
Stress

We have talked about internal forces, distributed them uniformly over an area and they became a *normal stress* acting *perpendicular* to some internal surface at a point, or a *shear stress* acting *tangentially, in plane*, at the point. Up to now, the choice of planes upon which these stress components act, their orientation within a solid, was dictated by the geometry of the solid and the nature of the loading. We have said nothing about how these stress components might change if we looked at a set of planes of another orientation at the point. And up to now, we have said little about how these normal and shear stresses might vary with position throughout a solid.¹

Now we consider a more general situation, namely an arbitrarily shaped solid which may be subjected to all sorts of externally applied loads - distributed or concentrated forces and moments. We are going to lift our gaze up from the world of crude structural elements such as truss bars in tension, shafts in torsion, or beams in bending to view these “solids” from a more abstract perspective. They all become special cases of a more general stuff we call a solid continuum.

Likewise, we develop a more general and more abstract representation of internal forces, moving beyond the notions of shear force, internal torque, uni-axial tension or compression and internal bending moment. Indeed, we have already done so in our representation of the internal force in a truss element as a normal stress, in our representation of torque in a thin-walled, circular shaft as the resultant of a uniformly distributed shear stress, in our representation of internal forces in a cylinder under internal pressure as a hoop stress (and as an axial stress). We want to develop our vocabulary and vision in order to speak intelligently about stress in its most general form.

We address two questions:

- How do the normal and shear components of stress acting on a plane at a given point change as we change the orientation of the plane at the point.
- How might stresses vary from one point to another throughout a continuum;

The first bullet concerns the *transformation of components of stress at a point*; the second introduces the notion of *stress field*. We take them in turn.

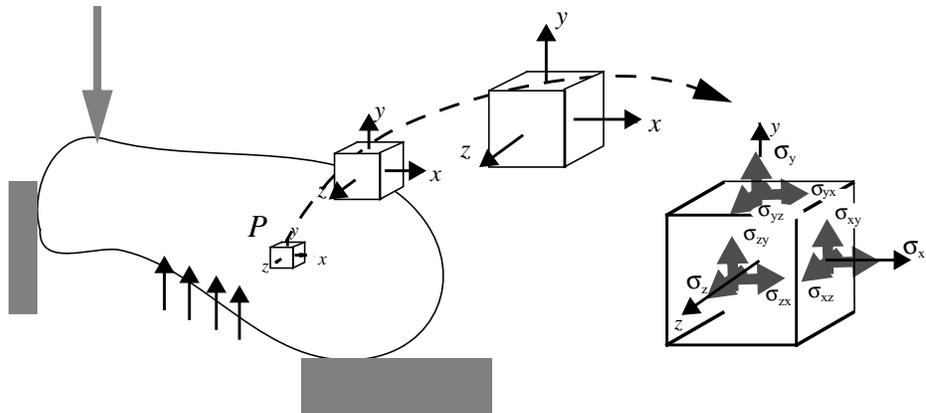
1. The beam is the one exception. There we explored how different normal stress distributions over a rectangular cross-section could be equivalent to a bending moment and zero resultant force.

4.1 Stress: The Creature and its Components

We first address what we need to know to fully define “stress at a point” in a solid continuum. We will see that the stress at a point in a solid continuum is defined by its scalar components. Just as a vector quantity, say the velocity of a projectile, is defined by its *three* scalar components, we will see that the stress at a point in a solid continuum is defined by its *nine* scalar components.

Now you are probably quite familiar with vector quantities - quantities that have three scalar components. But you probably have not encountered a quantity like stress that require more than three scalar values to fix its value at a point. This is a new kind of animal in our menagerie of variables; think of it as a new species, a new creature in our zoo. But don't let the number nine trouble you. It will lead to some algebraic complexity, compared to what we know how to do with vectors, but we will find that stress, a second order tensor, behaves as well as any vector we are familiar with.

The figure below is meant to illustrate the more general, indeed, the most general state of stress at a point. It requires some explanation:



The odd looking structural element, fixed to the ground at bottom and to the left, and carrying what appears to be a uniformly distributed load over a portion of its bottom and a concentrated load on its top, is meant to symbolize an arbitrarily loaded, arbitrarily constrained, arbitrarily shaped solid continuum. It could be a beam, a truss, a thin-walled cylinder though it looks more like a potato — which too is a solid continuum. At any arbitrarily chosen point inside this object we can ask about the value of the stress at the point, say the point P . But what do we mean by “value”; value of what at that point?

Think about the same question applied to a vector quantity: What do we mean when we say we know the value of the velocity of a projectile at a point in its trajectory? We mean we know its magnitude - its “speed” a scalar - and its direction. Direction is fully specified if we know two more scalar quantities, e.g., the direction the vector makes with respect to the axes as measured by the cosine of the angles it makes with each axis. More simply, we have fully defined the velocity at

a point if we specify its three scalar components with respect to some reference coordinate frame - say its x , y and z components.

Now how do we know this fully defines the vector quantity? We take as our criterion that anyone, anyone in the world (of mathematical physicists and engineers), would agree that they have in hand the same thing, no matter what coordinate frame they favor, no matter how they viewed the motion of the projectile. (We do insist that they are not displacing or rotating relative to one another, i.e., they all reside in the same inertial frame). This is assured if, after transforming the scalar components defined with respect to one reference frame to another, we obtain values for the components any other observer sees.

It is then the equations which transform the values of the components of the vector from one frame to another which define what a vector is. This is like defining a thing by what it does, e.g., “you are what you eat”, a behaviorists perspective - which is really all that matters in mathematical physics and in engineering.

Reid: Hey Katie: what do you think of all this talk about components? Isn't he going off the deep end here?

Katie: What do you mean, “...off the deep end”?

Reid: I mean why don't we stick with the stuff we were doing about beams and trusses? I mean that is the useful stuff. This general, abstract continuum business does nothing for me.

Katie: There must be a reason, Reid, why he is doing this. And besides, I think it is interesting; I mean have you ever thought about what makes a vector a vector?

Reid: I know what a vector is; I know about force and velocity; I know they have direction as well as a magnitude. So big deal. He is maybe just trying to snow us with all this talk about transformations.

Katie: But the point is what makes force and velocity the same thing?

Reid: Their not the same thing!

Katie: He is saying they are - at a more abstract, general level. Like....like robins and bluebirds are both birds.

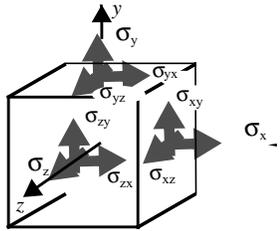
Reid: And so stress is like tigers, is that it?

Katie: Yeah, yeah - he said a new animal in the zoo.

Reid: Huh... pass me the peanuts, will you?

