ as much radiocarbon as a modern skull?

- Information about radioactive dating is on pages 243-245. Since the half-life of carbon 14 is 5568 years, the amount left at time t is $y_0 e^{ct}$ with exponent $c = \frac{\ln 1/2}{5568} = \frac{-\ln 2}{5568}$. We do not know the initial amount y_0 . But we can use $y_0 = 1$ (100% at the start) and $y = \frac{1}{5} = 0.2$ at the unknown age t . Then

$$0.2 = e^{\frac{-\ln 2}{5568}t} \text{ yields } t = \frac{(\ln 0.2)5568}{-\ln 2} = 31,425 \text{ years.}$$

11. (Problem 6.3.37) What value $y = \text{constant}$ solves $\frac{dy}{dt} = 4 - y$? Show that $y(t) = Ae^{-t} + 4$ is also a solution. Find $y(1)$ and y_∞ if $y_0 = 3$.

- If y is constant, then $\frac{dy}{dt} = 0$. Therefore $y - 4 = 0$. The steady state y_∞ is the constant $y = 4$.
- A non-constant solution is $y(t) = Ae^{-t} + 4$. Check: $\frac{dy}{dt} = -Ae^{-t}$ equals $4 - y = 4 - (Ae^{-t} + 4)$.
- If we know $y(0) = A + 4 = 3$, then $A = -1$. In this case $y(t) = -1e^{-t} + 4$ gives $y(1) = 4 - \frac{1}{e}$.
- To find y_∞ , let $t \rightarrow \infty$. Then $y = -e^{-t} + 4$ goes to $y_\infty = 4$, the expected steady state.

12. (Problem 6.3.46) (a) To have \$50,000 for college tuition in 20 years, what gift y_0 should a grandparent make now? Assume $c = 10\%$. (b) What continuous deposit should a parent make during 20 years to save \$50,000? (c) If the parent saves $s = \$1000$ per year, when does the account reach \$50,000?

- Part (a) is a question about the present value y_0 , if the gift is worth \$50,000 in 20 years. The formula $y = y_0e^{ct}$ turns into $y_0 = ye^{-ct} = (50,000)e^{-0.1(20)} = \6767 .
- Part (b) is different because there is a continuous deposit instead of one lump sum. In the formula $y_0e^{ct} + \frac{s}{c}(e^{ct} - 1)$ we know $y_0 = 0$ and $c = 10\% = 0.1$. We want to choose s so that $y = 50,000$ when $t = 20$. Therefore $50,000 = \frac{s}{0.1}(e^{0.1 \cdot 20} - 1)$. This gives $s = 782.59$. The parents should continuously deposit \$782.59 per year for 20 years.
- Part (c) asks how long it would take to accumulate \$50,000 if the deposit is $s = \$1000$ per year.

$$50,000 = \frac{1000}{0.1}(e^{0.1t} - 1) \text{ leads to } 5 = e^{0.1t} - 1 \text{ and } t = \frac{\ln 6}{0.1} = 17.9 \text{ years.}$$

This method takes 17.9 years to accumulate the tuition. The smaller deposit $s = \$782.59$ took 20 years.

13. (Problem 6.3.50) For how long can you withdraw \$500/year after depositing \$5000 at a continuous rate of 8%? At time t you run dry: and $y(t) = 0$.

- This situation uses both terms of the formula $y = y_0e^{ct} + \frac{s}{c}(e^{ct} - 1)$. There is an initial value $y_0 = 5000$ and a *sink* (negative source) of $s = -500/\text{year}$. With $c = .08$ we find the time t when $y = 0$:

$$\text{Multiply } 0 = 5000e^{.08t} - \frac{500}{.08}(e^{.08t} - 1) \text{ by } .08 \text{ to get } 0 = 400e^{.08t} - 500(e^{.08t} - 1).$$

Then $e^{.08t} = \frac{500}{100} = 5$ and $t = \frac{\ln 5}{.08} = 20.1$. You have 20 years of income.

14. Your Thanksgiving turkey is at 40°F when it goes into a 350° oven at 10 o'clock. At noon the meat thermometer reads 110° . When will the turkey be done (195°)?

- Newton's law of cooling applies even though the turkey is warming. Its temperature is approaching $y_\infty = 350^\circ$ from $y_0 = 40^\circ$. Using method 3 (page 250) we have $(y - 350) = (40 - 350)e^{ct}$. The value of c varies from turkey to turkey. To find c for your particular turkey, substitute $y = 110$ when $t = 2$:

$$110 - 350 = -310e^{2c} \Rightarrow \frac{-240}{-310} = e^{2c} \Rightarrow \ln \frac{24}{31} = 2c \Rightarrow c = \frac{1}{2} \ln \frac{24}{31} = -.128.$$

The equation for y is $350 - 310e^{-.128t}$. The turkey is done when $y = 195$:

$$195 = 350 - 310e^{-.128t} \text{ or } \frac{195 - 350}{-310} = e^{-.128t} \text{ or } -.128t = \ln \frac{-155}{-310} = \ln \frac{1}{2}.$$

This gives $t = 5.4$ hours. You can start making gravy at 3 : 24.

Read-throughs and selected even-numbered solutions :

If $y' = cy$ then $y(t) = y_0 e^{ct}$. If $dy/dt = 7y$ and $y_0 = 4$ then $y(t) = 4e^{7t}$. This solution reaches 8 at $t = \frac{\ln 2}{7}$. If the doubling time is T then $c = \frac{\ln 2}{T}$. If $y' = 3y$ and $y(1) = 9$ then y_0 was $9e^{-3}$. When c is negative, the solution approaches zero as $t \rightarrow \infty$.

The constant solution to $dy/dt = y + 6$ is $y = -6$. The general solution is $y = Ae^t - 6$. If $y_0 = 4$ then $A = 10$. The solution of $dy/dt = cy + s$ starting from y_0 is $y = Ae^{ct} + B = (y_0 + \frac{s}{c})e^{ct} - \frac{s}{c}$. The output from the source is $\frac{s}{c}(e^{ct} - 1)$. An input at time T grows by the factor $e^{c(t-T)}$ at time t .

At $c = 10\%$, the interest in time dt is $dy = .01y dt$. This equation yields $y(t) = y_0 e^{.01t}$. With a source term instead of y_0 , a continuous deposit of $s = 4000/\text{year}$ yields $y = 40,000(e - 1)$ after ten years. The deposit required to produce 10,000 in 10 years is $s = yc/(e^{ct} - 1) = 1000/(e - 1)$. An income of 4000/year forever (!) comes from $y_0 = 40,000$. The deposit to give 4000/year for 20 years is $y_0 = 40,000(1 - e^{-2})$. The payment rate s to clear a loan of 10,000 in 10 years is $1000e/(e - 1)$ per year.

The solution to $y' = -3y + s$ approaches $y_\infty = s/3$.

