
Strain

The study of the elastic behavior of statically determinate or indeterminate truss structures serves as a paradigm for the modeling and analysis of **all** structures in so far as it illustrates

- the isolation of a region of the structure prerequisite to imagining internal forces;
- the application of the equilibrium requirements relating the internal forces to one another and to the applied forces;
- the need to consider the displacements and deformations if the structure is redundant;¹
- and how, if displacements and deformations are introduced, then the constitution of the material(s) out of which the structure is made must be known so that the internal forces can be related to the deformations.

We are going to move on, with these items in mind, to study the elastic behavior of shafts in torsion and of beams in bending with the aim of completing the task we started in an earlier chapter – among other objectives, to determine when they might fail. To prepare for this, we step back and dig in a bit deeper to develop more complete measures of deformation, ones that are capable of taking us beyond uniaxial extension or contraction. We then must relate these measures of deformation, the *components of strain at a point* to the *components of stress at a point* through some stress-strain equations. We address that task in the next chapter.

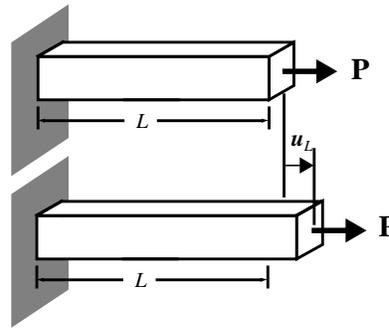
We will proceed without reference to truss members, beams, shafts in torsion, shells, membranes or whatever structural element might come to mind. We consider an arbitrarily shaped body, a *continuous solid body*, a *solid continuum*. We put on another special pair of eyeglasses, a pair that enables us to imagine what transpires *at a point* in a solid subjected to a load which causes it to deform and engenders strain along with some internal forces - the stresses of chapter 4. In our derivations that follow, we limit our attention to two dimensions: We first construct a set of strain measures in terms of the x, y (and z) components of displacement at a point. We then develop a set of stress/strain equations for a *linear, isotropic, homogenous, elastic solid*.

1. We of course must consider the deformations even of a determinate structure if we wish to estimate the displacements of points in the structure when loaded.

6.1 Strain: The Creature and its Components.

When a body displaces as a *rigid* body, points etched on the body will move through space but any arbitrarily chosen point will maintain the same distance from any other point—just as the stars in the sky maintain their position relative to other all other stars, night after night, as the heavens rotate about the earth. Except, of course, for certain “wandering stars” which do not maintain fixed distances among themselves or from the others.

But when a body *deforms*, points move relative to one another and distances between points change. For example, when the bar shown below is pulled with an end load P along its axis we know that a point at the end will displace to the right, say a distance u_L , relative to a point at the fixed, left end of the bar.



Assuming the bar is *homogenous*, that is, its constitution does not change as we move in from the end of the bar, we anticipate that the displacement relative to the fixed end will decrease. At the wall it must be zero; at the mid point we might anticipate it will be $u_L/2$. Indeed, this was the essence of our story about the behavior of an elastic rod in a uniaxial tension test.

$$\text{There we had } P = (AE/L) \cdot \delta = k \cdot \delta$$

The stiffness k is inversely proportional to the length of the rod so that, if the *same* end load is applied to bars of different length, the displacement of the ends will be proportional to their lengths, and the ratio of δ to L will be constant.

In our mind, then, we can imagine the horizontal rod shown above cut through at its midpoint. As far as the remaining, left portion is concerned, it is fixed at its left end and sees an a load P at its right end. Now since it has but half the length, its *end* will displace to the right but $u_L/2$.

We can continue this thought experiment from now to eternity; each time we make a cut we will obtain a midpoint displacement which is one-half the displacement at the right end of the previously imagined section. This of course assumes the bar is uniform in its cross-sectional area and material properties—that is, the bar is *homogeneous*. We summarize this result neatly by writing

$$u(x) = (u_L/L) \cdot x$$

where the factor, (u_L/L) , is a measure of the extensional strain of the bar, defined as the ratio of the change in length of the bar to its original length.

This brief thought experiment gives us a way to define a measure of extensional strain *at a point*. We say, at any point in the bar, that is, at any x ,

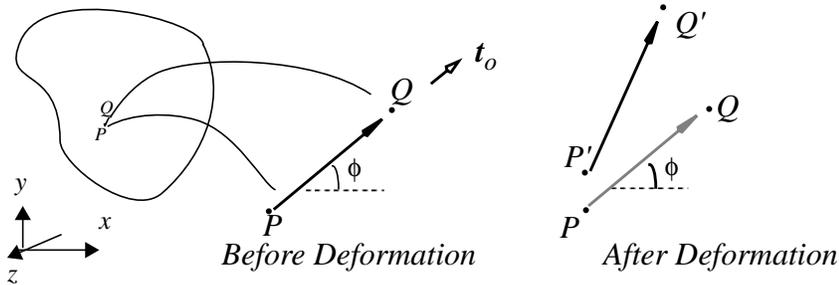
$$\epsilon_x = \lim_{\Delta x \rightarrow 0} (\Delta u / \Delta x) \Rightarrow \epsilon_x = \frac{\partial u}{\partial x}$$

For the homogenous bar under end load P we see that ϵ_x is a constant; it does not vary with x . We might claim that the end displacement, u_L is *uniformly distributed* over the length; that is, the relative displacement of any two points, equidistant apart in the undeformed state, is a constant; but this is not the usual way of speaking nor, other than for a truss element, is it usually the case.