We envision the components of stress as coming in sets of three: One set acts upon what we call an x plane, another upon a y plane, a third set upon a z plane. Which plane is which is defined by its *normal*: An x plane has its normal in the x direction, etc. Each set includes three scalar components, one *normal stress component* acting perpendicular to its reference plane, with its direction along one

coordinate axis, and two *shear stress components* acting *in plane* in the direction of the other two coordinate axes.

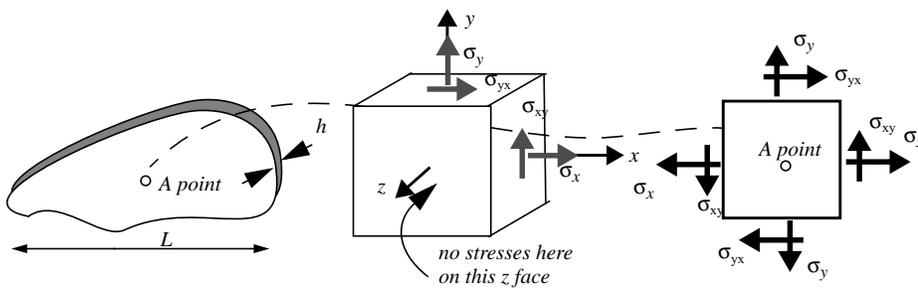


That's a grand total of *nine stress components to define the stress at a point*. To fully define the stress field throughout a continuum you need to specify how these nine scalar components. Fortunately, equilibrium requirements applied to a differential element of the continuum, what we will call a "micro-equilibrium" consideration, will reduce the number of independent stress components at a point from nine to six. We will find that the shear stress component σ_{xy} acting on the x face must equal its neighbor around the corner σ_{yx} acting on the y face and that $\sigma_{zy} = \sigma_{yz}$ and $\sigma_{xz} = \sigma_{zx}$ accordingly.

Fortunately too, in most of the engineering structures you will encounter, diagnose or design, only two or three of these now six components will matter, that is, will be *significant*. Often variations of the stress components in one, or more, of the three coordinate directions may be uniform. But perhaps the most important simplification is a simplification in modeling, made at the outset of our encounter. One particularly useful model, applicable to many structural elements is called *Plane Stress* and, as you might infer from the label alone, it restricts our attention to variations of stress in two dimensions.

Plane Stress

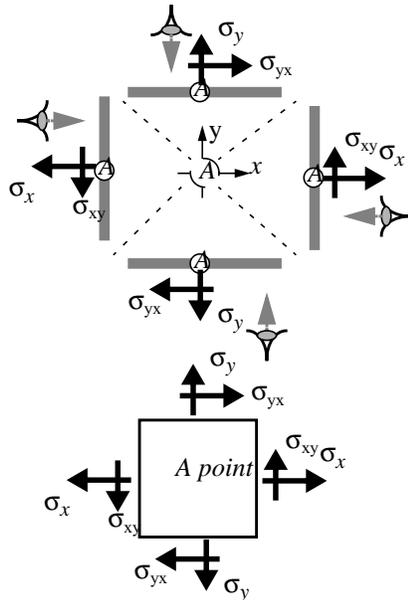
If we assume our continuum has the form of a thin plate of uniform thickness but of arbitrary closed contour in the x - y plane, our previously arbitrarily loaded, arbitrarily constrained continuum (we don't show these again) takes the planar form below.



Because the plate is thin in the z direction, $(h/L \ll 1)$ we will assume that variations of the stress components with z is uniform or, in other words, our stress components will be at most functions of x and y . We also take it that the z *boundary planes are unloaded, stress-free*. These two assumptions together imply that

the set of three “z” stress components that act upon *any* arbitrarily located z plane within the interior must also vanish. We will also take advantage of the micro-equilibrium consequences, yet to be explored but noted previously, and set σ_{yz} and σ_{xz} to zero. Our state of stress at a point is then as it is shown on the exploded view of the point - the block in the middle of the figure - and again from the point of view of looking normal to a z plane at the far right. This special model is called **Plane Stress**.

A Word about Sign Convention:



The figure at the far right seems to include more stress components than necessary; after all, if, in modeling, we eliminate the stress components acting on a z face and σ_{yz} and σ_{xz} as well, that should leave, at most, four components acting on the x and y faces. Yet there appear to be eight in the figure. No, there are only at most four components; we must learn to read the figure.

To do so, we make use of another sketch of *stress at a point*, the point A. The figure at the top is meant to indicate that we are looking at four faces or planes simultaneously. When we look at the *x face* from the right  we are looking at the stress components on a *positive x face* — it has its *outward normal in the positive x direction* — and a *positive normal stress*,

by convention, is *directed in the positive x direction*. A *positive shear stress component*, acting in plane, also acts, by convention, *in a positive coordinate direction* - in this case the positive y direction.

On the positive y face, we follow the same convention; a *positive σ_y* acts on a *positive y face in the positive y coordinate direction*; a *positive σ_{yx}* acts on a *positive y face in the positive x coordinate direction*.

We emphasize that we are looking at a point, point A, in these figures. More precisely we are looking at two mutually perpendicular planes intersecting at the point and from two vantage points in each case. We draw these two views of the two planes as four planes in order to more clearly illustrate our sign convention. But you ought to imagine the square having zero height and width: the σ_x acting to the *left*, in the *negative x direction*, upon the *negative x face* at the left, with its *outward normal pointing in the negative x direction* is a *positive component* at the point, the **equal and opposite reaction to the σ_x** acting to the right, in the *positive x direction*, upon the *positive x face* at the right, with its *outward normal pointing in the positive x direction*. Both are positive as shown; **both are the same quan-**

tity. So too the shear stress component σ_{xy} shown acting down, in the negative y direction, on the negative x face is the equal and opposite internal reaction to σ_{xy} shown acting up, in the positive y direction, on the positive x face

A general statement of our sign convention, which holds for all nine components of stress, even in 3D, is as follows:

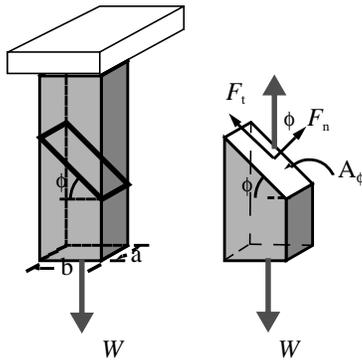
A positive component of stress acts on a positive face in a positive coordinate direction or on a negative face in a negative coordinate direction.

Transformation of Components of Stress

Before constructing the equations which fix how the components of stress transform in general, we consider a simple example of a bar suspended vertically and illustrate how components change when we change our reference frame at a point. In this example, we take the weight of the bar to be negligible relative to the weight suspended at its free end and explore how the normal and shear stress components at a point vary as we change the orientation of a plane.

Exercise 4.1

The solid column of rectangular cross section measuring $a \times b$ supports a weight W . Show that both a normal stress and a shear stress must act on any inclined interior face. Determine their respective values assuming that both are uniformly distributed over the area of the inclined face. Express your estimates in terms of the ratio (W/ab) and the angle ϕ .