- 12 To multiply again by 10 takes ten more hours, a total of **20 hours**. If $e^{10c} = 10$ (and $e^{20c} = 100$) then $10c = \ln 10$ and $c = \frac{\ln 10}{10} \approx .23$.
- 16 $8e^{.01t} = 6e^{.014t}$ gives $\frac{8}{6} = e^{.004t}$ and $t = \frac{1}{.004} \ln \frac{8}{6} = 250 \ln \frac{4}{3} = 72$ years.
- 24 Go from 4 mg back down to 1 mg in T hours. Then $e^{-.01T} = \frac{1}{4}$ and $-.01T = \ln \frac{1}{4}$ and $T = \frac{\ln \frac{1}{4}}{-.01} = 139$ hours (not so realistic).
- 28 Given $mv = mv - v\Delta m + m\Delta v - (\Delta m)\Delta v + \Delta m(v - 7)$; cancel terms to leave $m\Delta v - (\Delta m)\Delta v = 7\Delta m$; divide by Δm and approach the limit $m \frac{dv}{dm} = 7$. Then $v = 7 \ln m + C$. At $t = 0$ this is $20 = 7 \ln 4 + C$ so that $v = 7 \ln m + 20 - 7 \ln 4 = 7 \ln \frac{m}{4} + 20$.
- 36 (a) $\frac{dy}{dt} = 3y + 6$ gives $y \rightarrow \infty$ (b) $\frac{dy}{dt} = -3y + 6$ gives $y \rightarrow 2$ (c) $\frac{dy}{dt} = -3y - 6$ gives $y \rightarrow -2$
(d) $\frac{dy}{dt} = 3y - 6$ gives $y \rightarrow -\infty$.
- 42 \$1000 changes by (\$1000) $(-.04dt)$, a decrease of $40dt$ dollars in time dt . The printing rate should be $s = 40$.
- 48 The deposit of $4dT$ grows with factor c from time T to time t , and reaches $e^{c(t-T)}4dT$. With $t = 2$ add deposits from $T = 0$ to $T = 1$: $\int_0^1 e^{c(2-T)}4dT = [\frac{4e^{c(2-T)}}{-c}]_0^1 = \frac{4e^c - 4e^{2c}}{-c}$.
- 58 If $\frac{dy}{dt} = -y + 7$ then $\frac{dy}{dt}$ is zero at $y_\infty = 7$ (this is $-\frac{s}{c} = \frac{7}{1}$). The derivative of $y - y_\infty$ is $\frac{dy}{dt}$, so the derivative of $y - 7$ is $-(y - 7)$. The decay rate is $c = -1$, and $y - 7 = e^{-t}(y_0 - 7)$.
- 60 All solutions to $\frac{dy}{dt} = c(y - 12)$ converge to $y = 12$ provided c is negative.
- 66 (a) The white coffee cools to $y_\infty + (y_0 - y_\infty)e^{ct} = 20 + 40e^{ct}$. (b) The black coffee cools to $20 + 50e^{ct}$. The milk warms to $20 - 10e^{ct}$. The mixture $\frac{5(\text{black coffee})+1(\text{milk})}{6}$ has $20 + \frac{250-10}{6}e^{ct} = 20 + 40e^{ct}$. So it doesn't matter when you add the milk!

6.4 Logarithms (page 258)

This short section is packed with important information and techniques – how to differentiate and integrate logarithms, logarithms as areas, approximation of logarithms, and logarithmic differentiation (LD). The examples cover each of these topics:

Derivatives The rule for $y = \ln u$ is $\frac{dy}{dx} = \frac{1}{u} \frac{du}{dx}$. With a different base b , the rule for $y = \log_b u = \frac{\ln u}{\ln b}$ is $\frac{dy}{dx} = \frac{1}{u \ln b} \frac{du}{dx}$. Find $\frac{dy}{dx}$ in Problems 1–4.

- $y = \ln(5 - x)$. • $u = 5 - x$ so $\frac{dy}{dx} = \left(\frac{1}{5-x}\right)(-1) = \frac{1}{x-5}$.
- $y = \log_{10}(\sin x)$. • Change to base e with $y = \frac{\ln(\sin x)}{\ln 10}$. Now $\frac{dy}{dx} = \frac{1}{\ln 10} \cdot \frac{1}{\sin x} \cdot \cos x$.
- $y = (\ln x)^3$. • This is $y = u^3$, so $\frac{dy}{dx} = 3u^2 \frac{du}{dx} = 3(\ln x)^2 \frac{1}{x}$.
- $y = \tan x \ln \sin x$. • The product rule gives

$$\frac{dy}{dx} = \tan x \cdot \frac{1}{\sin x} \cdot \cos x + \sec^2 x (\ln \sin x) = 1 + \sec^2 x (\ln \sin x).$$

- (This is 6.4.53) Find $\lim_{x \rightarrow 0} \frac{\log_b(1+x)}{x}$.

• This limit takes the form $\frac{0}{0}$, so turn to l'Hôpital's rule (Section 3.8). The derivative of $\log_b(1+x)$ is $\left(\frac{1}{\ln b}\right)\left(\frac{1}{1+x}\right)$. The derivative of x is 1. The ratio is $\frac{1}{(\ln b)(1+x)}$ which approaches $\frac{1}{\ln b}$.

Logarithms as areas

- (This is 6.4.56) Estimate the area under $y = \frac{1}{x}$ for $4 \leq x \leq 8$ by four trapezoids. What is the exact area?
 - Each trapezoid has base $\Delta x = 1$, so four trapezoids take us from $x = 4$ to $x = 8$. With $y = \frac{1}{x}$ the sides of the trapezoids are the heights $y_0, y_1, y_2, y_3, y_4 = \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}$. The total trapezoidal area is

$$\Delta x \left(\frac{1}{2} y_0 + y_1 + y_2 + y_3 + \frac{1}{2} y_4 \right) = 1 \left(\frac{1}{8} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{16} \right) = 0.6970.$$

To get the exact area we integrate $\int_4^8 \frac{1}{x} dx = \ln 8 - \ln 4 = \ln \frac{8}{4} = \ln 2 \approx 0.6931$.

It is interesting to compare with the trapezoidal area from $x = 1$ to $x = 2$. The exact area $\int_1^2 \frac{1}{x} dx$ is still $\ln 2$. Now $\Delta x = \frac{1}{4}$ and the heights are $\frac{1}{1}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}$. The total trapezoidal area comes from the same rule:

$$\frac{1}{4} \left(\frac{1}{2} \cdot \frac{1}{1} + \frac{1}{5} + \frac{4}{6} + \frac{4}{7} + \frac{1}{2} \cdot \frac{1}{2} \right) = \left(\frac{1}{8} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{16} \right) = 0.6970 \text{ as before.}$$

The sum is not changed! This is another way to see why $\ln 8 - \ln 4$ is equal to $\ln 2 - \ln 1$. The area stays the same when we integrate $\frac{1}{x}$ from any a to $2a$.

Questions 7 and 8 are about approximations going as far as the x^3 term:

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} \quad \text{and} \quad e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

- Approximate $\ln(.98)$ by choosing $x = -.02$. Then $1 + x = .98$.

- $\ln(1 - .02) \approx (-.02) - \frac{(-.02)^2}{2} + \frac{(-.02)^3}{3} = -.0202026667.$

The calculator gives $\ln .98 = -.0202027073$. Somebody is wrong by $4 \cdot 10^{-8}$.

8. Find a quadratic approximation (this means x^2 terms) near $x = 0$ for $y = 2^x$.

- 2^x is the same as $e^{x \ln 2}$. Put $x \ln 2$ into the series. The approximation is $1 + x \ln 2 + \frac{(\ln 2)^2}{2} x^2$.