The partial derivative implies that, u , the displacement component in the x direction, can be a function of spatial dimensions other than x alone; that is, for an arbitrary solid, with things changing as one moves in any of the three coordinate directions, we would have $u = u(x, y, z)$. We turn to this more general situation now.

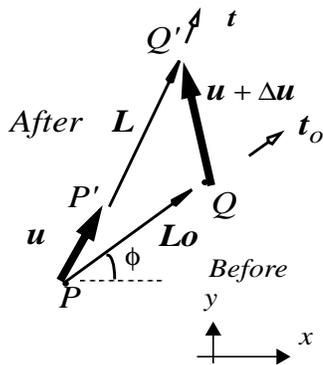
Exercise 6.1

What do I need to know about the displacements of points in a solid in order to compute the extensional strain at the point P , arbitrarily taken, in the direction of t_o , also arbitrarily chosen, as the body deforms from the state indicated at the left to that at the right?



We designate the extensional strain at P in the direction of t_o by ϵ_{PQ} . Our task is to see what we need to know in order to evaluate the limit

$$\epsilon_{PQ} = \lim_{PQ \rightarrow 0} (P'Q' - PQ) / (PQ)$$



To do this, we draw another picture of the undeformed and deformed differential line element, PQ , together with the displacements of its endpoints. Point P 's displacement to P' is shown as the vector, u , while the displacement of point Q , some small distance away, is designated by $u + \Delta u$.

This now looks very much like the representation used in the last chapter to illustrate and construct an expression for the extension of a truss member as a function of the horizontal and vertical components of displacement at its two ends. That's why I have introduced the

vectors L_o , and L for the *directed line segments* PQ , $P'Q'$ respectively though

they are in fact meant to be small, differential lengths. Proceeding in the same way as we did in our study of the truss, we write, as a consequence of vector addition,

$$\mathbf{u} + \mathbf{L} = \mathbf{u} + \Delta\mathbf{u} + \mathbf{L}_0$$

which yields an expression for $\Delta\mathbf{u}$ in terms of the vector difference of the two directed line segments, namely $\Delta\mathbf{u} = \mathbf{L} - \mathbf{L}_0$

We now introduce a most significant constraint. We assume, as we did with the truss, that *displacements and rotations are small* – displacements relative to some characteristic length of the solid, rotations relative to a radian. This should not be read as implying our analysis is of limited use. Most structures behave, i.e., deform, according to this constraint and, as we have seen in our study of a truss structure, it is entirely consistent with our writing the *equilibrium equations with respect to the undeformed configuration*. In fact *not* to do so would be erroneous.

Explicitly this means we will take

$$\mathbf{t} \approx \mathbf{t}_0 \quad \text{so that} \quad \mathbf{L} = \mathbf{t} \cdot \mathbf{L} \approx \mathbf{t}_0 \cdot \mathbf{L}$$

With this we can claim that the change in length of the directed line segment, PQ , in moving to $P'Q'$, is given by the projection of $\Delta\mathbf{u}$ upon PQ that is, since

$$P'Q' - PQ = \mathbf{L} - \mathbf{L}_0$$

we have

$$P'Q' - PQ = \mathbf{t}_0 \cdot \mathbf{L} - \mathbf{t}_0 \cdot \mathbf{L}_0 = \mathbf{t}_0 \cdot (\mathbf{L} - \mathbf{L}_0) = \mathbf{t}_0 \cdot \Delta\mathbf{u}$$

where \mathbf{t}_0 is, as before, a unit vector in the direction of PQ .

$$\mathbf{t}_0 = \cos\phi \cdot \mathbf{i} + \sin\phi \cdot \mathbf{j}$$

From here on in, constructing an expression for ϵ_{PQ} requires the machine-like evaluation of the scalar product, $\mathbf{t}_0 \cdot \Delta\mathbf{u}$, the introduction of the partial derivatives of the scalar components of the displacement taken with respect to position, and the manipulation of all of this into a form which reveals what's needed in order to compute the relative change in length of the arbitrarily oriented, differential line segment, PQ . We work with respect to a rectangular cartesian coordinate frame, x,y , and define the horizontal and vertical components of the displacement vector \mathbf{u} to be u,v respectively². That is, we set

$$\mathbf{u} = u(x, y) \cdot \mathbf{i} + v(x, y) \mathbf{j}$$

where the coordinates x,y label the position of the point P . The differential change in the displacement vector in moving from P to Q , a small distance which in the limit will go to zero, may then be written

2. In the following be careful to distinguish between the scalar u and the vector \mathbf{u} ; the former is the x component of the latter.

$$\Delta \mathbf{u} = \Delta u(x, y) \cdot \mathbf{i} + \Delta v(x, y) \cdot \mathbf{j}$$

Carrying out the scalar product, we obtain for the change in length of PQ :

$$P'Q' - PQ = \mathbf{t}_0 \cdot \Delta \mathbf{u} = (\Delta u) \cdot \cos \phi + (\Delta v) \cdot \sin \phi$$

We next approximate the small changes in the horizontal and vertical, scalar components of displacement by the products of their slopes at P taken with the appropriate differential lengths along the x and y axes as we move to point Q . That

$$\Delta u(x, y) \approx \left(\frac{\partial u}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y}\right)\Delta y \quad \text{and} \quad \Delta v(x, y) \approx \left(\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial v}{\partial y}\right)\Delta y$$

We have then

$$(P'Q' - PQ)/(PQ) \approx \left[\left(\frac{\partial u}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y}\right)\Delta y\right](\cos \phi/L_o) + \left[\left(\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial v}{\partial y}\right)\Delta y\right](\sin \phi/L_o)$$

where I have introduced L_o for the original length PQ .