For equilibrium of the isolation of a section of the column shown at the right, a force equal to the suspended weight (we neglect the weight of the column itself) must act upward. We show an equivalent force system — or, if you like, its components consisting of two perpendicular forces, one directed normal to the inclined plane, the other with its line of action in the plane inclined at the angle ϕ . We have

$$F_n = W \cdot \cos\phi \quad \text{and} \quad F_t = W \cdot \sin\phi$$

Now if we assume these are distributed uniformly over the section, we can construct an estimate of the normal stress and the shear stress acting on the inclined face. But first we must establish the area of the inclined face A_ϕ . From the geometry of the figure we see that the length of the inclined plane is $b/\cos\phi$ so the area is

$$A_\phi = (ab)/(\cos\phi)$$

With this we write the normal and shear stress components as

$$\sigma_n = F_n/A_\phi = \left(\frac{W}{ab}\right) \cdot \cos\phi^2 \quad \text{and} \quad \sigma_t = F_t/A_\phi = \left(\frac{W}{ab}\right) \cdot \cos\phi \sin\phi$$

These results clearly illustrate how the values for the normal and shear stress components of a force distributed over a plane inside of an object depends upon how you look at the point inside the object in the sense that **the values of the shear and normal stresses at a point within a continuum depend upon the orientation of the plane you have chosen to view.**

Why would anyone want to look at some arbitrarily oriented plane in an object, seeking the normal and shear stresses acting on the plane? Why do we ask you to learn how to figure out what the stress components on such a plane might be?

The answer goes as follows: One of our main concerns as a designer of structures is failure —fracture or excessive deformation of what we propose be built and fabricated. Now many kinds of failures initiate at a local, microscopic level. A minute imperfection at a point in a beam where the local stress is very high initiates fracture or plastic deformation, for example. Our quest then is to figure out where, at what points in a structural element, the normal and shear stress components achieve their maximum values. But we have just seen how these values depend upon the way we look at a point, that is, upon the orientation of the plane we choose to inspect. To ensure we have found the maximum normal stress at a point for example, we would then have to inspect every possible orientation of a plane passing through the point.²

This seems a formidable task. But before taking it on, we pose a prior question:

Exercise 4.2

What do you need to know in order to determine the normal and shear stress components acting upon an arbitrarily oriented plane at a point in a fully three dimensional object?

The answer is what we might anticipate from our original definition of six stress components for if we know these six scalar quantities³, the three normal stress components σ_x , σ_y , and σ_z , and the three shear stress components σ_{xy} , σ_{yz} , and σ_{xz} , then we can find the normal and shear stress components acting upon an arbitrarily oriented plane at the point. That is the answer to our need to know question.

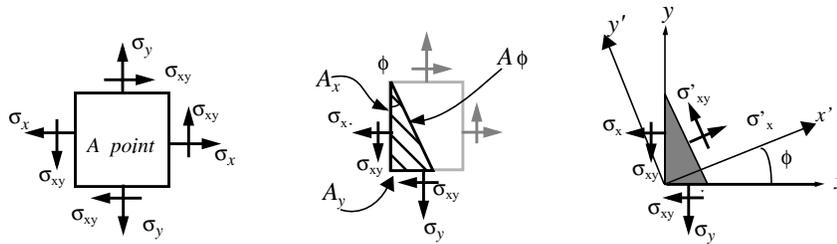
To show this, we derive a set of equations that will enable you to do this. But note: we take the six stress components relative to the three orthogonal, let's call

2. Much as we have done in the preceding exercise. Our analysis shows that the maximum normal stress acts on the horizontal plane, defined by $\phi = 0$. The maximum shear stress, on the other hand acts on a plane oriented at 45° to the horizontal. The factor $\cos\phi \sin\phi$ has a maximum at $\phi = 45^\circ$.

3. We take advantage of moment equilibrium and take $\sigma_{yx} = \sigma_{xy}$, $\sigma_{zx} = \sigma_{xz}$, and $\sigma_{zy} = \sigma_{yz}$.

them, x, y, z planes as given, as known quantities. Furthermore, again we restrict our attention to two dimensions - the case of Plane Stress. That is we say that the components of stress acting on one of the planes at the point - we take the z planes - are zero. This is a good approximation for certain objects — those which are *thin* in the z direction relative to structural element's dimensions in the $x-y$ plane. It also, makes our derivation a bit less tedious, though there is nothing conceptual complex about carrying it through for three dimensions, once we have it for two.

In two dimensions we can draw a simpler picture of *the state of stress at a point*. We are not talking differential element here but of stress at a point. The figure below shows an arbitrarily oriented plane, defined by its normal, the x' axis, inclined at an angle ϕ to the horizontal. In this two dimensional state of stress we



have but three scalar components to specify to fully define the state of stress at a point: σ_x , σ_y and $\sigma_{yx} = \sigma_{xy}$. Knowing these three numbers, we can determine the normal and shear stress components acting on any plane defined by the orientation ϕ as follows.

Consider equilibrium of the shaded wedge shown. Here we let A_ϕ designate the area of the inclined face at a point, A_x and A_y the areas of the x face with its outward normal pointing in the $-x$ direction and of the y face with its outward normal pointing in the $-y$ direction respectively. In this we take a unit depth into the paper. We have

$$A_x = A_\phi \cdot \cos\phi \quad \text{and} \quad A_y = A_\phi \cdot \sin\phi$$

That takes care of the relative areas. Now for *force* equilibrium, in the x and y directions we must have:

$$-\sigma_x \cdot A_x - \sigma_{xy} \cdot A_y + (\sigma'_x \cdot \cos\phi - \sigma'_{xy} \cdot \sin\phi) \cdot A_\phi = 0$$

and

$$-\sigma_{xy} \cdot A_x - \sigma_y \cdot A_y + (\sigma'_x \cdot \sin\phi + \sigma'_{xy} \cdot \cos\phi) \cdot A_\phi = 0$$

If we multiply the first by $\cos\phi$, the second by $\sin\phi$ and add the two we can eliminate σ'_{xy} . We obtain

$$\sigma'_x A_\phi - \sigma_x \cos\phi A_x - \sigma_{xy} \cos\phi A_y - \sigma_{xy} \sin\phi A_x - \sigma_y \sin\phi A_y = 0$$

which, upon expressing the areas of the x,y faces in terms of the area of the inclined face, can be written (noting $A\phi$ becomes a common factor).

$$\sigma'_x = \sigma_x \cos^2 \phi + \sigma_y \sin^2 \phi + 2\sigma_{xy} \sin \phi \cos \phi$$

In much the same way, multiplying the first equilibrium equation by $\sin \phi$, the second by $\cos \phi$ but subtracting rather than adding you will obtain eventually

$$\sigma'_{xy} = (\sigma_y - \sigma_x) \sin \phi \cos \phi + \sigma_{xy} (\cos^2 \phi - \sin^2 \phi)$$