Integration The basic rule is $\int \frac{du}{u} = \ln |u| + C$. Why not just $\ln u + C$? Go back to the definition of $\ln u = \text{area under the curve } y = \frac{1}{x} \text{ from } x = 1 \text{ to } x = u$. Here u must be positive since we cannot cross $x = 0$, where $\frac{1}{x}$ blows up. However if u stays negative, there is something we can do. Write $\int \frac{du}{u} = \int \frac{-du}{-u}$. The denominator $-u$ is positive and the numerator is its derivative! In that case, $\int \frac{-du}{-u} = \ln(-u) + C$. The expression $\int \frac{du}{u} = \ln |u| + C$ covers both cases. When you *know* u is positive, as in $\ln(x^2 + 1)$, leave off the absolute value sign.

For definite integrals, the limits of integration should tell you whether u is negative or positive. Here are two examples with $u = \sin x$:

$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} dx = \ln(\sin x) \Big|_{\pi/4}^{\pi/2} \qquad \int_{\pi/2}^{3\pi/4} \frac{\cos x}{\sin x} dx = \ln |\sin x| \Big|_{\pi/2}^{3\pi/4}.$$

The integral $\int_0^\pi \frac{\cos x}{\sin x} dx$ is illegal. It starts and ends with $u = \sin x = 0$

9. Integrate $\int \frac{x dx}{1-x^2}$.

- Let $1 - x^2$ equal u . Then $du = -2x dx$. The integral becomes $-\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \ln |u| + C$. This is $-\frac{1}{2} \ln |1 - x^2| + C$. Avoid $x = \pm 1$ where $u = 0$.

10. Integrate $\int \frac{x dx}{1-x}$. This is *not* $\int \frac{du}{u}$. But we can write $\frac{x}{1-x}$ as $-1 + \frac{1}{1-x}$:

- $\int \frac{x dx}{1-x} = \int (-1 + \frac{1}{1-x}) dx = -x - \ln |1-x| + C$.

11. (This is 6.4.18) Integrate $\int_2^e \frac{dx}{x(\ln x)^2}$.

- A sneaky one, not $\frac{du}{u}$. Set $u = \ln x$ and $du = \frac{dx}{x}$:

$$\int_{\ln 2}^1 \frac{du}{u^2} = -\frac{1}{u} \Big|_{\ln 2}^1 = -1 + \frac{1}{\ln 2}.$$

Logarithmic differentiation (LD) greatly simplifies derivatives of powers and products, and quotients. To find the derivative of $x^{1/x}$, LD is the best way to go. (Exponential differentiation in Problem 6.5.70 amounts to the same thing.) The secret is in decomposing the original expression. Here are examples:

12. $y = \frac{(x^2+7)^3}{\sqrt{x^3-9}} (4x^8)$ leads to $\ln y = 3 \ln(x^2 + 7) + \ln 4 + 8 \ln x - \frac{1}{2} \ln(x^3 - 9)$.

Multiplication has become addition. Division has become subtraction. The powers 3, 8, $\frac{1}{2}$ now multiply. This is as far as logarithms can go. *Do not try to separate* $\ln x^2$ *and* $\ln 7$. Take the derivative of $\ln y$:

- $\frac{1}{y} \frac{dy}{dx} = 3 \frac{2x}{x^2+7} + 0 + \frac{8}{x} - \frac{3x^2}{2(x^3-9)}$.

If you substitute back for y then $\frac{dy}{dx} = \frac{(x^2+7)^3 \cdot 4x^8}{\sqrt{x^3-9}} \left[\frac{6x}{x^2+7} + \frac{8}{x} - \frac{3x^2}{2(x^3-9)} \right]$.

13. $y = (\sin x)^{x^2}$ has a function $\sin x$ raised to a functional power x^2 . LD is necessary.

- First take logarithms: $\ln y = x^2 \ln \sin x$. Now take the derivative of both sides. Notice especially the left side: $\frac{1}{y} \frac{dy}{dx} = x^2 \frac{\cos x}{\sin x} + 2x \ln \sin x$. Multiply by y to find $\frac{dy}{dx}$.

14. Find the tangent line $y^2(2-x) = x^3$ at the point (1,1). **ID** and **LD** are useful but not necessary.

- We need to know the slope dy/dx at (1,1). Taking logarithms gives

$$\ln y^2 + \ln(2-x) = \ln x^3 \text{ or } 2 \ln y + \ln(2-x) = 3 \ln x.$$

Now take the x derivative of both sides: $\frac{2}{y} \frac{dy}{dx} + \frac{-1}{2-x} = \frac{3}{x}$. Plug in $x = 1, y = 1$ to get $2 \frac{dy}{dx} + \frac{-1}{1} = 3$ or $\frac{dy}{dx} = 2$. The tangent line through (1,1) with slope 2 is $y - 1 = 2(x - 1)$.

Read-throughs and selected even-numbered solutions :

The natural logarithm of x is $\int_1^x \frac{dt}{t}$ (or $\int_1^x \frac{dx}{x}$). This definition leads to $\ln xy = \ln x + \ln y$ and $\ln x^n = n \ln x$. Then e is the number whose logarithm (area under $1/x$ curve) is 1. Similarly e^x is now defined as the number whose natural logarithm is x . As $x \rightarrow \infty$, $\ln x$ approaches infinity. But the ratio $(\ln x)/\sqrt{x}$ approaches zero. The domain and range of $\ln x$ are $0 < x < \infty, -\infty < \ln x < \infty$.

The derivative of $\ln x$ is $\frac{1}{x}$. The derivative of $\ln(1+x)$ is $\frac{1}{1+x}$. The tangent approximation to $\ln(1+x)$ at $x = 0$ is x . The quadratic approximation is $x - \frac{1}{2}x^2$. The quadratic approximation to e^x is $1 + x + \frac{1}{2}x^2$.

The derivative of $\ln u(x)$ by the chain rule is $\frac{1}{u(x)} \frac{du}{dx}$. Thus $(\ln \cos x)' = -\frac{\sin x}{\cos x} = -\tan x$. An antiderivative of $\tan x$ is $-\ln \cos x$. The product $p = xe^{5x}$ has $\ln p = 5x + \ln x$. The derivative of this equation is $p'/p = 5 + \frac{1}{x}$. Multiplying by p gives $p' = xe^{5x}(5 + \frac{1}{x}) = 5xe^{5x} + e^{5x}$, which is **LD** or logarithmic differentiation.

The integral of $u'(x)/u(x)$ is $\ln u(x)$. The integral of $2x/(x^2+4)$ is $\ln(x^2+4)$. The integral of $1/cx$ is $\frac{\ln x}{c}$. The integral of $1/(ct+s)$ is $\frac{\ln(ct+s)}{c}$. The integral of $1/\cos x$, after a trick, is $\ln(\sec x + \tan x)$. We should write $\ln|x|$ for the antiderivative of $1/x$, since this allows $x < 0$. Similarly $\int du/u$ should be written $\ln|u|$.