This is an approximate relationship because the changes in the horizontal and vertical components of displacement are only approximately represented by the first partial derivatives. *In the limit*, however, as the distance PQ , and hence as Δx , Δy approaches zero, the approximation may be made as accurate as we like. Note also, that the ratios $\Delta x/L_o$, $\Delta y/L_o$ approach $\cos \phi$ and $\sin \phi$ respectively.

We obtain, finally, letting PQ go to zero, the following expression for the extensional strain at the point P in the direction PQ :

$$\varepsilon_{PQ} = \left(\frac{\partial u}{\partial x}\right)\cos^2 \phi + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos \phi \sin \phi + \left(\frac{\partial v}{\partial y}\right)\sin^2 \phi$$

It appears that in order to compute ε_{PQ} in the direction ϕ we *need to know* the four first partial derivatives of the scalar components of the displacement at the point P . In fact, however, we do not need to know all four partial derivatives since it is enough to know the **three** bracketed terms appearing above. Think of computing ε_{PQ} for different values of ϕ ; knowing the values for the three bracketed terms will enable you to do this.

The relationship above is a very important piece of machinery. It tells us how to compute the extensional strain in any direction, defined by ϕ , at any point, defined by x, y , in a body. In what follows, we call the three quantities within the brackets the *three scalar components of strain at a point*. But first observe:

- If we set ϕ equal to zero in the above, which is equivalent to setting PQ out along the x axis, we obtain, as we would expect, that $\varepsilon_{PQ} = \varepsilon_x$, the extensional strain at P in the x direction, i.e.,

$$\varepsilon_x = \left(\frac{\partial u}{\partial x}\right)$$

3. It is easy to be confused in the midst of all these partial derivatives. It's worth taking five minutes to try to sort them out.

- Our machinery is thus consistent with our previous definition of ϵ_x for uniaxial loading of a bar fixed at one end and lying along the x axis.
- If, in the same way, we set ϕ equal to a right angle, we obtain

$$\epsilon_{PQ} = \left(\frac{\partial v}{\partial y}\right)$$

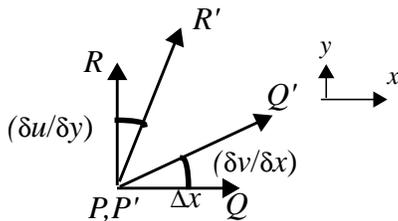
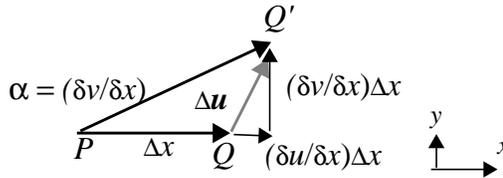
which can be read as the extensional strain at P in the direction of a line segment along the y axis. We call this ϵ_y . That is

$$\epsilon_y = \left(\frac{\partial v}{\partial y}\right)$$

- The meaning of the term $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ is best extracted from a sketch; below we show how the term $\frac{\partial v}{\partial x}$ can be interpreted as the angle of rotation, about the z axis, of a line segment PQ along the x axis. For small rotations we can claim

$$\alpha \sim \tan \alpha = \frac{\left(\frac{\partial v}{\partial x}\right)\Delta x}{\Delta x}$$

Similarly, the term $\delta u/\delta y$ can be interpreted as the angle of rotation of a line segment along the y axis, but now, if positive, about the *negative* z axis. The figure below shows the meaning of both terms.



The sum of the two terms is the change in the right angle, PQR at point P . If it is a positive quantity, the right angle of the first quadrant has decreased. We define this sum to be a *shear strain* component at point P and label it with the symbol γ_{xy} .

- Building on the last figure, we define a *rotation* at the point P as the **average** of the rotations of the two, x,y , line segments. That is we define

$$\omega_{xy} = (1/2)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Note the negative sign to account for the different directions of the two line segment rotations. If, for example, $\delta v/\delta x$ is positive, and $\delta u/\delta y = -\delta v/\delta x$ then there is *no shear strain*, no change in the right angle, but there *is* a *rotation*, of magnitude $\delta v/\delta x$ positive

about the z axis at the point P . These three quantities $\epsilon_x, \gamma_{xy}, \epsilon_y$ are the three components of strain at a point.

$$\epsilon_x \equiv \frac{\partial u}{\partial x} \quad \gamma_{xy} \equiv \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \quad \epsilon_y \equiv \frac{\partial v}{\partial y}$$