We deduce the normal stress component acting on the y' face of this rotated frame by replacing ϕ in our equation for σ'_x by $\phi + \pi/2$. We obtain in this way:

$$\sigma'_y = \sigma_y \cos^2 \phi + \sigma_x \sin^2 \phi - 2\sigma_{xy} \sin \phi \cos \phi$$

The three transformation equations for the three components of stress at a point can be expressed, using the double angle formula for the cosine and the sine, as

$$\begin{aligned} \sigma'_x &= \left[\frac{(\sigma_x + \sigma_y)}{2} \right] + \left[\frac{(\sigma_x - \sigma_y)}{2} \right] \cdot \cos 2\phi + \sigma_{xy} \sin 2\phi \\ \sigma'_y &= \left[\frac{(\sigma_x + \sigma_y)}{2} \right] - \left[\frac{(\sigma_x - \sigma_y)}{2} \right] \cdot \cos 2\phi - \sigma_{xy} \sin 2\phi \\ \sigma'_{xy} &= - \left[\frac{(\sigma_x - \sigma_y)}{2} \right] \cdot \sin 2\phi + \sigma_{xy} \cos 2\phi \end{aligned}$$

Here we have the equations to do what we said we could do. Think of the set as a machine: You input the three components of stress at a point defined relative to an $x-y$ coordinate frame, then give me the angle ϕ , and I will crank out -- not only the normal and shear stress components acting on the face with its outward normal inclined at the angle ϕ with respect to the x axis, but the normal stress on the y' face as well. In fact I could draw a square tilted at an angle ϕ to the horizontal and show the stress components σ'_x, σ'_y and σ'_{xy} acting on the x' and y' faces.

To show the utility of these relationships consider the following scenario:

Exercise 4.3

An solid circular cylinder made of some brittle material is subject to pure torsion — a torque M_t . If we assume that a shear stress $\tau(r)$ acts within the cylinder, distributed over any cross section, varying with r according to

$$\tau(r) = c \cdot r^n$$

where n is a positive integer, then the maximum value of τ , will occur at the outer radius of the shaft.

But is this the maximum value? That is, while certainly r^n is maximum at the outermost radius, $r=R$, it may very well be that the maximum shear stress acts on some other plane at that point in the cylinder.

Show that the maximum shear stress is indeed that which acts on a plane normal to the axis of the cylinder at a point on the surface of the shaft.

Show too, that the maximum normal stress in the cylinder acts

- at a point on the surface of the cylinder
- on a plane whose normal is inclined 45° to the x axis and its value is $\sigma'_x|_{max} = \tau(R)$

We put to use our machinery for computing the stress components acting upon an arbitrarily oriented plane at a point. Our initial set of stress components for this particular state of stress is

$$\sigma_x = 0$$

$$\sigma_y = 0$$

and

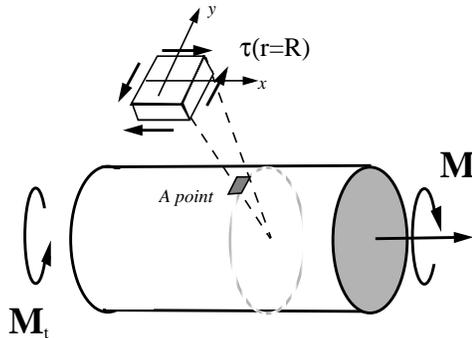
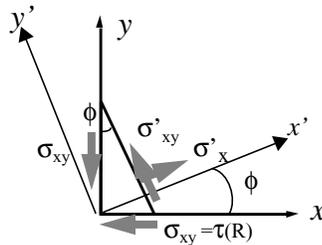
$$\sigma_{xy} = \tau(R)$$

defined relative to the x - y coordinate frame shown top right. Our equations defining the transformation of components of stress at the point take the simpler form

$$\sigma'_x = \tau \cdot \sin 2\phi$$

$$\sigma'_y = -\tau \cdot \sin 2\phi$$

$$\sigma'_{xy} = \tau \cdot \cos 2\phi$$



To find the maximum value for the shear stress component with respect to the plane defined by ϕ , we set the derivative of σ'_{xy} to zero. Since there are no “boundaries” on ϕ to worry about, this ought to suffice.

So, for a maximum, we must have

$$\frac{d\sigma'_{xy}}{d\phi} = -2\tau \cdot \sin 2\phi = 0$$

Now there are many values of ϕ which satisfy this requirement, $\phi=0, \phi=\pi/2, \dots$. But all of these roots just give the orientation of the of our initial two mutually perpendicular, x - y planes. Hence the maximum shear stress within the shaft is just τ at $r=R$.

To find the extreme, including maximum, values for the normal stress, σ'_x we proceed in much the same way; differentiating our expression above for σ'_x with respect to ϕ yields

$$\frac{d\sigma'_x}{d\phi} = 2\tau \cdot \cos 2\phi = 0 \tag{EQ 1}$$

Again there is a string of values of ϕ , each of which satisfies this requirement.

We have $2\phi = \pi/2, 3\pi/2 \dots$ or $\phi = \pi/4, 3\pi/4$

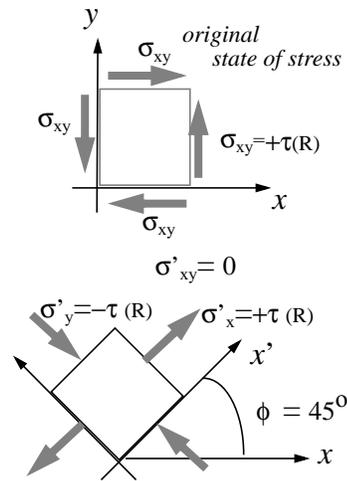
At $\phi = \pi/4$ ($= 45^\circ$), the value of the normal stress is $\sigma'_x = +\tau \sin 2\phi = \tau$. So the maximum normal stress acting at the point on the surface is equal in magnitude to the maximum shear stress component.

Note too that our transformation relations say that the normal stress component acting on the y' plane, with $\phi=\pi/4$ is negative and equal in magnitude to τ . Finally we find that the shear stress acting on the x' - y' planes is zero! We illustrate the state of stress at the point relative to the x' - y' planes below right.

Backing out of the woods in order to see the trees, we claim that if our cylinder is made of a brittle material, it will fracture across the plane upon which the maximum tensile stress acts. If you go now and take a piece of chalk and subject it to a torque until it breaks, you should see a fracture plane in the form of a helical surface inclined at 45 degrees to the axis of the cylinder. Check it out.

Of course it's not enough to know the orientation of the fracture plane when designing brittle shafts to carry torsion. We need to know the *magnitude of the torque* which will cause fracture. In other words we need to know how the shear stress does *in fact* vary throughout the cylinder.