- 4** $\frac{x(\frac{1}{x}) - (\ln x)}{x^2} = \frac{1 - \ln x}{x^2}$ **6** Use $(\log_e 10)(\log_{10} x) = \log_e x$. Then $\frac{d}{dx}(\log_{10} x) = \frac{1}{\log_e 10} \cdot \frac{1}{x} = \frac{1}{x \ln 10}$.
- 16** $y = \frac{x^3}{x^2+1}$ equals $x - \frac{x}{x^2+1}$. Its integral is $[\frac{1}{2}x^2 - \frac{1}{2} \ln(x^2+1)]_0^2 = 2 - \frac{1}{2} \ln 5$.
- 20** $\int \frac{\sin x}{\cos x} dx = \int \frac{-du}{u} = -\ln u = -\ln(\cos x)|_0^{\pi/4} = -\ln \frac{1}{\sqrt{2}} + 0 = \frac{1}{2} \ln 2$.
- 24** Set $u = \ln \ln x$. By the chain rule $\frac{du}{dx} = \frac{1}{\ln x} \cdot \frac{1}{x}$. Our integral is $\int \frac{du}{u} = \ln u = \ln(\ln(\ln x)) + C$.
- 28** $\ln y = \frac{1}{2} \ln(x^2+1) + \frac{1}{2} \ln(x^2-1)$. Then $\frac{1}{y} \frac{dy}{dx} = \frac{x}{x^2+1} + \frac{x}{x^2-1} = \frac{2x^2}{x^4-1}$. Then $\frac{dy}{dx} = \frac{2x^3 y}{x^4-1} = \frac{2x^3}{\sqrt{x^4-1}}$.
- 36** $\ln y = -\ln x$ so $\frac{1}{y} \frac{dy}{dx} = \frac{-1}{x}$ and $\frac{dy}{dx} = -\frac{e^{-\ln x}}{x}$. Alternatively we have $y = \frac{1}{x}$ and $\frac{dy}{dx} = -\frac{1}{x^2}$.
- 40** $\frac{d}{dx} \ln x = \frac{1}{x}$. Alternatively use $\frac{1}{x^2} \frac{d}{dx}(x^2) - \frac{1}{x} \frac{d}{dx}(x) = \frac{1}{x}$.
- 54** Use l'Hôpital's Rule: $\lim_{x \rightarrow 0} \frac{b^x \ln b}{1} = \ln b$. We have redone the derivative of b^x at $x = 0$.
- 62** $\frac{1}{x} \ln \frac{1}{x} = -\frac{\ln x}{x} \rightarrow 0$ as $x \rightarrow \infty$. This means $y \ln y \rightarrow 0$ as $y = \frac{1}{x} \rightarrow 0$. (Emphasize: The factor $y \rightarrow 0$ is "stronger" than the factor $\ln y \rightarrow -\infty$.)
- 70 LD:** $\ln p = x \ln x$ so $\frac{1}{p} \frac{dp}{dx} = 1 + \ln x$ and $\frac{dp}{dx} = p(1 + \ln x) = x^x(1 + \ln x)$. Now find the same answer by **ED:** $\frac{d}{dx}(e^{x \ln x}) = e^{x \ln x} \frac{d}{dx}(x \ln x) = x^x(1 + \ln x)$.

6.5 Separable Equations Including the Logistic Equation (page 266)

Separation of variables works so well (when it works) that there is a big temptation to use it often and wildly. I asked my class to integrate the function $y(x) = \frac{d}{dx}(e^{1+x^2})$ from $x = 0$ to $x = 3$. The point of this question is that you don't have to take the derivative of e^{1+x^2} . When you integrate, that brings back the original function. So the answer is

$$\int_0^3 y(x) dx = [e^{1+x^2}]_0^3 = e^{10} - e.$$

One mistake was to write that answer as e^9 . The separation of variables mistake was in $y dy$:

$$\text{from } y = \frac{d}{dx}(e^{1+x^2}) \text{ the class wrote } \int y dy = \int \frac{d}{dx}(e^{1+x^2}) dx.$$

You can't multiply one side by dy and the other side by dx . This mistake leads to $\frac{1}{2}y^2$ which shouldn't appear. Separation of variables starts from $\frac{dy}{dx} = u(y)v(x)$ and does *the same thing to both sides*. Divide by $u(y)$, multiply by dx , and integrate. Then $\int dy/u(y) = \int v(x)dx$. Now a y -integral equals an x -integral.

Solve the differential equations in Problems 1 and 2 by separating variables.

1. $\frac{dy}{dx} = \sqrt{xy}$ with $y_0 = 4$ (which means $y(0) = 4$.)

- First, move dx to the right side and \sqrt{y} to the left: $\frac{dy}{y^{1/2}} = x^{1/2} dx$. Second, integrate both sides: $2y^{1/2} = \frac{2}{3}x^{3/2} + C$. (This constant C combines the constants for each integral.) Third, solve for $y = (\frac{1}{3}x^{3/2} + C)^2$. Here $C/2$ became C . Half a constant is another constant. This is the general solution. Fourth, use the starting value $y_0 = 4$ to find C :

$$4 = (\frac{1}{3}(0)^{3/2} + C)^2 \text{ yields } C = \pm 2. \text{ Then } y = (\frac{1}{3}x^{3/2} \pm 2)^2.$$

2. Solve $(x-3)t dt + (t^2+1)dx = 0$ with $x = 5$ when $t = 0$.

- Divide both sides by $(x-3)(t^2+1)$ to separate t from x :

$$\frac{t dt}{t^2+1} + \frac{dx}{x-3} = 0 \text{ or } -\int \frac{t dt}{t^2+1} = \int \frac{dx}{x-3}.$$

Integrating gives $-\frac{1}{2}\ln(t^2+1) = \ln(x-3) + C$ or $(t^2+1)^{-1/2} = e^C(x-3)$. Since $x = 5$ when $t = 0$ we have $1 = 2e^C$. Put $e^C = \frac{1}{2}$ into the solution to find $x-3 = 2(t^2+1)^{-1/2}$ or $x = 3 + 2(t^2+1)^{-1/2}$.

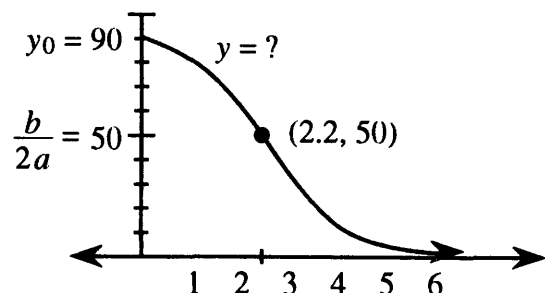
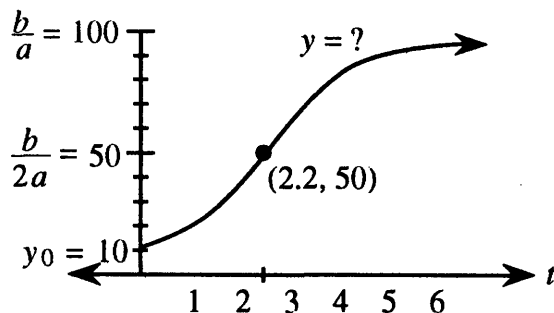
Problems 3 - 5 deal with the logistic equation $y' = cy - by^2$.

3. (This is 6.5.15.) Solve $\frac{dz}{dt} = -z + 1$ with $z_0 = 2$. Turned upside down, what is $y = \frac{1}{z}$? Graph y and z .

- Separation of variables gives $\frac{dz}{-z+1} = dt$ or $-\ln|-z+1| = t + C$. Put in $z = 2$ when $t = 0$ to find $C = 0$. Also notice that $-2+1$ is *negative*. The absolute value is reversing the sign. So we have

$$-\ln(z-1) = t \quad \text{or} \quad z-1 = e^{-t} \quad \text{or} \quad z = e^{-t} + 1.$$

Now $y = \frac{1}{z} = \frac{1}{1+e^{-t}}$. According to Problem 6.3.15, this y solves the logistic equation $y' = y - y^2$.



4. Each graph above is an S-curve that solves a logistic equation $y' = \pm y \pm by^2$ with $c = 1$ or $c = -1$. Each has an inflection point at $(2.2, 50)$. Find the differential equations and the solutions.