If we know the way $\epsilon_x(x,y)$, $\gamma_{xy}(x,y)$, and $\epsilon_y(x,y)$ vary, we say we know the *state of strain* at any point in the body. We can then write our equation for computing the extensional strain in any arbitrary direction in terms of these three strain components associated with the x,y frame at a point as:

$$\epsilon_{PQ} = \epsilon_x \cdot \cos^2 \phi + \gamma_{xy} \cdot \cos \phi \sin \phi + \epsilon_y \cdot \sin^2 \phi$$

Finally, note that if we are given the displacement components as continuous functions u and v we can, by taking the appropriate partial derivatives, compute a set of strain functions, also continuous in x,y . On the other hand, going the other way, given the three strain components, $\epsilon_x, \gamma_{xy}, \epsilon_y$ as continuous functions of position, we cannot be assured that we can determine unique, continuous functions for the two displacement components from an integration of the strain-displacement relations. We say that the strains represent a *compatible* state of deformation only if we can do so, that is, only if we can construct a continuous *displacement field* from the strain components.

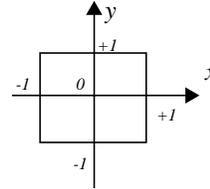
Exercise 6.2

For the planar displacement field defined by

$$u(x, y) = -\kappa \cdot xy \quad v(x, y) = \kappa \cdot x^2/2$$

where $\kappa = 0.25$, sketch the locus of the edges of a 2×2 square, centered at the origin, after deformation and construct expressions for the strain components ϵ_x, ϵ_y ,

and γ_{xy}



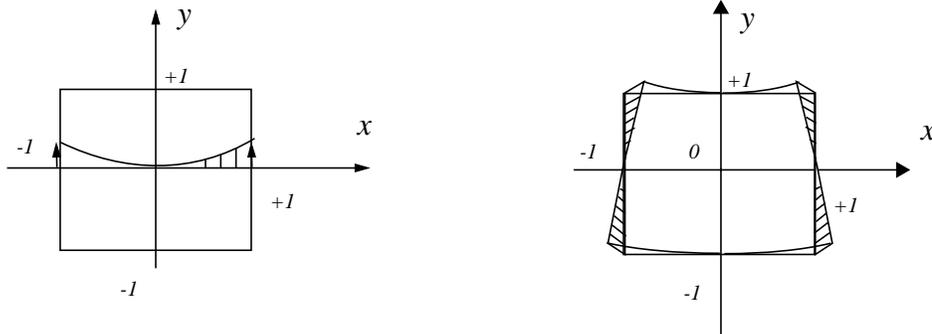
We start by evaluating the components of strain; we obtain

$$\epsilon_x \equiv \frac{\partial u}{\partial x} = -\kappa y \quad \gamma_{xy} \equiv \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) = -\kappa x + \kappa x = 0 \quad \epsilon_y \equiv \frac{\partial v}{\partial y} = 0$$

We see that the only non zero strain is the extensional strain in the x direction at every point in the plane. In particular, right angles formed by the intersection of a line segment in the x direction with another in the y direction remain right angles since the shear strain vanishes. The average rotation of these intersecting line segments at each and every point is found to be

$$\omega_{xy} = (1/2)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \kappa x$$

We sketch the locus of selected points and line segments below:



Focus first, on the figure at the left above which shows the deformed position of the points that originally lay along the x axis, at $y=0$. The vertical component of displacement v describes a parabola in the deformed state. Furthermore, the points along the x axis experience no horizontal displacement.

On the other hand, the points off the x or the y axis all have a horizontal component of displacement - as well as vertical. Consider now the figure above right. For example the point $(1,1)$ moves to the left a distance 0.25 while moving up a distance 0.125 . Below the x axis, however, the point originally at $(1,-1)$ moves to the right 0.25 while it still displaces upward the same 0.125 . The shaded lines are meant to indicate the u at each point.

Observe

- The state of strain does not vary with x , but does so with y .
- Right angles formed by x - y line segments remain right angles, that is the *shear strain* is zero.
- The average rotations of these right angles *does* vary with x but not with y . Note too that we have seemingly violated the assumption of small rotations. We did so in order to better illustrate the deformed pattern.

6.2 Transformation of Components of Strain

The axial stress in a truss member is related to the extensional strain in the member through an equation that looks very much like that which relates the force in a spring to its deflection. We shall relate all stress and strain components through some more general constitutive relations — equations which bring the specific properties of the material into the picture. But stress and strain are “relations” in another sense, in a more abstract, mathematical way: They are both the **same kind of mathematical entity**. The criterion and basis for this claim is the following: **The components of stress and strain at a point transform according to the**

same equations. By transform we mean change; by change we mean change due to a rotation of our reference axis at the point.

Our study of how the components of strain and stress transform is motivated as much by the usefulness of this knowledge in engineering practice as by visions of mathematical elegance and sophistication⁴. For, although this section could have been labeled *the transformation of symmetric, second-order tensors*, we have already seen an example, back in our study of stress, an example suggesting the potential utility of the component transformation machinery. We do an exercise very similar to that we tackled before to refresh our memory.