This remains an unanswered question. So too for the beam: How do the normal stress (and shear stress) components vary over a cross section of the beam? In a subsequent section, we explore how far we can go with equilibrium consideration in responding. But, in the end, we will find that the problem remains statically indeterminate; we will have to go beyond the concept of stress and consider the deformation and displacement of points in the continuum. But first, a special technique for doing the transformation of components of stress at a point. "Mohr's



Circle” is a graphical technique which, while offering no new information, does provide a different and useful perspective on our subject.

Studying Mohr’s Circle is customarily the final act in this first stage of indoctrination into concept of stress. Your uninitiated colleagues may be able to master the idea of a truss member in tension or compression, a beam in bending, a shaft in torsion using their common sense knowledge of the world around them, but Mohr’s Circle will appear as a complete mystery, an unfathomable ritual of signs, circles, and greek symbols. Although it does not tell us anything new, over and above all that we have done up to this point in the chapter, once you’ve mastered the technique it will set you apart from the crowd and shape your very well being. It may also provide you with a useful aid to understanding the transformation of stress and strain at a point on occasion.

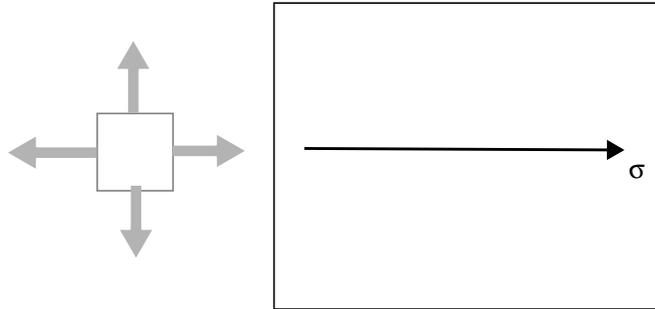
Mohr’s Circle

Our working up of the transformation relations for stress and our exploration of their implications for determining extreme values has required considerable mathematical manipulation. We turn now to a graphical rendering of these relationships. I will set out the rules for constructing the circle for a particular state of stress, show how to read the pattern, then comment about its legitimacy. I first repeat the transformation equations for a two-dimensional state of stress.

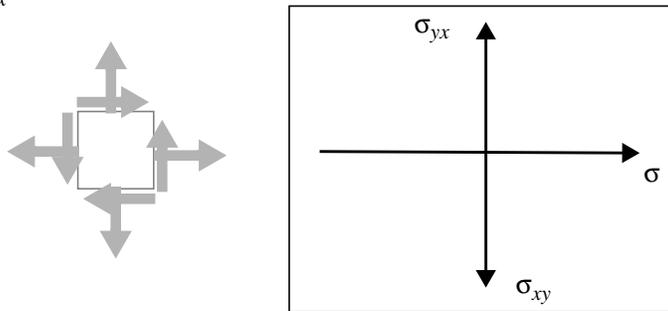
$$\begin{aligned}\sigma'_x &= \left[\frac{(\sigma_x + \sigma_y)}{2} \right] + \left[\frac{(\sigma_x - \sigma_y)}{2} \right] \cdot \cos 2\phi + \sigma_{xy} \sin 2\phi \\ \sigma'_y &= \left[\frac{(\sigma_x + \sigma_y)}{2} \right] - \left[\frac{(\sigma_x - \sigma_y)}{2} \right] \cdot \cos 2\phi - \sigma_{xy} \sin 2\phi \\ \sigma'_{xy} &= - \left[\frac{(\sigma_x - \sigma_y)}{2} \right] \cdot \sin 2\phi + \sigma_{xy} \cos 2\phi\end{aligned}$$

To construct Mohr’s Circle, given the state of stress $\sigma_x = 7$, $\sigma_{xy} = 4$, and $\sigma_y = 1$ we proceed as follows: Note that I have dropped all pretense of reality in this choice of values for the components of stress. As we shall see, it is their relative magnitudes that is important to this geometric construction. Everything will scale by any common factor you please to apply. You could think of these as $\sigma_x = 7 \times 10^3 \text{ KN/m}^2 \dots$ etc., if you like.

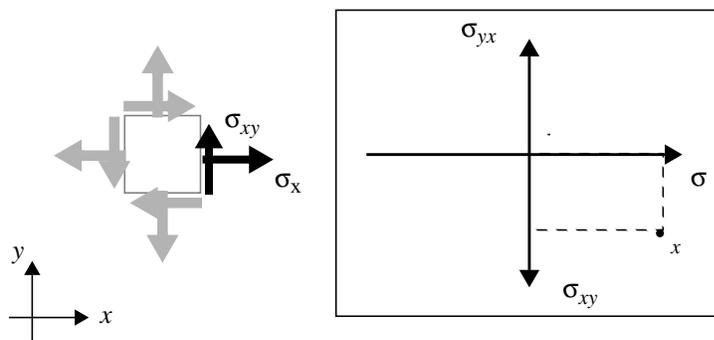
- Lay out a horizontal axis and label it σ positive to the right.



- Lay out an axis perpendicular to the above and label it σ_{xy} positive down and σ_{yx} positive up⁴.

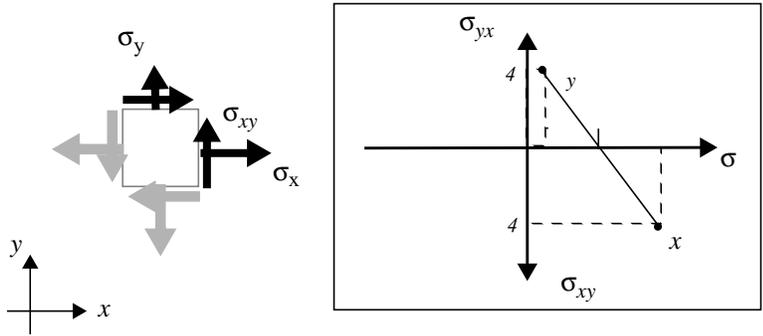


- Plot a point associated with the stress components acting on an x face at the coordinates $(\sigma_x, \sigma_{xy}) = (7, 4_{down})$. Label it x_{face} , or x if you are cramped for space.

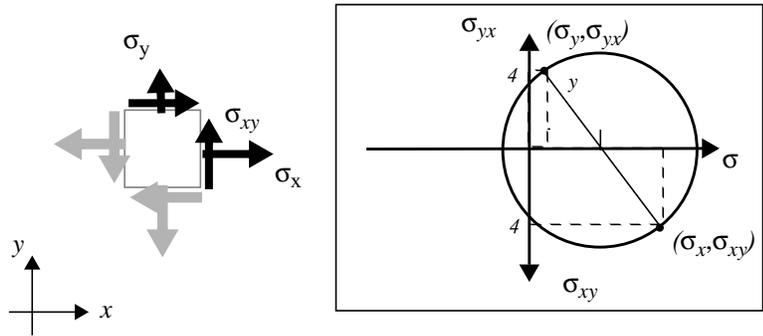


4. WARNING: Different authors and engineers use different conventions in constructing the Mohr's circle.

- Plot a second point associated with the stress components acting on an y face at the coordinates $(\sigma_y, \sigma_{yx}) = (1, 4)$. Label it y , or y if you are cramped for space. Connect the two points with a straight line. Note the order of the subscripts on the shear stress.