- The first graph shows $y_0 = 10$. The inflection point is at height $\frac{c}{2b} = 50$. Then $c = 1$ and $b = \frac{c}{100} = .01$. The limiting value $y_\infty = \frac{c}{b}$ is twice as high at $y_\infty = 100$. The differential equation is $dy/dt = y - .01y^2$. The solution is given by equation (12) on page 263:

$$y = \frac{c}{b + de^{-ct}} \text{ where } d = \frac{c - by_0}{y_0} = \frac{1 - (.01)(10)}{10} = .09. \text{ Then } y = \frac{1}{.01 - .09e^{-t}}.$$

The second graph must solve the differential equation $\frac{dy}{dt} = -y + by^2$. Its slope is just the opposite of the first. Again we have $\frac{c}{2b} = 50$ and $b = 0.01$. Substitute $c = -1$ and $y_0 = 90$:

$$d = \frac{c - by_0}{y_0} = \frac{-1 + .01(90)}{90} = \frac{-1}{900} \text{ and } y(t) = \frac{-1}{-.01 - \frac{e^t}{900}} = \frac{900}{9 + e^t}.$$

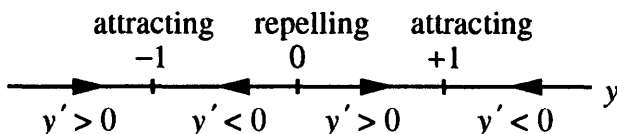
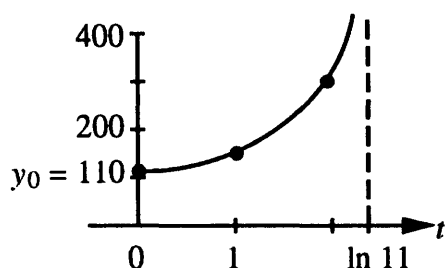
This is a case where *death wins*. Since $y_0 < \frac{c}{b} = 100$ the population dies out before the cooperation term $+by^2$ is strong enough to save it. See Example 6 on page 264 of the text.

5. Change y_0 in Problem 4 to 110. Then $y_0 > \frac{c}{b} = 100$. Find the solution $y(t)$ and graph it.

- As in 4(b) the equation is $\frac{dy}{dt} = -y + .01y^2$. Since y_0 is now 110, the solution has

$$d = \frac{-1 + .01(110)}{110} = \frac{1}{1100} \text{ and } y(t) = \frac{-1}{-.01 + \frac{e^t}{1100}} = \frac{1100}{11 - e^t}.$$

The graph is sketched below. After a sluggish start, the population blows up at $t = \ln 11$.



6. Draw a y -line for $y' = y - y^3$. Which steady states are approached from which initial values y_0 ?

- Factor $y - y^3$ to get $y' = y(1 - y)(1 + y)$. A steady state has $y' = 0$. This occurs at $y = 0, 1$, and -1 . Plot those points on the straight line. They are not all attracting.

Now consider the sign of $y(1 - y)(1 + y) = y'$. If y is below -1 , y' is positive. (Two factors y and $1 + y$ are negative but their product is positive.) If y is between -1 and 0 , y' is negative and y decreases. If y is between 0 and 1 , all factors are positive and so is y' . Finally, if $y > 1$ then y' is negative.

The signs of y' are $+ - + -$. The curved line $f(y)$ is sketched to show those signs. A positive y' means an increasing y . So the solution moves toward -1 and also toward $+1$. It moves away from $y = 0$, because y is increasing on the right of zero and decreasing on the left of zero.

The arrows in the y -line point to the left when y' is negative. The sketch shows that $y = -1$ and $y = +1$ are *stable steady states*. They are attracting, while $y = 0$ is an unstable (or *repelling*) stationary point. The solution approaches -1 from $y_0 < 0$, and it approaches $+1$ from $y_0 > 0$.

Read-throughs and selected even-numbered solutions :

The equations $dy/dt = cy$ and $dy/dt = cy + s$ and $dy/dt = u(y)v(t)$ are called **separable** because we can separate y from t . Integration of $\int dy/y = \int c dt$ gives $\ln y = ct + \text{constant}$. Integration of $\int dy/(y + s/c) = \int c dt$ gives $\ln(y + \frac{s}{c}) = ct + C$. The equation $dy/dx = -x/y$ leads to $\int y dy = -\int x dx$. Then $y^2 + x^2 = \text{constant}$ and the solution stays on a circle.

The logistic equation is $dy/dt = cy - by^2$. The new term $-by^2$ represents **competition** when cy represents growth. Separation gives $\int dy/(cy - by^2) = \int dt$, and the y -integral is $1/c$ times $\ln \frac{y}{c-by}$. Substituting y_0 at $t = 0$ and taking exponentials produces $y/(c - by) = e^{ct}y_0/(c - by_0)$. As $t \rightarrow \infty$, y approaches $\frac{c}{b}$. That is the steady state where $cy - by^2 = 0$. The graph of y looks like an S, because it has an inflection point at $\frac{1}{2} \frac{c}{b}$.

In biology and chemistry, concentrations y and z react at a rate proportional to y times z . This is the **Law of Mass Action**. In a model equation $dy/dt = c(y)z$, the rate c depends on y . The MM equation is

$dy/dt = -cy/(y + K)$. Separating variables yields $\int \frac{y+K}{y} dy = \int -c dt = -ct + C$.

- 6 $\frac{dy}{\tan y} = \cos x dx$ gives $\ln(\sin y) = \sin x + C$. Then $C = \ln(\sin 1)$ at $x = 0$. After taking exponentials $\sin y = (\sin 1)e^{\sin x}$. No solution after $\sin y$ reaches 1 (at the point where $(\sin 1)e^{\sin x} = 1$).
- 8 $e^y dy = e^t dt$ so $e^y = e^t + C$. Then $C = e^e - 1$ at $t = 0$. After taking logarithms $y = \ln(e^t + e^e - 1)$.
- 10 $\frac{d(\ln y)}{d(\ln x)} = \frac{dy/y}{dx/x} = n$. Therefore $\ln y = n \ln x + C$. Therefore $y = (x^n)(e^C) = \text{constant times } x^n$.
- 16 Equation (14) is $z = \frac{1}{c}(b + \frac{c-by_0}{y_0}e^{-ct})$. Turned upside down this is $y = \frac{c}{b+de^{-ct}}$ with $d = \frac{c-by_0}{y_0}$.
- 20 $y' = y + y^2$ has $c = 1$ and $b = -1$ with $y_0 = 1$. Then $y(t) = \frac{1}{-1+2e^{-t}}$ by formula (12). The denominator is zero and y blows up when $2e^{-t} = 1$ or $t = \ln 2$.
- 26 At the middle of the S-curve $y = \frac{c}{2b}$ and $\frac{dy}{dt} = c(\frac{c}{2b}) - b(\frac{c}{2b})^2 = \frac{c^2}{4b}$. If b and c are multiplied by 10 then so is this slope $\frac{c^2}{4b}$, which becomes steeper.
- 28 If $\frac{cy}{y+K} = d$ then $cy = dy + dK$ and $y = \frac{dK}{c-d}$. At this steady state the maintenance dose replaces the aspirin being eliminated.
- 30 The rate $R = \frac{cy}{y+K}$ is a decreasing function of K because $\frac{dR}{dK} = \frac{-cy}{(y+K)^2}$.
- 34 $\frac{d[A]}{dt} = -r[A][B] = -r[A](b_0 - \frac{n}{m}(a_0 - [A]))$. The changes $a_0 - [A]$ and $b_0 - [B]$ are in the proportion m to n ; we solved for $[B]$.