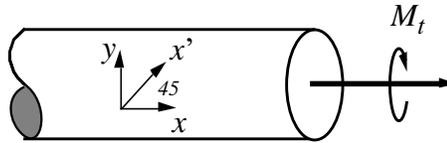
Exercise 6.3

Three strain gages, attached to the surface of a solid shaft in torsion in the directions x , y , and x' measure the three extensional strains

$$\varepsilon_x = 0 \quad \varepsilon_y = 0 \quad \text{and} \quad \varepsilon_{x'} = 0.00032$$

Estimate the shear strain γ_{xy} .

Let's work backwards. No one says you have to work forward from the "givens" straight through to the answer⁵.



We are given the values of three extensional strains measured at a point on the surface of the shaft⁶. The task is to determine the shear strain at the point from the three, measured extensional strains.

From the previous section we know that the extensional strain in the x' direction - thinking of that direction as "PQ" - can be expressed as

$$\varepsilon_{PQ} = \varepsilon_x \cdot \cos^2\phi + \gamma_{xy} \cdot \cos\phi \sin\phi + \varepsilon_y \cdot \sin^2\phi,$$

which tells me how to compute the extensional strain in some arbitrarily oriented direction at a point, as defined by the angle ϕ , given the state of strain at the point as defined by the three components of strain with respect to an x, y axis.

Working backwards, I will use this to compute the shear strain γ_{xy} given knowledge of the extensional strain ε_{PQ} where PQ is read as the direction of the gage x'

4. Katie: See Reid...I told you so!

5. This is characteristic of most work, not only in engineering but in science as well. The desired end state - the answer to the problem, the basic form of a design, the theorem to be proven, the character of the data to be collected - is usually known at the outset. There are really very few surprises in science or engineering in this respect. What is surprising, and exciting, and rewarding is that you can manage to construct things to come out right and they work according to your expectations.

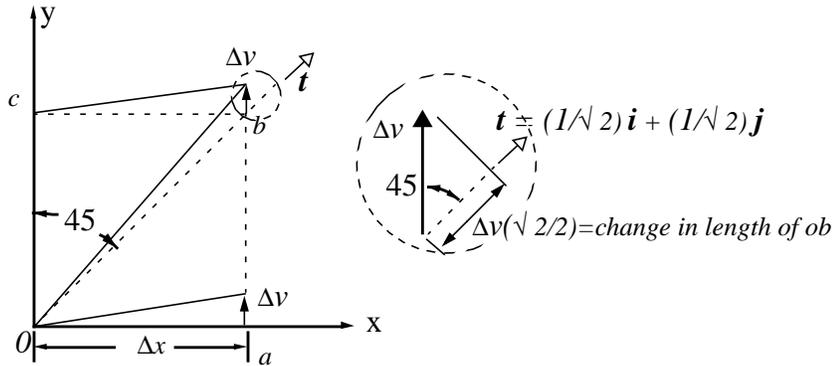
6. It's not really a point but a region about the size of a small coin.

oriented at 45° to the axis of the shaft and pasted to its surface. Now both ϵ_x and ϵ_y are zero⁷ so this equation gives

$$0.000032 = \gamma_{xy} \cdot (1/2) \quad \text{or} \quad \gamma_{xy} = 0.00064$$

Observe:

- If the strains ϵ_x and ϵ_y were different from zero we would still use this relationship to obtain an estimate of the shear strain. The former would provide us with direct estimates of any axial or hoop strain.
- I can graphically interpret this equation for determining the shear strain by constructing a compatible (continuous) displacement field from the strain components ϵ_x , ϵ_y , and γ_{xy} . Note this is not the only displacement field I might generate that is consistent with these strain components but it will serve to illustrate the relationship.



With the shaft oriented horizontally and twisted as shown, I take the displacement component, $u(x,y)$ to be zero and $v(x,y)$ to be proportional to x but independent of y . Then the points a and b both displace vertically a distance Δv with respect to points O and c . The extension of the diagonal Ob is, for small displacements and rotations, the projection of Δv at b upon the diagonal itself. So the change in length is given by $\Delta v/(\sqrt{2})$. Its original length is $\sqrt{2} \cdot \Delta x$ so we can

$$\text{write } \epsilon_{x'} = \epsilon_{ob} = \frac{(\Delta v/\Delta x)}{2}$$

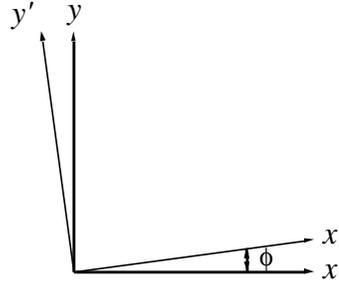
But, again for small rotations, $\Delta v/\Delta x = \gamma_{xy}$ the decrease in the right angle, the shear strain. Thus, as before,

$$\epsilon_{x'} = \gamma_{xy} \cdot (1/2)$$

7. More realistic values would be some small, insignificant numbers due to *noise* or slight imbalance in the apparatus used to measure, condition, and amplify the signal produced by the strain gage. Even so, if the shaft was subject to forms of loading other than, and in addition to the torque we seek to estimate, and these engendered significant strains in the a and c directions we would still make use of this relationship in estimating the shear strain.

This exercise illustrates an application of the rules governing the transformation of the components of strain at a point. That's now the way we read the equation we in the previous section – as a way to obtain the **extensional strain** along one axis of an arbitrarily oriented coordinate frame at a point in terms of the strain components known with respect to some reference coordinate frame.