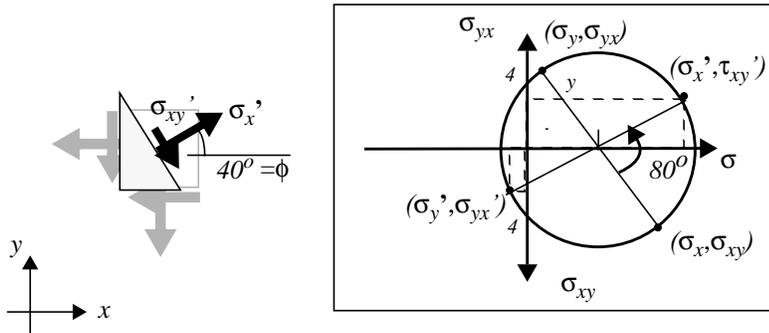


- Chanting “similar triangles”, note that the center of the line must necessarily lie on the horizontal, σ axis since $\sigma_{xy} = \sigma_{yx}$, $4 = 4$. Draw a circle with the line as a diagonal.



- Note that the radius of this circle is $R_{Mohr's} = \sqrt{(\sigma_{xy})^2 + [(\sigma_x - \sigma_y)/2]^2}$ which for the numbers we are using is just $R_{Mohr's} = 5$, and its center lies at $(\sigma_x + \sigma_y)/2 = 4$.
- To find the stress components acting on a plane whose normal is inclined at an angle of ϕ degrees, positive counterclockwise, to the x axis **in the physical plane**, rotate the diagonal 2ϕ **in the Mohr's Circle plane**. We illustrate this for $\phi = 40^\circ$. Note that the shear stress on the new x' face is

negative according to the convention we have chosen for our Mohr's Circle.⁵

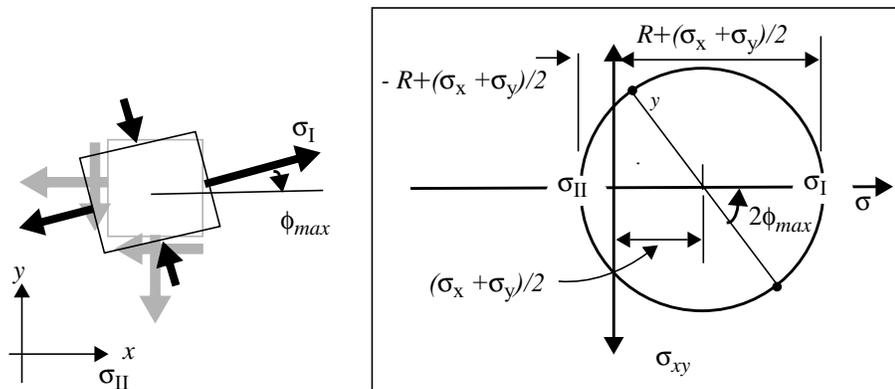


- The stress components acting on the y' face, at $\phi + \pi/2 = 130^\circ$ around in the **physical plane** are $2\phi + \pi = 240^\circ$ around in the **Mohr's Circle plane**, just 2ϕ around from the y face in the Mohr's Circle plane.

We establish the legitimacy of this graphical representation of the transformation equations for stress making the following observations:

- The extreme values of the normal stress lie at the two intersections of the circle with the σ axis. The angle of rotation from the x_{face} to the principal plane I on the Mohr's Circle is related to the stress components by the equation previously derived:

$$\tan 2\phi = 2\sigma_{xy} / (\sigma_x - \sigma_y).$$



5. WARNING, again: Other texts use other conventions.

- Note that on the principal planes the shear stress vanishes.
- The values of the two principal stresses can be written in terms of the radius of the circle.

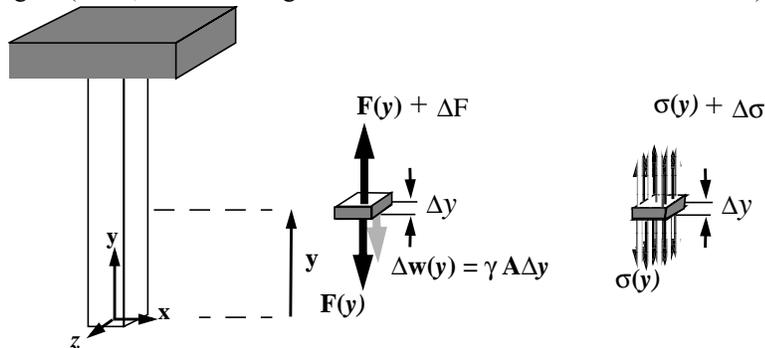
$$\sigma_{I,II} = [(\sigma_x + \sigma_y)/2] \pm \sqrt{(\sigma_{xy})^2 + [(\sigma_x - \sigma_y)/2]^2}$$

- The orientation of the planes upon which an extreme value for the shear stress acts is obtained from a rotation of 90° around from the σ axis on the Mohr's Circle. The corresponding rotation in the physical plane is 45° .
- The sum $\sigma_x + \sigma_y$ is an invariant of the transformation. The center of the Mohr's Circle does not move. This result too can be obtained from the equations derived simply by adding the expression for σ_x' to that obtained for σ_y' .
- So too the radius of the Mohr's Circle is an invariant. This takes a little more effort to prove.

Enough. Now onto the second topic of the chapter - the variation of stress components as we move throughout the continuum. This is prerequisite if we seek to find extreme values of stress.

4.2 The Variation of Stress (Components) in a Continuum

To begin, we re-examine the case of a bar suspended vertically but now consider the state of stress at each and every point in the continuum engendered by its own weight. (Note, I have changed the orientation of the reference axes). We will



construct a differential equation which governs how the axial stress varies as we move up and down the bar. We will solve this differential equation, not forgetting to apply an appropriate *boundary condition* and determine the axial stress field.

