

Problem Set #8 Solution 1.050 Solid Mechanics Fall 2004

Problem 8.1

A solid aluminum, circular shaft has length 0.35 m and diameter 6 mm. How much does one end rotate relative to the other if a torque about the shaft axis of 10 N-m is applied?

We turn to the stiffness relationship, $M_t = \frac{GJ}{L} \cdot \phi$ in order to determine the relative angle of twist, ϕ , over the length of the shaft, L. For a solid circular shaft, the moment of inertia, $J = \pi R^4/2$. The shear modulus G is related to the material's elastic modulus and Poisson's ratio by $G = \frac{E}{2(1 + \nu)}$ and the elastic modulus for Aluminum, found in the table on page 212 is $70 \text{ E } 09 \text{ N/m}^2$.

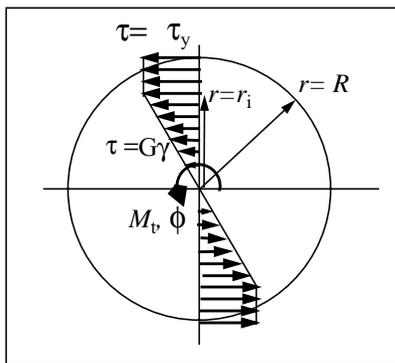
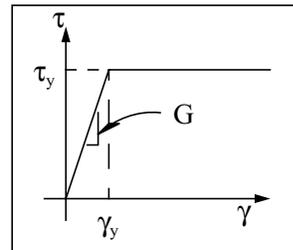
Taking Poisson's ratio as 1/3, G then evaluates to $26 \text{ E } 09 \text{ N/m}^2$. Putting all of this together gives an angle of rotation $\phi = \frac{L}{GJ} \cdot M_t = 1.06 \text{ radians}$. The maximum shear stress is obtained from $\tau_{|_{max}} = M_t R / J$.

This evaluates to $\tau_{|_{max}} = 2.36 \times 10^8 \text{ N/m}^2$ (This is the change 12 Nov. 04).

What follows is an excursion which considers the possibility of plastic flow in the shaft.

We see that the maximum shear stress is near or greater than one-half the yield stress (in tension) of the three types of Aluminum given in the table in chapter 7. So what will ensue?

To proceed, we need a model for the constitutive behavior that includes the relationship between shear stress and shear strain beyond the elastic range. We assume "elastic - perfectly plastic" behavior, i.e., that shown at the right. Beyond the yield point, no further increase in stress is required to produce an increase in strain¹.



Consider the shear stress distribution at some instance *beyond* the onset of yield. Onset of yield will occur when the shear stress at the outer radius of the shaft, $r = R$, attains the value τ_y . With some additional loading, some additional torque applied, the shear stress distribution over the radius might appear as at the left.

We show the material yielding in the region of the cylinder beyond some radius $r = r_i$ where the subscript "i" denotes the radius at the interface of the elastic and perfectly plastic regions. In that region of plastic flow, the shear stress is everywhere equal to the shear stress at yield, τ_y . At $r = r_i$, the shear stress and strain are just τ_y and γ_y respectively. The material within the core of the shaft, $r < r_i$ continues to deform elastically.

Within this elastic region, we have a linear distribution of both stress and strain with r , which can be written:

$$\tau = \tau_y \cdot \left(\frac{r}{r_i}\right) \quad \text{and} \quad \gamma = \gamma_y \cdot \left(\frac{r}{r_i}\right)$$

This stress distribution results in a torque, which we obtain from an integration over the area of the entire cross-section. We have:

$$M_t = \int_A r \cdot \tau dA = \int_0^{r_i} r \cdot \left(\tau_y \frac{r}{r_i}\right) \cdot 2\pi r dr + \int_{r_i}^R r \cdot (\tau_y) \cdot 2\pi r dr$$

where we have taken account of the rotationally symmetry of the problem, carrying out the integration with respect to theta, giving the factor 2π . Carrying out the integration with respect to r yields:

1. If a maximum shear stress criterion applies, the shear stress at yield is one-half the yield stress in a tension test.

$$M_t = 2\pi\tau_y \cdot \left[\frac{r_i^3}{4} + \frac{R^3}{3} - \frac{r_i^3}{3} \right] = \frac{2R^3}{3}\pi\tau_y \cdot \left[1 - \frac{r_i^3}{4R^3} \right]$$

This relationship, which holds only after some outer portion of the shaft has gone plastic, shows how the torque, M_t , increases as r_i decreases - as more and more of the shaft enters the plastic zone. The onset of plastic flow happens when $r_i = R$. We label the torque when this happens M_{yield} .

$$M_{yield} \equiv \frac{2R^3}{3}\pi\tau_y \cdot \left[1 - \frac{1}{4} \right] = \frac{\pi}{2}R^3\tau_y$$

Likewise, we can define the angle of twist at the onset of yield. We have,

$$\gamma = r \cdot \left(\frac{\phi}{L} \right) \quad \text{so at the onset of yield} \quad \gamma_y = R \cdot \left(\frac{\phi_{yield}}{L} \right).$$

Now, past the point of the onset of plastic flow at the outermost radius, when the shear stress distribution appears as shown in the figure above, at $r = r_i < R$, we have

$$\gamma = r \cdot \left(\frac{\phi}{L} \right) \quad \text{so at the interface} \quad \gamma_y = r_i \cdot \left(\frac{\phi}{L} \right)$$

These two expressions for γ_y enable us to write $\frac{r_i}{R} = \left(\frac{\phi_{yield}}{\phi} \right)$

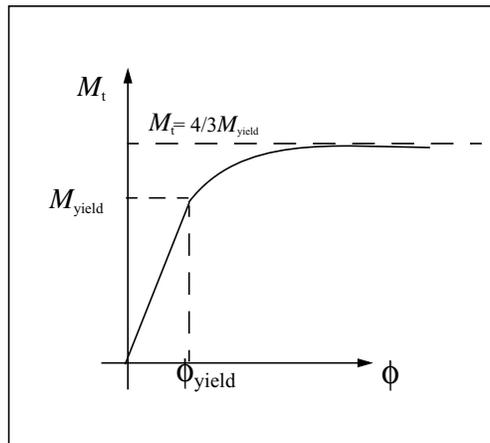
We can now eliminate the yield shear stress from our express for the torque beyond onset of yield, replacing it with the torque required at the onset of yield and also eliminate the radius to the interface, replacing it with the angle of twist at the onset of yield. We have, for $\phi \geq \phi_{yield}$:

$$M_t = \left(\frac{4}{3} \right) M_{yield} \left[1 - \frac{r_i^3}{4R^3} \right] = \left(\frac{4}{3} \right) M_{yield} \left[1 - \frac{1}{4} \left(\frac{\phi_{yield}}{\phi} \right)^3 \right]$$

While for $\phi \leq \phi_{yield}$ we have linear elastic behavior. At the interface, $M_t = \frac{GJ}{L} \cdot \phi = M_{yield} \cdot \frac{\phi}{\phi_{yield}}$.

We plot and see, from the above expression for M_t , letting ϕ get very large, that the shaft can sustain a torque of up to four thirds the torque for onset of yield.

Of course the angle of twist increases without limit as we approach this "ultimate" torque.



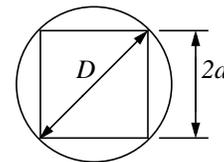
Problem 8.2

What if a solid circular shaft is replaced by a square shaft whose **diagonal** is equal to the diameter of the original circular shaft; How does the torsional stiffness change; For the same torque, how does the maximum shear stress change?

What if the solid circular shaft is replaced by a square shaft whose **side** is equal to the diameter of the original circular shaft; How do these change?