For example, if I let the arbitrarily oriented frame be labeled $x'-y'$, then the **extensional** strain components relative to this new axis system can be written in terms of the strain components associated with the original, $x-y$ frame as

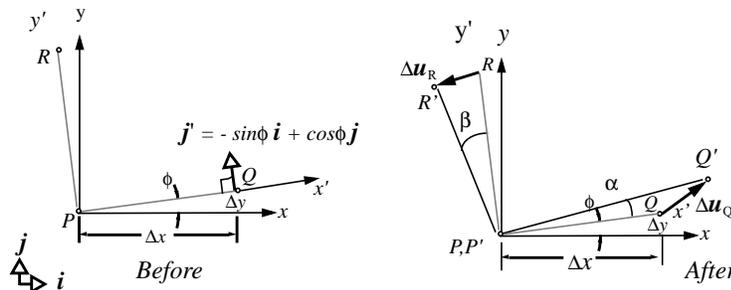


$$\begin{aligned} \epsilon'_{x'} &= \epsilon_x \cdot \cos^2 \phi + \gamma_{xy} \cdot \cos \phi \sin \phi + \epsilon_y \cdot \sin^2 \phi \\ \epsilon'_{y'} &= \epsilon_x \cdot \sin^2 \phi - \gamma_{xy} \cdot \cos \phi \sin \phi + \epsilon_y \cdot \cos^2 \phi \end{aligned}$$

In obtaining the expression for the extensional strain in the y' direction, I substituted $\phi + \pi/2$ for ϕ in the first equation.

But there is more to the story. I must construct an equation that allows me to compute the **shear strain**, $\gamma'_{x'y'}$ relative to the arbitrarily oriented frame, $x'y'$. To do so I make use of the same graphical methods of the previous section.

The figure below left shows the orientation of my reference $x-y$ axis and the orientation of an arbitrarily oriented frame $x'-y'$. PQ is a differential line element lying along the x' axis. \mathbf{t} is a unit vector along PQ ; \mathbf{e} is a unit vector perpendicular to PQ in the sense shown. Δx , Δy are the horizontal and vertical coordinates of Q relative to the origin of the reference frame.



On the right we show the position of PQ in the deformed state as $P'Q'$. The displacement of point Q relative to P is shown as $\Delta \mathbf{u}_Q$. The angle α is the (small) rotation of the line element PQ . This is what we seek to express in terms of the strain components ϵ_x , ϵ_y and γ_{xy} at the point. We will also determine the rotation of a line element along the y' axis. Knowing these we can compute the change in

the right angle QPR , the shear strain component with respect to the $x'-y'$ system which we will mark with a "prime", γ'_{xy} .

The angle α is given approximately by $\alpha = \Delta \mathbf{u} \cdot \mathbf{j}' / (PQ)$ where \mathbf{j}' is perpendicular to PQ .

The displacement vector we write as $\Delta \mathbf{u} = \Delta u \cdot \mathbf{i} + \Delta v \cdot \mathbf{j}$ which, to first order may be written in terms of the partial derivatives of the scalar components of the relative displacement of Q .

$$\Delta \mathbf{u} = \left(\Delta x \frac{\partial u}{\partial x} + \Delta y \frac{\partial u}{\partial y} \right) \mathbf{i} + \left(\Delta x \frac{\partial v}{\partial x} + \Delta y \frac{\partial v}{\partial y} \right) \mathbf{j}$$

and the unit vector is $\mathbf{j}' = -\sin\phi \cdot \mathbf{i} + \cos\phi \cdot \mathbf{j}$

Carrying out the scalar, dot product, noting that

$$\Delta y / PQ = \cos\phi \quad \text{and} \quad \Delta x / PQ = \sin\phi$$

we obtain

$$\alpha \approx -\sin\phi \left(\cos\phi \frac{\partial u}{\partial x} + \sin\phi \frac{\partial u}{\partial y} \right) + \cos\phi \left(\cos\phi \frac{\partial v}{\partial x} + \sin\phi \frac{\partial v}{\partial y} \right)$$

Or collecting terms

$$\alpha \approx \sin\phi \cos\phi \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) + \cos^2\phi \frac{\partial v}{\partial x} - \sin^2\phi \frac{\partial u}{\partial y}$$

I obtain the angle β the rotation of a line segment PR originally oriented along the y' axis most simply by letting ϕ go to $\phi + \pi/2$ in the above equation for the angle α . Thus

$$\beta \approx -\sin\phi \cos\phi \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) + \sin^2\phi \frac{\partial v}{\partial x} - \cos^2\phi \frac{\partial u}{\partial y}$$

the diminution in the right angle QPR is just $\alpha - \beta$ so I obtain:

$$\gamma'_{xy} \approx 2\sin\phi \cos\phi \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) + (\cos^2\phi - \sin^2\phi) \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

which, in terms of the strain components associated with the x, y axes becomes

$$\gamma'_{xy} = 2(\epsilon_y - \epsilon_x) \cdot \sin\phi \cos\phi + \gamma_{xy} \cdot (\cos^2\phi - \sin^2\phi)$$

With this I have all the machinery I need to compute the components of strain with respect to one orientation of axes at a point **given** their values with respect to another. I summarize below, making use of the *double angle* identities for the $\cos\phi$ and the $\sin\phi$, namely, $\cos 2\phi = \cos^2\phi - \sin^2\phi$ and $\sin 2\phi = 2\sin\phi \cos\phi$.