We see that for equilibrium of the differential element of the bar, of planar cross-sectional area A and of weight density γ , we have

$$F + \Delta F - \gamma \cdot A \cdot \Delta y - F = 0$$

If we assume the tensile force is uniformly distributed over the cross-sectional area, and dividing by the area (which does not change with the independent spatial coordinate y) we can write

$$\sigma + \Delta\sigma - \gamma \cdot \Delta y - \sigma = 0 \quad \text{where} \quad \sigma \equiv F/A$$

Chanting “...going to the limit, letting Δy go to zero”, we obtain a differential equation fixing how $\sigma(y)$, a function of y , varies throughout our continuum, namely

$$\frac{d\sigma}{dy} - \gamma = 0$$

We solve this ordinary differential equation easily, integrating once and obtain

$$\sigma(y) = \gamma \cdot y + \text{Constant}$$

The *Constant* is fixed by a prescribed condition at some y surface; If the end of the bar is *stress free*, we indicate this writing

$$\text{at } y = 0 \quad \sigma = 0$$

so

$$\sigma(y) = \gamma \cdot y$$

If, on another occasion, a weight of magnitude P_0 is suspended from the free end, we would have

$$\text{at } y = 0 \quad \sigma = P_0/A$$

and

$$\sigma(y) = \gamma \cdot y + P_0/A$$

Here then are two stress fields for two different loading conditions⁶. Each stress field describes how the normal stress $\sigma(x,y,z)$ varies throughout the continuum at every point in the continuum. I show the stress as a function of x and z as well as y to emphasize that we can evaluate its value at *every point* in the continuum, although it only varies with y . That the stress does not vary with x and z was implied when we stipulated or assumed that the internal force, F , acting upon any y plane was uniformly distributed over that plane. This example is a special case in another way; not only is it one-dimensional in its dependence upon spatial position, but it is the simplest example of stress at a point in that it is described fully by a single component of stress, the normal stress acting on a plane perpendicular to the y axis.

6. A third loading condition is obtained by setting the weight density γ to zero; our bar then is assumed weightless relative to the end-load P_0 .

Stress Fields & “Micro” Equilibrium

In our analysis of how the normal stress varied throughout the vertically suspended bar, we considered a differential element of the bar and constructed a differential equation which described how the normal stress component varied in one direction, in one spatial dimension. We can call this picture of equilibrium “micro” in nature and distinguish it from the “macro” equilibrium considerations of the last chapter. There we isolated large chunks of structure e.g., when we cut through the beam to see how the shear force and bending moment varied with distance along the beam.

Now we look with finer resolution and attempt to determine how the normal and shear stress components vary at the micro level throughout the beam. The question may be put this way: Knowing the shear force and bending moment at any section along the beam, how do the normal and shear stress components vary over the section?

To proceed, we make some appropriate assumptions about the nature of the beam and build upon the conjectures we made in the last chapter about how the stress components might vary.

We model the end-loaded cantilever with relatively thin rectangular cross-section as a plane stress problem. In this, b is the “thin” dimension, i.e., $b/L < 1$.

If we assume a normal stress distribution over an x face is proportional to some odd power of y , as we did in Chapter 3, our state of stress at a point might look like that shown in figure (c). In this, σ_x would have the form

$$\sigma_x(x, y) = C(n, b, h) \cdot W(L-x)y^n$$

where $C(n,b,h)$ is a constant which depends upon the cross-sectional dimensions of the beam and the odd exponent n . The factor $W(L-x)$ is the magnitude of the internal bending moment at the location x measured from the root. See figure (b).

But this is only one component of our stress field. What are the other components of stress at point A?

Our plane stress model allows us to claim that the three z face components are zero and if we take σ_{yz} and σ_{xz} to be zero, that still leaves σ_{xy} and σ_y in addition to σ_x .

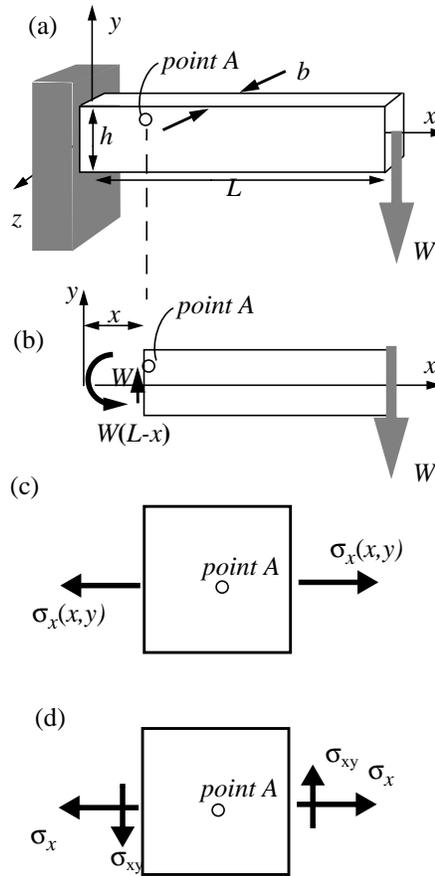
To continue our estimation process, we make the most of what we already know: For example, we know that a shear force of magnitude W acts at any x section. For the end-loaded cantilever, neglecting the weight of the beam itself, it does not vary with x . We might assume, then, that the shear force is uniformly distributed over the cross-section and set

$$\sigma_{xy} = -W/(bh) \quad \text{Our stress at a point at point A would then look like}$$

figure (d).

We could, of course, posit other shear stress distributions at any x station, e.g., some function like $\sigma_{xy} = \text{Constant} \cdot y^m$ where m is an integer and the constant is determined from the requirement that the resultant force due to this shear stress distribution over the cross section must be W .

The component σ_y - how it varies with x and y - remains a complete unknown. We will argue that it is small, relative to the normal and shear stress components, moved by the observation that the normal stress on the top and the



bottom surfaces of the beam is zero; we say the top and bottom surfaces are “stress free”. Continuing, if σ_y vanishes there at the boundary, then it probably will not grow to be significant in the interior. This indeed can be shown to be the case if $h/L \ll 1$, as it is for a beam. So we estimate $\sigma_y=0$.

But there is something more we can do. We can look at equilibrium of a differential element within the beam and, as we did in the case of a bar hung vertically, construct a differential equation whose solution (supplemented with suitable boundary conditions) defines how the normal and shear stress components vary throughout the plane, with x and y . Actually we construct more than a single differential equation: We obtain two, coupled, first-order, partial differential equations for the normal and shear stress components.

Think now, of a differential element in 2D at any point within the cantilever beam: We show such on the right. Note now we are no longer focused on two intersecting, perpendicular planes at a point but on a differential element of the continuum. Now we see that the stress components may very well be different on the two x faces and on the two y faces.

We allow the x face components, and those on the two y faces to change as we move from x to $x+\Delta x$ (holding y constant) and from y to $y+\Delta y$ (holding x constant).

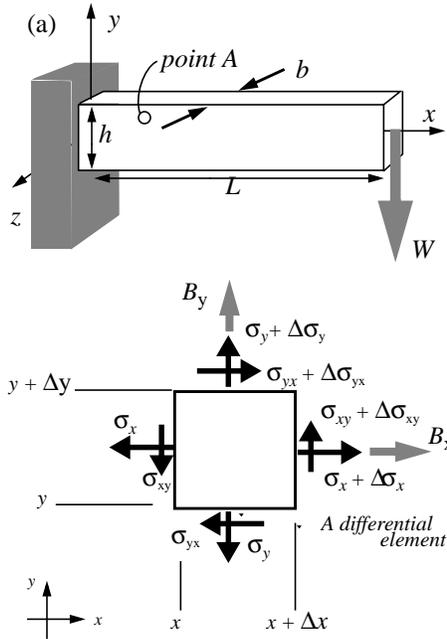
We show two other arrows on the figure, B_x and B_y . These are meant to represent the x and y components of what is called a *body force*. A body force is any externally applied force acting on each element of volume of the continuum. It is thus a force per unit volume. For example, if we need consider the weight of the beam, B_y would be just

$$B_y = -\gamma \quad \text{where} \quad \gamma = \text{the weight density}$$

where the negative sign is necessary because we take a positive component of the body force vector to be in a positive coordinate direction.