The figure at the right shows the first case: Here the length of side of the inscribed square, which we call $2a$, expressed in terms of the diameter of the circle is

$$2a = \frac{D}{\sqrt{2}} \quad \text{or, in terms of the radius, } R \quad a = \frac{R}{\sqrt{2}}$$



The torsional stiffness of the solid circular shaft is $\pi R^4/2$. That for the square shaft is obtained from the textbook, page 232. There we find $M_t = K_t \cdot \phi/L$ where $K_t = G \cdot k_1 \cdot (2a)^3 (2b)$. In terms of the radius of the circle, putting a (and b) in terms of the radius R , gives

$$K_t = G \cdot k_1 \cdot \left(2 \frac{R}{\sqrt{2}}\right)^4 = G \cdot \left(\frac{8k_1}{\pi}\right) \cdot \frac{\pi R^4}{2}$$

A table gives the value for k_1 , for the case when $2a=2b$, as 0.1406. So we obtain $K_t = 0.36GJ$ where J is the polar moment of inertia of the solid circular cylinder. The inscribed square shaft has a torsional stiffness of 36% of the solid circular shaft.

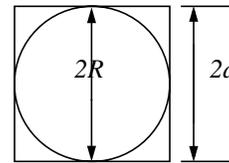
The maximum shear stress is obtained from a second relationship found on page 232: Now we have $\tau|_{max} = M_t \left(\frac{1}{k_2 \cdot (2a)^2(2b)} \right)$ where k_2 for our case when $a=b$ is found in the table, i.e., $k_2 = 0.208$.

Carrying out the same substitutions, we obtain: $\tau|_{max} = M_t \left(\frac{1}{\left[\frac{8k_2}{\pi\sqrt{2}}\right] \cdot (\pi R^3)/2} \right)$.
The factor $\left[\frac{8k_2}{\pi\sqrt{2}}\right]$ gives $\left[\frac{8k_2}{\pi\sqrt{2}}\right] = 0.374$.

This factor is 1.0 for the circular shaft. The maximum shear stress in the inscribed square shaft is 2.67 times greater than the maximum shear stress in the solid circular shaft.

For the square shaft circumscribing the cylinder, we have $a=R$. We find in this case that the square shaft is stiffer in torsion than the circular shaft by a factor of 1.43.

And the maximum shear stress in the square shaft is just a bit smaller than the maximum shear stress in the circular shaft - a factor of 0.94.



Problem 8.3

A thin Aluminum tube, whose wall thickness is 1 mm, carries a torque 80% of the torque required for the onset of yield. The radius of the tube is 20 mm.

Show that an estimation of shear stress, based upon the assumption that it is uniformly distributed over the thickness of the tube (and using an estimate of J that is linear in the wall thickness), gives a value within 10% of that computed using the full expressions for shear stress and polar moment of inertia.

For a hollow shaft, the polar moment of inertia J , is given by $J = \frac{\pi R_o^4}{2} - \frac{\pi R_i^4}{2}$ where R_o and R_i are the outer and inner radii of the shaft.

Letting t be the thickness, we have $t = R_o - R_i$ so J is $J = \frac{\pi}{2} \cdot ((R_i + t)^4 - R_i^4)$. Expanding the first term we have, assuming t/R is small

$J = \frac{\pi}{2} \cdot (R_i^4 + 4R_i^3 t + O(t^2) - R_i^4) \cong \frac{\pi}{2} \cdot (4R_i^3 t) = 2\pi R_i^3 t$ which is the basis for our approximate solution.

The shear stress is given by $\tau|_{max} = M_t R_o / J$. So the ratio of the two is

$$\frac{\tau|_{max}}{\tau|_{maxapprox}} = \left[\frac{M_t R_o}{\frac{\pi}{2} \cdot (R_o^4 - R_i^4)} \right] \cdot \left[\frac{2\pi R_i^3 t}{M_t R_o} \right] = 0.92$$

(exact) (approx.)

Thus, the approximate shear stress is within 10%.