$$\begin{aligned}\epsilon'_x &= \left[\frac{(\epsilon_x + \epsilon_y)}{2} \right] + \left[\frac{(\epsilon_x - \epsilon_y)}{2} \right] \cdot \cos 2\phi + (\gamma_{xy}/2) \sin 2\phi \\ \epsilon'_y &= \left[\frac{(\epsilon_x + \epsilon_y)}{2} \right] - \left[\frac{(\epsilon_x - \epsilon_y)}{2} \right] \cdot \cos 2\phi - (\gamma_{xy}/2) \sin 2\phi \\ (\gamma'_{xy}/2) &= - \left[\frac{(\epsilon_x - \epsilon_y)}{2} \right] \cdot \sin 2\phi + (\gamma_{xy}/2) \cos 2\phi\end{aligned}$$

I have introduced a common factor of $(1/2)$ in the equation for the shear strain for the following reasons: If you compare these transformation relationships with those we derived for the components of stress, back in chapter 4, you will see they are identical in form if we identify the normal strain components with their corresponding normal stress components but we must identify τ_{xy} with $\gamma_{xy}/2$.

One additional relationship about deformation follows from our analysis: If I average the angular rotations of the two orthogonal line segments PQ and PR , I obtain an expression for what we define as the *rotation* of the x' - y' axes at the point. This produces

$$\omega'_{xy} = \left(\frac{1}{2} \right) (\alpha + \beta) = \left(\frac{1}{2} \right) \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \omega_{xy}$$

This, we note, is identical to ω_{xy} which is what justifies labeling this measure of deformation a *rigid body rotation*. It is also *invariant* of the transformation; regardless of the orientation of the coordinate frame at the point, you will always get the same number for this measure of rotation.

Exercise 6.4

A “bug” in my graphics software distorts the image appearing on my monitor. Horizontal lines are stretched 1%; vertical lines are compressed 5% and there is a distortion of the right angles formed by the intersection of horizontal and vertical lines of approximately 3° – a decrease in right angle in the first quadrant. Estimate the maximum extensional distortion I can anticipate for an arbitrarily oriented line drawn by my software. What is the orientation of this particular line relative to the horizontal?

I seek a maximum value for the extensional strain at a point — the extensional strain of an arbitrarily oriented line segment which is maximum. Any point on the screen will serve; we are working with a *homogeneous* state of strain, one which does not vary with position. I also of course want to know the direction of this line segment. The equation above for ϵ'_x shows the extensional strain as a function of

ϕ ; we differentiate with respect to ϕ seeking the value for the angle which will give a maximum (or minimum) extensional strain. I have:

$$\frac{d\epsilon'_x}{d\phi} = -(\epsilon_y - \epsilon_x)\sin 2\phi + \gamma_{xy} \cdot \cos 2\phi = 0$$

which I manipulate to

$$\tan 2\phi = \gamma_{xy}/(\epsilon_y - \epsilon_x)$$

Now the three x,y components of strain are $\epsilon_x=0.01$, $\epsilon_y = -0.05$, and $\gamma_{xy} = 3/57.3 = 0.052$. The above relationship, because of the behavior of the tangent function, will give me two roots within the range $0 < \phi < 360^\circ$, hence two values of ϕ .

I obtain two possibilities for the angle of orientation of maximum (or minimum) extensional strain, $\phi = 20.6^\circ$ and $\phi = 20.6 + 90^\circ = 110.6^\circ$. One of these will correspond to a maximum extensional strain, the other to a minimum. Note that we can read the second root as an extensional strain in a direction perpendicular to that associated with the first root. In other words, if we evaluate both ϵ'_x and ϵ'_y for a rotation of $\phi = 20.6^\circ$ we will find one a maximum the other a minimum. This we do now.

Taking then, $\phi = 20.6^\circ$ I obtain for the extensional strain in that direction, $\epsilon_I = 0.0197$

about two percent extension. The extensional strain at right angles to this I obtain from the equation for ϵ'_y , a strain along an axis 110.6° around from the horizontal, $\epsilon_{II} = -0.0597$,

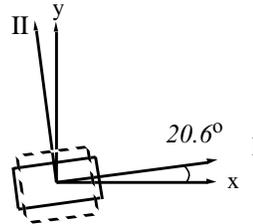
about six percent contraction. This latter is the maximum extensional distortion, a contraction of 5.97%. We illustrate the situation below.

Observe

- We call this pair of extreme values of extensional strain at a point, one a maximum, the other a minimum, *the principal strains*; the axes they are associated with are called *the principal axes*.
- The **shear strain associated with the principal axes is zero, always**. This follows from comparing the equation we derived by setting the derivative of the arbitrarily oriented extensional strain with respect to angle of rotation, namely

$$\tan(2\phi) = \gamma_{xy}/(\epsilon_x - \epsilon_y)$$

with the equation for the transformed component γ'_{xy} . If the former is satisfied then the shearmust vanish.