B_x would be taken as zero.

We now consider force and moment equilibrium for this differential element, our micro isolation. We sum forces in the x direction which will include the shear stress component σ_{yx} , acting on the y face in the x direction as well as the normal



stress component σ_x acting on the x faces. But note that these components are not forces; to figure their contribution to the equilibrium requirement, we must factor in the areas upon which they act.

I present just the results of the limiting process which, we note, since all components may be functions of both x and y , brings partial derivatives into the picture.

\sum Forces in the x direction \rightarrow	$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + B_x = 0$
\sum Forces in the y direction \rightarrow	$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + B_y = 0$
\sum Moments about the center of the element \rightarrow	$\sigma_{yx} = \sigma_{xy}$

For example, the change in the stress component $\Delta\sigma_x$ may be written

$$\Delta\sigma_x = \frac{\partial \sigma_x}{\partial x} \cdot \Delta x$$

and the force due to this “unbalanced” component in the x direction is

$$\frac{\partial \sigma_x}{\partial x} \cdot \Delta x \cdot (\Delta y \Delta z)$$

where the product, $\Delta y \Delta z$, is just the differential area of the x face.

The contribution of the body force (per unit volume) to the sum of force components in the x direction will be $B_x(\Delta x \Delta y \Delta z)$ where the product of deltas is just the differential volume of the element. We see that this product will be a common factor in all terms entering into the equations of force equilibrium in the x and y directions.

The last equation of moment equilibrium shows that, as we forecast, the shear stress component on the y face must equal the shear stress component acting on the x face. The differential changes in the shear stress components are of lower order and drop out of consideration in the limiting process, as we take Δx and Δy to zero.

We might now try to solve this system of differential equations for σ_{xy} and σ_y and σ_x but, in fact, we are doomed from the start. Even with the simplification afforded by moment equilibrium we are left with two coupled, linear, first-order

partial differential equations for these three unknowns. The problem is statically indeterminate so we are not going to be able to construct a unique solution to the equilibrium requirements.

To answer these questions we must go beyond the concepts and principles of static equilibrium. We have to consider the requirements of continuity of displacement and compatibility of deformation. This we do in the next chapter, looking first at simple indeterminate systems, then on to the indeterminate truss, the beam in bending and the torsion of shafts.

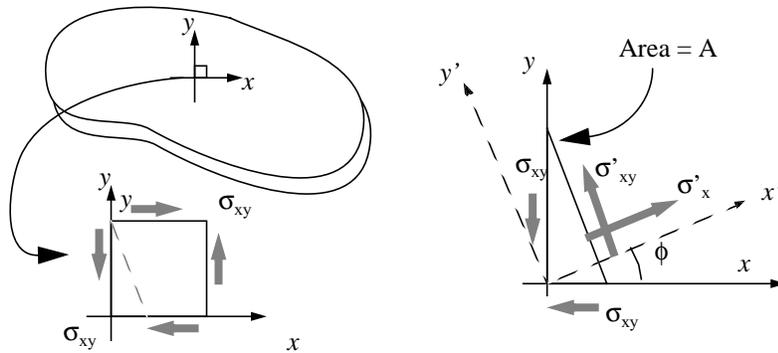
4.3 Problems

4.1 A fluid can be defined as a continuum which - unlike a solid body - is unable to support a shear stress and remain at rest. The state of stress at any point, within a fluid column for example, we label “hydrostatic”; the normal stress components are equal to the negative of the static pressure at the point and the shear stress components are all zero. $\sigma_x = \sigma_y = \sigma_z = -p$ and $\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$.

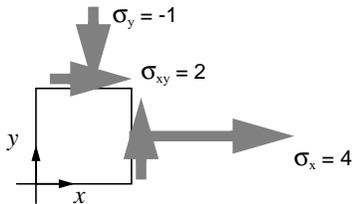
Using the two dimensional transformation relations (the existence of σ_z does not affect their validity) show that the shear stress on any arbitrarily oriented plane is zero and the normal stress is again $-p$.

4.2 Estimate the compressive stress at the base of the Washington Monument - the one on the Mall in Washington, DC.

4.3 The stress at a point in the plane of a thin plate is shown. Only the shear stress component is not zero relative to the x - y axis. From equilibrium of a section cut at the angle ϕ , deduce expressions for the normal and shear stress components acting on the inclined face of area A . NB: stress is a force per unit area so the areas of the faces the stress components act upon must enter into your equilibrium considerations.



4.4 Construct Mohr's circle for the state of stress of exercise 4.3, above. Determine the "principle stresses" and the orientation of the planes upon which they act relative to the xy frame.



4.5 Given the components of stress relative to an x - y frame at a point in plane stress are:
 $\sigma_x = 4, \quad \sigma_{xy} = 2 \quad \sigma_y = -1$

What are the components with respect to an axis system rotated 30 deg. counter clockwise at the point?

Determine the orientation of axis which yields maximum and minimum normal stress components. What are their values?

4.6 A thin walled glass tube of radius $R = 1$ inch, and wall thickness $t = 0.010$ inches, is closed at both ends and contains a fluid under pressure, $p = 100$ psi. A torque, M_t , of 300 inch-lbs, is applied about the axis of the tube.

Compute the stress components relative to a coordinate frame with its x axis in the direction of the tube's axis, its y axis circumferentially directed and tangent to the surface.

Determine the maximum tensile stress and the orientation of the plane upon which it acts.

4.7 What if we change our sign convention on stress components so that a normal, compressive stress is taken as a positive quantity (a tensile stress would then be negative). What becomes of the transformation relations? How would you alter the rules for constructing and using a Mohr's circle to find the stress components on an arbitrarily oriented plane?

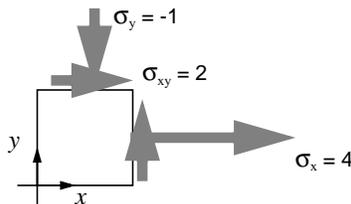
What if you changed your sign convention on shear stress as well; how would things change?

4.8 Given the components of stress relative to an x - y frame at a point in plane stress are:

$$\sigma_x = 4, \quad \sigma_{xy} = 2 \quad \sigma_y = -1$$

What are the components with respect to an axis system rotated 30 deg. counter clockwise at the point?

Determine the orientation of axis which yields maximum and minimum normal stress components. What are their values?



4.9 Estimate the "hoop stress" within an un-opened can of soda.