6.6 Powers Instead of Exponentials (page 276)

1. Write down a power series for $y(x)$ whose derivative is $\frac{1}{2}y(x)$. Assume that $y(0) = 1$.

- *First method:* Look for $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$, and choose the a 's so that $y' = \frac{1}{2}y$. Start with $a_0 = 1$ so that $y(0) = 1$. Then take the derivative of each term:

$$y' = 0 + a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots + na_nx^{n-1} + \dots$$

Matching this series with $\frac{1}{2}y$ gives $a_1 = \frac{1}{2}a_0$ and $2a_2 = \frac{1}{2}a_1$. Therefore $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{8}$. Similarly $3a_3$ matches $\frac{1}{2}a_2$ and na_n matches $\frac{1}{2}a_{n-1}$. The pattern continues with $a_3 = \frac{1}{3} \cdot \frac{1}{2} \cdot a_2$ and $a_4 = \frac{1}{4} \cdot \frac{1}{2} \cdot a_3$. The typical term is $a_n = \frac{1}{n!2^n}$:

$$\text{The series is } y(x) = 1 + \frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 2^2} + \frac{x^3}{3!2^3} + \dots + \frac{x^n}{n!2^n} + \dots$$

- *Second method:* We already know the solution to $y' = \frac{1}{2}y$. It is $y_0e^{\frac{1}{2}x}$. Starting from $y_0 = 1$, the solution is $y = e^{\frac{1}{2}x}$. We also know the exponential series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$. So just substitute the new exponent $\frac{1}{2}x$ in place of x :

$$y = e^{\frac{1}{2}x} = 1 + \frac{1}{2}x + \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \dots + \frac{1}{n!}\left(\frac{x}{2}\right)^n + \dots = \text{same answer.}$$

2. (This is 6.6.19) Solve the difference equation $y(t+1) = 3y(t) + 1$ with $y_0 = 0$.

- Follow equations 8 and 9 on page 271. In this problem $a = 3$ and $s = 1$. Each step multiplies the previous y by 3 and adds 1. From $y_0 = 0$ we have $y_1 = 1$ and $y_2 = 4$. Then $y_3 = 13$ and $y_4 = 40$. The solution is

$$y(t) = 3^t \cdot 0 + 1 \frac{(3^t - 1)}{3 - 1} \quad \text{or} \quad y(t) = \frac{3^t - 1}{2}.$$

- If prices rose $\frac{3}{10}\%$ in the last month, what is the equivalent annual rate of inflation?
 - The answer is not 12 times $\frac{3}{10} = 3.6\%$. The monthly increases are *compounded*. A \$1 price at the beginning of the year would be $(1 + .003)^{12} \approx 1.0366$ at the end of the year. The annual rate of inflation is .0366 or 3.66%.
- If inflation stays at 4% a year, find the present value that yields a dollar after 10 years.
 - Use equation 2 on page 273 with $n = 1$ and $y = 1$. The rate is .04 instead of .05, for 10 years instead of 20. We get $y_0 = (1 + \frac{.04}{1})^{-10} = 0.6755$. In a decade a dollar will be worth what 67.55 cents is worth today.
- Write the difference equation and find the steady state for this situation: Every week 80% of the cereal is sold and 400 more boxes are delivered to the supermarket.
 - If $C(t)$ represents the number of cereal boxes after t weeks, the problem states that $C(t + 1) = 0.2C(t) + 400$. The reason for 0.2 is that 80% are sold and 20% are left. The difference equation has $a = 0.2$ and $s = 400$. Since $|a| < 1$, a steady state is approached: $C_\infty = \frac{s}{1-a} = \frac{400}{.8} = 500$. At that steady state, 80% of 500 boxes are sold (that means 400) and they are replaced by 400 new boxes.

Read-throughs and selected even-numbered solutions :

The infinite series for e^x is $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$. Its derivative is e^x . The denominator $n!$ is called "n factorial" and is equal to $n(n-1) \cdots (1)$. At $x = 1$ the series for e is $1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots$.

To match the original definition of e , multiply out $(1 + 1/n)^n = 1 + n(\frac{1}{n}) + \frac{n(n-1)}{2}(\frac{1}{n})^2$ (first three terms). As $n \rightarrow \infty$ those terms approach $1 + 1 + \frac{1}{2}$ in agreement with e . The first three terms of $(1 + x/n)^n$ are $1 + n(\frac{x}{n}) + \frac{n(n-1)}{2}(\frac{x}{n})^2$. As $n \rightarrow \infty$ they approach $1 + x + \frac{1}{2}x^2$ in agreement with e^x . Thus $(1 + x/n)^n$ approaches e^x . A quicker method computes $\ln(1 + x/n)^n \approx nx$ (first term only) and takes the exponential.

Compound interest (n times in one year at annual rate x) multiplies by $(1 + \frac{x}{n})^n$. As $n \rightarrow \infty$, continuous compounding multiplies by e^x . At $x = 10\%$ with continuous compounding, \$1 grows to $e^{.1} \approx \$1.105$ in a year.

The difference equation $y(t+1) = ay(t)$ yields $y(t) = a^t$ times y_0 . The equation $y(t+1) = ay(t) + s$ is solved by $y = a^t y_0 + s[1 + a + \dots + a^{t-1}]$. The sum in brackets is $\frac{1-a^t}{1-a}$ or $\frac{a^t-1}{a-1}$. When $a = 1.08$ and $y_0 = 0$, annual deposits of $s = 1$ produce $y = \frac{1.08^t - 1}{.08}$ after t years. If $a = \frac{1}{2}$ and $y_0 = 0$, annual deposits of $s = 6$ leave $12(1 - \frac{1}{2^t})$ after t years, approaching $y_\infty = 12$. The steady equation $y_\infty = ay_\infty + s$ gives $y_\infty = s/(1-a)$.

When $i =$ interest rate per period, the value of $y_0 = \$1$ after N periods is $y(N) = (1+i)^N$. The deposit to produce $y(N) = 1$ is $y_0 = (1+i)^{-N}$. The value of $s = \$1$ deposited after each period grows to $y(N) =$

$\frac{1}{i}((1+i)^N - 1)$. The deposit to reach $y(N) = 1$ is $s = \frac{1}{i}(1 - (1+i)^{-N})$.

Euler's method replaces $y' = cy$ by $\Delta y = cy\Delta t$. Each step multiplies y by $1 + c\Delta t$. Therefore y at $t = 1$ is $(1 + c\Delta t)^{1/\Delta t}y_0$, which converges to y_0e^c as $\Delta t \rightarrow 0$. The error is proportional to Δt , which is too large for scientific computing.