6.3 Mohr's Circle

Our working up of the transformation relations for stress and for strain and our exploration of their meaning in terms of extreme values has required considerable mathematical manipulation. We turn again to our graphical rendering of these relationships called *Mohr's Circle*. I have set out the rules for constructing the circle for a particular state of stress. What I seek now is to show the “sameness” of the transformation relations for strain components.

First, I repeat the transformation equations for a two-dimensional state of stress.

$$\begin{aligned}\sigma'_x &= \left[\frac{(\sigma_x + \sigma_y)}{2} \right] + \left[\frac{(\sigma_x - \sigma_y)}{2} \right] \cdot \cos 2\phi + \sigma_{xy} \sin 2\phi \\ \sigma'_y &= \left[\frac{(\sigma_x + \sigma_y)}{2} \right] - \left[\frac{(\sigma_x - \sigma_y)}{2} \right] \cdot \cos 2\phi - \sigma_{xy} \sin 2\phi \\ \sigma'_{xy} &= - \left[\frac{(\sigma_x - \sigma_y)}{2} \right] \cdot \sin 2\phi + \sigma_{xy} \cos 2\phi\end{aligned}$$

and now the transformation equations for a two-dimensional state of strain:

$$\begin{aligned}\epsilon'_x &= \left[\frac{(\epsilon_x + \epsilon_y)}{2} \right] + \left[\frac{(\epsilon_x - \epsilon_y)}{2} \right] \cdot \cos 2\phi + (\gamma_{xy}/2) \sin 2\phi \\ \epsilon'_y &= \left[\frac{(\epsilon_x + \epsilon_y)}{2} \right] - \left[\frac{(\epsilon_x - \epsilon_y)}{2} \right] \cdot \cos 2\phi - (\gamma_{xy}/2) \sin 2\phi \\ (\gamma'_{xy}/2) &= - \left[\frac{(\epsilon_x - \epsilon_y)}{2} \right] \cdot \sin 2\phi + (\gamma_{xy}/2) \cos 2\phi\end{aligned}$$

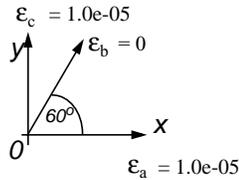
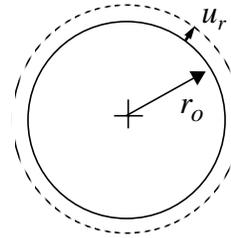
Comparing the two sets, we see they are the same **if we compare half the shear strain with the corresponding shear stress**. This means we can use the same Mohr's Circle as for stress when doing strain transformation problems. All we need do is think of the vertical axis as being a measure of $\gamma/2$.

6.4 Problems

6.1 Show that for the thin circular hoop subject to an axi-symmetric, radial extension u_r , that the circumferential extensional strain, can be expressed as

$$\epsilon_\theta = (L - L_0)/L_0 = u_r/r_0$$

where L_0 is the original, undeformed circumference.



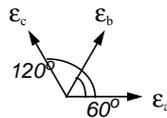
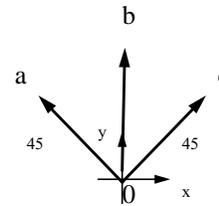
6.2 Three strain gages are mounted in the directions shown on the surface of a thin plate. The values of the extensional strain each measures is also shown in the figure.

- i) Determine the shear strain component γ_{xy} at the point with respect to the xy axes shown.
- ii) What orientation of axes gives extreme values for the extensional strain components at the point.
- iii) What are these values.

6.3 Three strain gages measure the extensional strain in the three directions 0a, 0b and 0c at “the point 0”. Using the relationship we derived in class

$$\epsilon_{PQ} = \epsilon_x \cos^2 \phi + \gamma_{xy} \cos \phi \sin \phi + \epsilon_y \sin^2 \phi$$

find the components of strain with respect to the xy axis in terms of ϵ_a , ϵ_b and ϵ_c



6.4 A strain gage rosette, fixed to a flat, thin plate, measures the following extensional strains

- $\epsilon_a = 1. \text{E-}04$
- $\epsilon_b = 1. \text{E-}04$
- $\epsilon_c = 2. \text{E-}04$

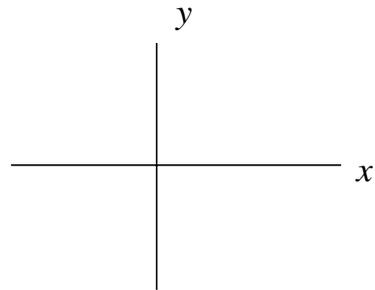
Determine the state of strain at the point, expressed in terms of components relative to the xy coordinate frame shown.

6.5 A two dimensional displacement field is defined by

$$u(x, y) = \frac{-\alpha}{2} \cdot y \quad \text{and} \quad v(x, y) = \frac{\alpha}{2} \cdot x$$

Sketch the position of the points originally lying along the x axis, the line $y = 0$, due to this displacement field.

Assume α is very much less than 1.0.



Likewise, on the same sketch, show the position of the points originally lying along the y axis, the line $x = 0$, due to this displacement field.

Likewise, on the same sketch, show the position of the points originally lying along the line $y=x$, due to this displacement field.

Calculate the state of strain at the origin; at the point, x, y .

Respond again but now with $u(x, y) = \frac{\alpha}{2} \cdot y$ and $v(x, y) = \frac{\alpha}{2} \cdot x$