- 4** A larger series is $1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 3$. This is greater than $1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots = e$.
- 8** The exact sum is $e^{-1} \approx .37$ (Problem 6). After five terms $1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{9}{24} = .375$.
- 14** $y(0) = 0, y(1) = 1, y(2) = 3, y(3) = 7$ (and $y(n) = 2^n - 1$). **24** Ask for $\frac{1}{2}y(0) - 6 = y(0)$. Then $y(0) = -12$.
- 30** The equation $-dP(t+1) + b = cP(t)$ becomes $-2P(t+1) + 8 = P(t)$ or $P(t+1) = -\frac{1}{2}P(t) + 4$. Starting from $P(0) = 0$ the solution is $P(t) = 4\left[\frac{(-\frac{1}{2})^t - 1}{-\frac{1}{2} - 1}\right] = \frac{8}{3}(1 - (-\frac{1}{2})^t) \rightarrow \frac{8}{3}$.
- 38** Solve $\$1000 = \$8000 \left[\frac{1}{1 - (1.1)^{-n}}\right]$ for n . Then $1 - (1.1)^{-n} = .8$ or $(1.1)^{-n} = .2$. Thus $1.1^n = 5$ and $n = \frac{\ln 5}{\ln 1.1} \approx 17$ years.
- 40** The interest is $(.05)1000 = \$50$ in the first month. You pay \$60. So your debt is now $\$1000 - \$10 = \$990$. Suppose you owe $y(t)$ after month t , so $y(0) = \$1000$. The next month's interest is $.05y(t)$. You pay \$60. So $y(t+1) = 1.05y(t) - 60$. After 12 months $y(12) = (1.05)^{12}1000 - 60\left[\frac{(1.05)^{12} - 1}{.05}\right]$. This is also $\frac{60}{.05} + (1000 - \frac{60}{.05})(1.05)^{12} \approx \841 .
- 44** Use the loan formula with $.09/n$ not $.09n$: payments $s = 80,000 \frac{.09/12}{[1 - (1 + \frac{.09}{12})^{-360}]} \approx \643.70 . Then 360 payments equal \$231,732.

6.7 Hyperbolic Functions (page 280)

1. Given $\sinh x = \frac{5}{12}$, find the values of $\cosh x, \tanh x, \coth x, \operatorname{sech} x$ and $\operatorname{csch} x$.

- Use the identities on page 278. The one to remember is similar to $\cos^2 x + \sin^2 x = 1$:

$$\cosh^2 x - \sinh^2 x = 1 \text{ gives } \cosh^2 x = 1 + \frac{25}{144} = \frac{169}{144} \text{ and } \cosh x = \frac{13}{12}.$$

Note that $\cosh x$ is always positive. Then $\tanh x = \frac{\sinh x}{\cosh x}$ is $\frac{\frac{5}{12}}{\frac{13}{12}} = \frac{5}{13}$. The others are upside down:

$$\coth x = \frac{1}{\tanh x} = \frac{13}{5} \text{ and } \operatorname{sech} x = \frac{1}{\cosh x} = \frac{12}{13} \text{ and } \operatorname{csch} x = \frac{1}{\sinh x} = \frac{12}{5}.$$

2. Find $\cosh(2 \ln 10)$. Substitute $x = 2 \ln 10 = \ln 100$ into the definition of $\cosh x$:

- $\cosh(2 \ln 10) = \frac{e^{\ln 100} + e^{-\ln 100}}{2} = \frac{100 + \frac{1}{100}}{2} = \frac{100.01}{2} = 50.005$.

3. Find $\frac{dy}{dx}$ when $y = \sinh(4x^3)$. Use the chain rule with $u = 4x^3$ and $\frac{du}{dx} = 12x^2$

- The derivative of $\sinh u(x)$ is $(\cosh u) \frac{du}{dx} = 12x^2 \cosh(4x^3)$.

4. Find $\frac{dy}{dx}$ when $y = \ln \tanh 2x$. • Let $y = \ln u$, where $u = \tanh 2x$. Then

$$\frac{dy}{dx} = \frac{1}{u} \frac{du}{dx} = \frac{2 \operatorname{sech}^2 2x}{\tanh 2x} = \frac{2}{\sinh 2x \cosh 2x}.$$

5. Find $\frac{dy}{dx}$ when $y = \operatorname{sech}^{-1} 6x$. • See equation (3) on page 279. If $u = 6x$ then

$$\frac{dy}{dx} = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx} = \frac{-6}{6x\sqrt{1-36x^2}} = \frac{-1}{x\sqrt{1-36x^2}}.$$

6. Find $\int \frac{dx}{\sqrt{x^2+9}}$. • Except for the 9, this looks like $\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x + C$ on page 279. Factoring out $\sqrt{9}$ leaves $\sqrt{x^2+9} = \sqrt{9}\sqrt{\frac{x^2}{9}+1}$. So the problem has $u = \frac{x}{3}$ and $du = \frac{1}{3}dx$:

$$\int \frac{dx}{3\sqrt{(\frac{x}{3})^2+1}} = \int \frac{du}{\sqrt{u^2+1}} = \sinh^{-1} u + C = \sinh^{-1}\left(\frac{x}{3}\right) + C.$$

7. Find $\int \cosh^2 x \sinh x \, dx$ (This is 6.7.53.) Remember that $u = \cosh x$ has $\frac{du}{dx} = +\sinh x$:

- The problem is really $\int u^2 du$ with $u = \cosh x$. The answer is $\frac{1}{3}u^3 + C = \frac{1}{3}\cosh^3 x + C$.

8. Find $\int \frac{\sinh x}{1+\cosh x} dx$. (This is 6.7.29.) The top is the derivative of the bottom!

- $\int \frac{du}{u} = \ln|u| + C = \ln(1+\cosh x) + C$.

The absolute value sign is dropped because $1+\cosh x$ is always positive.

9. (This is Problem 6.7.54) A falling body with friction equal to velocity squared obeys $\frac{dv}{dt} = g - v^2$.
 (a) Show that $v(t) = \sqrt{g} \tanh \sqrt{gt}$ satisfies the equation. (b) Derive this yourself by integrating $\frac{dv}{g-v^2} = dt$.
 (c) Integrate $v(t)$ to find the distance $f(t)$.

- (a) The derivative of $\tanh x$ is $\operatorname{sech}^2 x$. The derivative of $v(t) = \sqrt{g} \tanh \sqrt{gt}$ has $u = \sqrt{g} t$. The chain rule gives $\frac{dv}{dt} = \sqrt{g}(\operatorname{sech}^2 u) \frac{du}{dt} = g \operatorname{sech}^2 \sqrt{gt}$. Now use the identity $\operatorname{sech}^2 u = 1 - \tanh^2 u$:

$$\frac{dv}{dt} = g(1 - \tanh^2 \sqrt{gt}) = g - v^2.$$

- (b) The differential equation is $\frac{dv}{dt} = g - v^2$. Separate variables to find $\frac{dv}{g-v^2} = dt$:

$$\int \frac{dv}{g-v^2} = \int \frac{dv}{g[1-(\frac{v}{\sqrt{g}})^2]} = \frac{1}{\sqrt{g}} \tanh^{-1} \frac{v}{\sqrt{g}} \text{ by equation (2), on page 279.}$$

The integral of dt is $t + C$. Assuming the body falls from rest ($v = 0$ at $t = 0$), we have $C = 0$. Then $t = \frac{1}{\sqrt{g}} \tanh^{-1} \frac{v}{\sqrt{g}}$ turns into $v = \sqrt{g} \tanh \sqrt{gt}$.

- (c) $\int v \, dt = \int \sqrt{g} \tanh \sqrt{gt} \, dt = \ln \cosh \sqrt{gt} + C$.

Read-throughs and selected even-numbered solutions :

$\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh^2 x - \sinh^2 x = 1$. Their derivatives are $\sinh x$ and $\cosh x$ and zero. The point $(x, y) = (\cosh t, \sinh t)$ travels on the hyperbola $x^2 - y^2 = 1$. A cable hangs in the shape of a catenary $y = a \cosh \frac{x}{a}$.

The inverse functions $\sinh^{-1} x$ and $\tanh^{-1} x$ are equal to $\ln|x + \sqrt{x^2 + 1}|$ and $\frac{1}{2} \ln \frac{1+x}{1-x}$. Their derivatives are $1/\sqrt{x^2 + 1}$ and $\frac{1}{1-x^2}$. So we have two ways to write the antiderivative. The parallel to $\cosh x + \sinh x = e^x$ is Euler's formula $\cos x + i \sin x = e^{ix}$. The formula $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ involves imaginary exponents. The parallel formula for $\sin x$ is $\frac{1}{2i}(e^{ix} - e^{-ix})$.

12 $\sinh(\ln x) = \frac{1}{2}(e^{\ln x} - e^{-\ln x}) = \frac{1}{2}(x - \frac{1}{x})$ with derivative $\frac{1}{2}(1 + \frac{1}{x^2})$.

16 $\frac{1+\tanh x}{1-\tanh x} = e^{2x}$ by the equation following (4). Its derivative is $2e^{2x}$. More directly the quotient rule gives $\frac{(1-\tanh x)\operatorname{sech}^2 x + (1+\tanh x)\operatorname{sech}^2 x}{(1-\tanh x)^2} = \frac{2\operatorname{sech}^2 x}{(1-\tanh x)^2} = \frac{2}{(\cosh x - \sinh x)^2} = \frac{2}{e^{-2x}} = 2e^{2x}$.

18 $\frac{d}{dx} \ln u = \frac{du/dx}{u} = \frac{\operatorname{sech} x \tanh x - \operatorname{sech}^2 x}{\operatorname{sech} x + \tanh x}$. Because of the minus sign we do not get $\operatorname{sech} x$. The integral of $\operatorname{sech} x$ is $\sin^{-1}(\tanh x) + C$.

30 $\int \coth x \, dx = \int \frac{\cosh x}{\sinh x} \, dx = \ln(\sinh x) + C$. **32** $\sinh x + \cosh x = e^x$ and $\int e^{nx} \, dx = \frac{1}{n} e^{nx} + C$.

36 $y = \operatorname{sech} x$ looks like a bell-shaped curve with $y_{\max} = 1$ at $x = 0$. The x axis is the asymptote. But note that y decays like $2e^{-x}$ and not like e^{-x^2} .

40 $\frac{1}{2} \ln \frac{1+x}{1-x}$ approaches $+\infty$ as $x \rightarrow 1$ and $-\infty$ as $x \rightarrow -1$. The function is *odd* (so is the \tanh function). The graph is an **S** curve rotated by 90° .

44 The x derivative of $x = \sinh y$ is $1 = \cosh y \frac{dy}{dx}$. Then $\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}} = \text{slope of } \sinh^{-1} x$.

50 Not hyperbolic! Just $\int (x^2 + 1)^{-1/2} x \, dx = (\mathbf{x^2 + 1})^{1/2} + C$.

58 $\cos ix = \frac{1}{2}(e^{i(ix)} + e^{-i(ix)}) = \frac{1}{2}(e^{-x} + e^x) = \cosh x$. Then $\cos i = \cosh 1 = \frac{e+e^{-1}}{2}$ (real!).

6 Chapter Review Problems

Graph Problems (Sketch the graphs and locate maxima, minima, and inflection points)

G1 $y = x \ln x$

G2 $y = e^{-x^2}$

G3 $y = e^{-x^3}$

G4 $y = x^2 - 72 \ln x$

G5 $y = x^6 e^{-x}$

G6 $y = e^{\ln x}$ (watch the domain)

G7 Sketch $\ln 3$ as an area under a curve. Approximate the area using four trapezoids.

G8 Sketch $y = \ln x$ and $y = \ln \frac{1}{x}$. Also sketch $y = e^x$ and $y = e^{-1/x}$.

G9 Sketch $y = 2 + e^x$ and $y = e^{x+2}$ and $y = 2e^x$ on the same axes.

Review Problems

- R1** Give an example of a linear differential equation and a nonlinear differential equation. If possible find their solutions starting from $y(0) = A$.
- R2** Give examples of differential equations that can and cannot be solved by separation of variables.
- R3** In exponential growth, the rate of change of y is directly proportional to _____. In exponential decay, dy/dt is proportional to _____. The difference is that _____.
- R4** What is a steady state? Give an example for $\frac{dy}{dt} = y + 3$.
- R5** Show from the definition that $d(\cosh x) = \sinh x \, dx$ and $d(\operatorname{sech} x) = \operatorname{sech} x \tanh x \, dx$.
- R6** A particle moves along the curve $y = \cosh x$ with $dx/dt = 2$. Find dy/dt when $x = 1$.
- R7** A chemical is decomposing with a half-life of 3 hours. Starting with 120 grams how much remains after 3 hours and how much after 9 hours?
- R8** A radioactive substance decays with a half-life of 10 hours. Starting with 100 grams, show that the average during the first 10 hours is $100/\ln 2$ grams.
- R9** How much money must be deposited now at 6% interest (compounded continuously) to build a nest egg of \$40,000 in 15 years?
- R10** Show that a continuous deposit of \$1645 per year at 6% interest yields more than \$40,000 after 15 years.

Drill Problems (Find dy/dx in **D1** to **D 12**.)

- | | | | |
|------------|--|------------|------------------------|
| D1 | $y = e^{\cos x}$ | D2 | $e^y + e^{-y} = 2x$ |
| D3 | $\sin x = e^y$ | D4 | $y = \pi^x + \pi^{-x}$ |
| D5 | $y = \frac{e^x}{x}$ | D6 | $y = \sec e^x$ |
| D7 | $y = \ln \frac{x-2}{x+2}$ | D8 | $y^2 = \ln(x^2 + y^2)$ |
| D9 | $y = \frac{\sqrt{x^2+5}(2x-3)^2}{\sqrt[3]{x^4(x+1)}} \text{ (use LD)}$ | D10 | $y = x^{\cos x}$ |
| D11 | $y = \ln(\tanh x^2)$ | D12 | $y = \cosh x \sinh x$ |

Find the integral in **D13** to **D20**.

D13 $\int 5^x dx$

D14 $\int x e^{x^2+1} dx$

D15 $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

D16 $\int \frac{e^x}{5+e^x} dx$

D17 $\int \frac{\cos x}{4+\sin x} dx$

D18 $\int \sinh x \cosh x dx$

D19 $\int \tanh^2 x \operatorname{sech}^2 x dx$

D20 $\int \frac{dx}{x \ln \frac{1}{x}}$

Solve the differential equations **D21** to **D26**

D21 $y' = -4y$ with $y(0) = 2$

D22 $\frac{dy}{dt} = 2 - 3y$ with $y_0 = 1$

D23 $\frac{dy}{dt} = t^2 \sqrt{y}$ with $y_0 = 9$

D24 $\frac{dy}{dt} = 2ty^2$ with $y_0 = 1$

D25 $\frac{dy}{dx} = e^{xy}$ with $y_0 = 10$

D26 $\frac{dy}{dt} = y - 2y^2$ with $y_0 = 100$

Solutions $y = 2e^{-4t}$ $y = \frac{1}{3}e^{-3t} + \frac{2}{3}$ $y = \left(\frac{t^3}{6} + 3\right)^2$ $y = \frac{-1}{t^2-1}$ $y = -\ln|e^{-10} - e^x|$ $y = \frac{1}{2-1.99e^{-t}}$

D27 If a population grows continuously at 2% a year, what is its percentage growth after 20 years?